Thick-walled Pressure Vessel

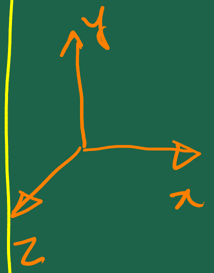
$t \not\ll R$

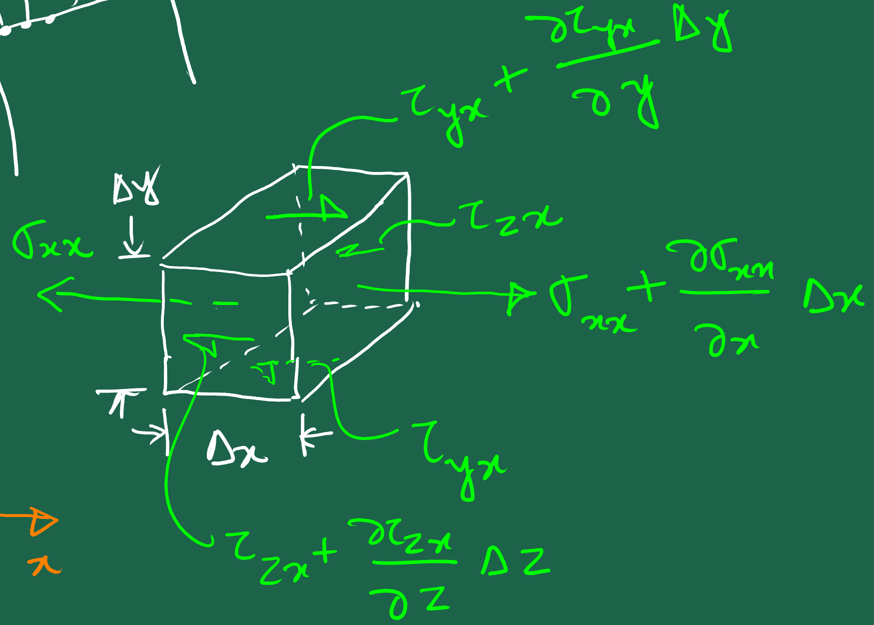
$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}$$

$$\rho \underbrace{\cancel{\frac{D\vec{v}}{Dt}}}_{0} = \nabla\cdot\underset{\approx}{\sigma} + \underbrace{\cancel{\rho\vec{b}}}_{0}$$

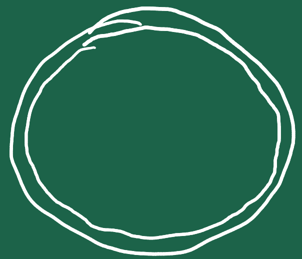
$$\Rightarrow \nabla\cdot\underset{\approx}{\sigma} = 0$$

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0$$





$$\left(\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}\Delta x\right) - \sigma_{xx} + \left(\tau_{yx} + \frac{\partial\tau_{yx}}{\partial y}\Delta y\right) - \tau_{yx}$$
$$+ \left(\tau_{zx} + \frac{\partial\tau_{zx}}{\partial z}\Delta z\right) - \tau_{zx} = 0$$



$$\left.\begin{aligned}\sigma_1 &= \frac{pr}{t}\\ \sigma_2 &= \frac{pr}{2t}\end{aligned}\right\}\ \text{Cyl}$$

$$\sigma = \frac{pr}{t}\ \Big\}\ \text{Sph}$$

$\sigma_{xx}(x+\Delta x)$

$$\left(\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}\Delta x\right)-\sigma_{xx}+\left(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\Delta y\right)-\tau_{yx}$$

$$+\left(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\Delta z\right)-\tau_{zx}=0$$

$$\Rightarrow\quad \frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0 \quad (\text{Force balance in } x \text{ dir}^n)$$

Similarly

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=0 \quad (\text{" " " } y \text{ "})$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0 \quad (\text{" " " } z \text{ "})$$

Stress eqb. eqns.

$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}$

Stress eqb. eqn in r-dir^n

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left[\frac{\partial \tau_{r\theta}}{\partial \theta} + \sigma_{rr} - \sigma_{\theta\theta}\right] + \frac{\partial \tau_{zr}}{\partial z} = 0$$

θ : - - - - - - - - - - - - - - -

z : - - - - - - - - - - - - - - -

3 eqns

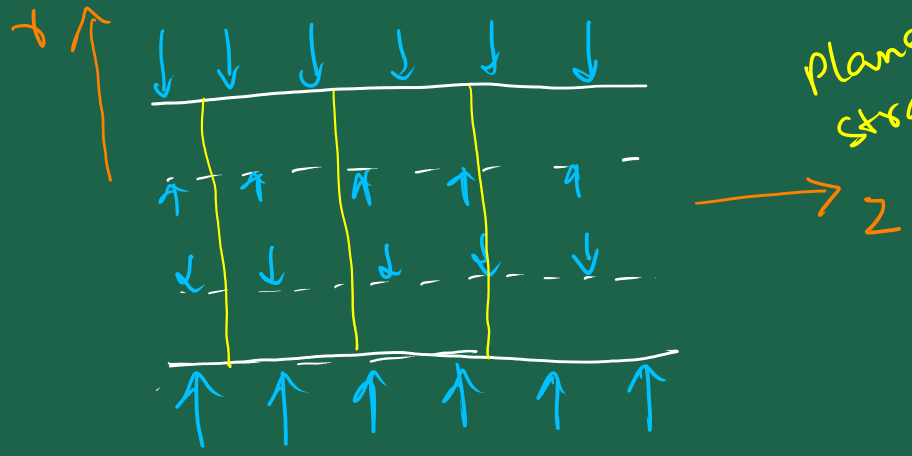
6 unknowns
(6 stress comp.)

Strain-Displacement } 6 eqns
6 ε , 3 : u, v, w → 9 unknowns

9 eqns
15 unknowns

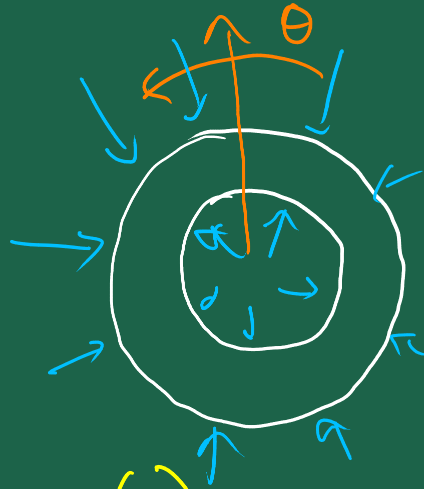
Generalized Hooke's Law } → 6 eqns

15 eqns
15 unknowns



plane strain

$$\begin{cases} u \equiv u(r,\theta,\cancel{z}) \equiv u(r,\theta) \\ v \equiv v(r,\theta,\cancel{z}) \equiv v(r,\theta) \\ w \equiv w(r,\theta,\cancel{z}) = 0 \end{cases}$$



Axisymmetry : $\frac{\partial}{\partial\theta}$ $\Leftrightarrow$ No θ-dependence $\Rightarrow$

$$u \equiv u(r,\cancel{\theta}) \equiv u(r)$$
$$v \equiv v(r,\cancel{\theta}) \equiv v(r)$$

Loading purely in radial dir^n $\Rightarrow$ $v = 0$

$$\boxed{u \equiv u(r) \; , \; v = 0 \; , \; w = 0}$$

Plane strain: $\varepsilon_{zz} = 0$, $\varepsilon_{z\theta} = 0$, $\varepsilon_{zr} = 0$

$$\Rightarrow \tau_{zr} = 2G\,\underbrace{\varepsilon_{zr}}_{=0}$$

Stress eqb. eqn in r-dir^n

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\left[\underbrace{\frac{\partial \tau_{r\theta}}{\partial \theta}}_{=0} + \sigma_{rr} - \sigma_{\theta\theta}\right] + \frac{\partial \cancelto{0}{\tau_{zr}}}{\partial z} = 0$$

$$\Rightarrow \boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0}$$

$$\cancelto{0}{\varepsilon_{zz}} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] \Rightarrow \sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$$

$$\varepsilon_{rr} = \frac{1}{E}\left[\sigma_{rr} - \nu(\sigma_{\theta\theta} + \sigma_{zz})\right]$$

$$\sigma_{zz} = E(\sigma_{rr} + \sigma_{\theta\theta})$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{zz})\right]$$

$$\varepsilon_{rr} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right]$$

$$\varepsilon_{\theta\theta} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right]$$

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right]$$

Now, we use the strain-displacement relations (simplified)

$$\varepsilon_{rr} = \frac{du}{dr}, \quad \varepsilon_{\theta\theta} = \frac{u}{r}$$

$u \equiv u(r)$

$$\sigma_{rr} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{du}{dr} + \nu\frac{u}{r}\right]$$

$$\sigma_{\theta\theta} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{u}{r} + \nu\frac{du}{dr}\right]$$

e^{mx}

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0$$

$$\Rightarrow \boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0}$$

← 2nd order, linear diff. eqn (ODE)

$\frac{du}{dr}\frac{du}{dr}$ X

$u\frac{du}{dr}$ X

$u = r^m$

$$m(m-1)r^{m-2} + m r^{m-2} - r^{m-2} = 0$$

$$m(m-1)r^{m-2} + m r^{m-2} - r^{m-2} = 0$$

$$\Rightarrow \quad m(m-1) + m - 1 = 0$$

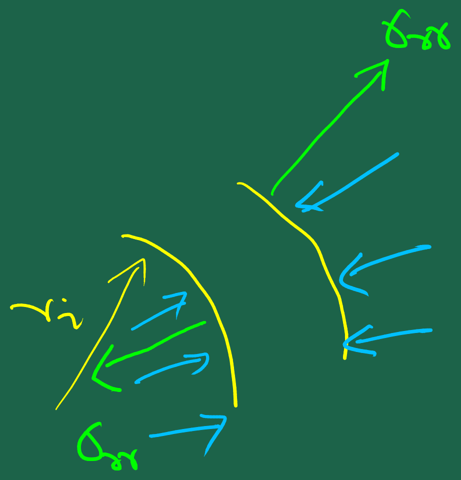
$$\Rightarrow \quad m^2 - 1 = 0$$

$$\Rightarrow \quad m = \pm 1$$

$$u = A_1 r^1 + A_2 r^{-1} \Rightarrow u = A_1 r + \frac{A_2}{r}$$

For the B.C.s reqd to solve for A_1 and A_2,

$$\sigma_{rr}\Big|_{r=r_i} = -p_i \quad ; \quad \sigma_{rr}\Big|_{r=r_o} = -p_o$$



$$
\left\{
\begin{aligned}
-p_i &= \frac{E}{(1+\nu)(1-2\nu)}\left[A_1 - (1-2\nu)\frac{A_2}{r_i^2}\right] \\
-p_o &= \frac{E}{(1+\nu)(1-2\nu)}\left[A_1 \pm (1-2\nu)\frac{A_2}{r_o^2}\right]
\end{aligned}
\right.
$$

$\rightarrow$ Find A_1 and A_2

$$\sigma_{rr} = C_1 - \frac{C_2}{r^2}, \quad \sigma_{\theta\theta} = C_1 + \frac{C_2}{r^2}$$

$\rightarrow \sigma_{rr} + \sigma_{\theta\theta} = 2C_1$
$= \text{const.}$

where

$$C_1 = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2}, \quad C_2 = \frac{(p_i - p_o)\, r_i^2 r_o^2}{r_o^2 - r_i^2}$$

$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$
$= \text{const.}$