Energy Methods

Strain Energy

1st law of Thermodynamics :

$$W + \cancel{Q} = \Delta U + \cancel{\Delta K}$$

Adiabatic : $Q = 0$

Static equilibrium : $\Delta K = 0$

$$W = \Delta U$$

Mechanical system is said to be elastic if <u>the internal forces are conservative</u>.

If forces are conservative then some kind of potential energy will be associated with it.

∴ The conservative internal forces in an elastic mechanical must also have a <u>potential energy</u> associated.

↳ strain energy

The strain energy differs from the internal energy only by an additive constant

$$\Delta \bar{U} = \Delta U$$

$$\bar{U} = U + C$$

$$W = \Delta U$$

↳ change in strain energy

$$W = \int_V \rho \vec{b} \cdot \vec{u} \, dV + \int_S \vec{T} \cdot \vec{u} \, dS$$

Consider simple cases:

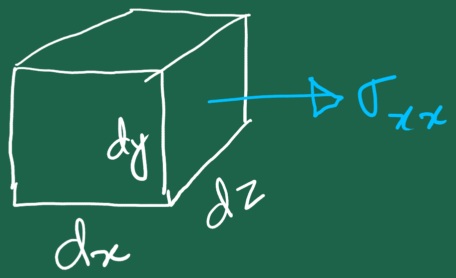
$$W = \int_0^{\Delta} F \, dx = \int_0^{\Delta} P \frac{x}{\Delta} \, dx$$

Assume force is linearly related to the deformation

$$F = P \frac{x}{\Delta}$$

$$W = \frac{P}{\Delta}\frac{\Delta^2}{2} = \frac{1}{2}P\Delta = \Delta U$$

Due to normal stresses



$$dU = \frac{1}{2}\underbrace{\sigma_{xx}\,dy\,dz}_{\text{force}}\;\underbrace{\varepsilon_{xx}\,dx}_{\text{displ.}}$$

$$= \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dx\,dy\,dz$$

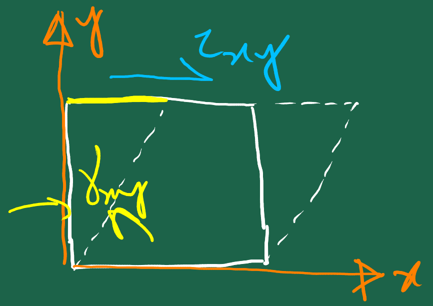
$$= \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dV$$

$$U_0 = \frac{dU}{dV} = \frac{1}{2}\sigma_{xx}\varepsilon_{xx}$$

↳ Strain energy density

Due to shear stress

$$dU = \frac{1}{2}\underbrace{\tau_{xy}\,dx\,dz}_{\text{force}}\ \underbrace{\gamma_{xy}\,dy}_{\text{displ.}}$$



$$= \frac{1}{2}\tau_{xy}\gamma_{xy}\,dx\,dy\,dz$$

$$= \frac{1}{2}\tau_{xy}\gamma_{xy}\,dV$$

$$U_0 = \frac{dU}{dV} = \frac{1}{2}\tau_{xy}\gamma_{xy}$$

Strain energy for various types of loading

Axial load

$$\Delta U = \int dU = \int \frac{dU}{dV}\, dV = \int U_0\, dV$$

$$= \int \frac{1}{2} \sigma_{xx} \varepsilon_{xx}\, dV$$

$$= \int \frac{\sigma_{xx}^2}{2E}\, dV$$

$$= \int \frac{N^2}{2A^2E}\, dV = \int_0^L \frac{N^2}{2EA^2} A\, dx = \int_0^L \frac{N^2}{2AE}\, dx$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\cancel{\sigma_{yy}} + \cancel{\sigma_{zz}}\right)\right]$$

$$= \frac{\sigma_{xx}}{E}$$

Bending moment

$$\Delta U = \int \frac{1}{2}\sigma_{xx}\varepsilon_{xx}\,dV = \int \frac{\sigma_{xx}^2}{2E}\,dV$$

$$= \int \frac{M^2 y^2}{2EI^2}\,dV = \int_0^L \int_A \frac{M^2 y^2}{2EI^2}\,dA\,dx$$

$$= \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2\,dA\right)dx$$

$$= \int_0^L \frac{M^2}{2EI^2}\,I\,dx$$

$$= \int_0^L \frac{M^2}{2EI}\,dx$$

Transverse shear

$$\Delta U = \int \frac{1}{2} \tau_{xy} \gamma_{xy} \, dV$$

$$= \int \frac{\tau_{xy}^2}{2G} \, dV$$

$$= \int \frac{V^2 Q^2}{2G I^2 t^2} \, dV$$

$$= \int_0^L \int_A \frac{V^2 Q^2}{2G I^2 t^2} \, dA \, dx$$

$$= \int_0^L \frac{V^2}{2G I^2} \left(\int_A \frac{Q^2}{t^2} \, dA \right) dx$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\tau_{xy} = \frac{VQ}{It}$$

Define a form factor for shear

$$f = \frac{A}{I^2} \int_A \frac{Q^2}{t^2} \, dA$$

$$\Rightarrow \int \frac{Q^2}{t^2} \, dA = f \frac{I^2}{A}$$

$$\therefore \Delta U = \int_0^L \frac{V^2}{2GI^2} f \frac{I^2}{A} dx = \int_0^L f \frac{V^2}{2GA} dx$$

For a rectangular c/s : $f = \frac{6}{5}$ (Check yourself)

Torsion

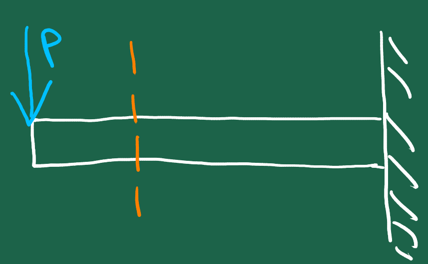
$$\Delta U = \int \frac{1}{2} \tau_{z\theta} \gamma_{z\theta} dV \qquad \tau_{z\theta} = \frac{Tr}{J}$$

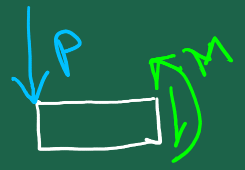
$$= \int \frac{\tau_{z\theta}^2}{2G} dV$$

$$= \int_0^L \int_0^R \frac{T^2 r^2}{2GJ^2} 2\pi r \, dr \, dz = \int_0^L \frac{T^2}{2GJ^2} \left(\int_0^R 2\pi r^3 dr \right) dz = \int_0^L \frac{T^2}{2GJ^2} \underbrace{\left(\pi \frac{R^4}{2} \right)}_{J} dz$$

$$= \int_0^L \frac{T^2}{2GJ} dz$$

Limitations of conservation of energy principle





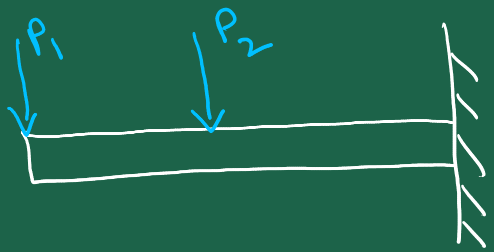
$$M = -Px$$

$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI}\,dx$$

$$= \int_0^L \frac{P^2x^2}{2EI}\,dx$$

$$= \frac{P^2}{2EI}\int_0^L x^2\,dx = \frac{P^2L^3}{6EI}$$

$$\therefore \Delta = \frac{PL^3}{3EI}$$



$$\frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 = \int_0^L \frac{M^2}{2EI}\,dx$$

We need another eqn to solve for Δ_1 & Δ_2

Principle of virtual work

$$W_e = \Delta U = -W_i$$

$$W = -\Delta U$$

$$W_e = -W_i$$

$$\Rightarrow W_e + W_i = 0 \rightarrow$$ more general

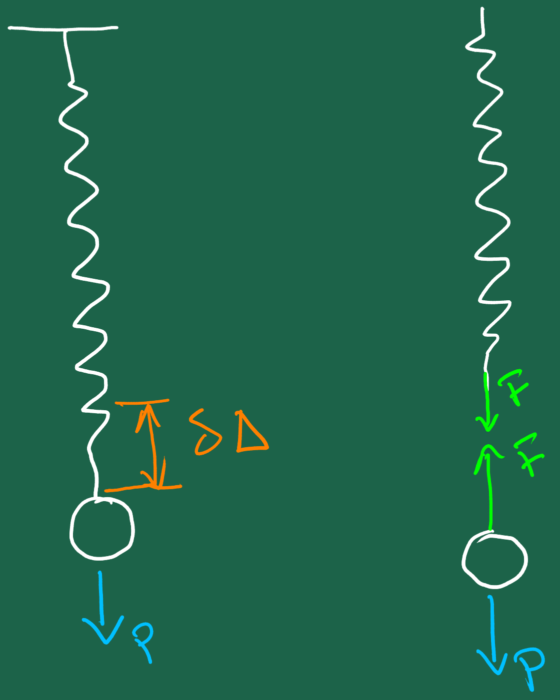
applicable even to non-elastic systems

Virtual work → Virtual displacements

Virtual work → Virtual forces

Virtual work based on virtual displacements

$W = \vec{F} \cdot \vec{r}$



$$\delta W_e = -\delta W_i$$

$$\Rightarrow P\delta\Delta = -(-F\delta\Delta)$$

$$\Rightarrow P\delta\Delta = F\delta\Delta$$

$$\Rightarrow (P-F)\delta\Delta = 0$$

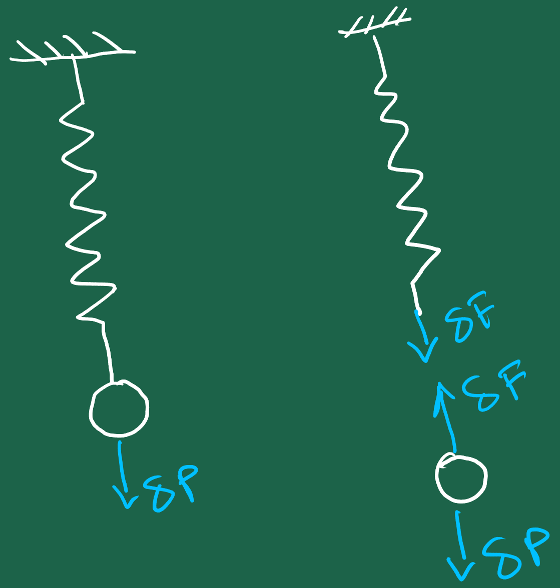
$$\Rightarrow P-F=0$$

$$\Rightarrow P=F \quad \text{(eqn of static eqb.)}$$

$$\delta W_e = -\delta W_i = \delta W_{ie}$$

$$P\delta F = F\delta\Delta$$

Virtual work based on virtual force



$$\delta W_e = -\delta W_i$$

Popov

$$\Rightarrow \delta P\,\Delta = -(-\delta F\,\Delta_{sp})$$

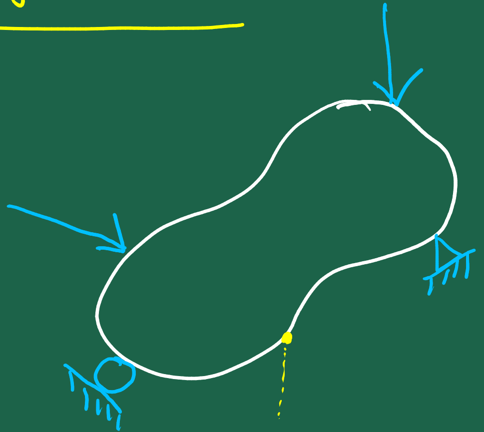
$$\Rightarrow \delta P\,\Delta = \delta F\,\Delta_{sp}$$

We know a priori that $\delta P = \delta F$

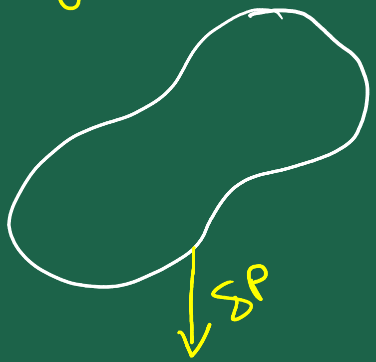
$\therefore\ \Delta = \Delta_{sp}$ (eqn of compatibility)

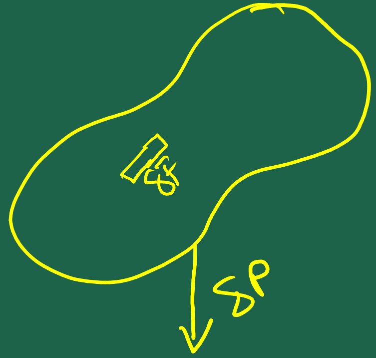
$$\delta W_e = -\delta W_i = \delta W_{ie}$$

Virtual work associated with virtual forces as a method for finding deflection

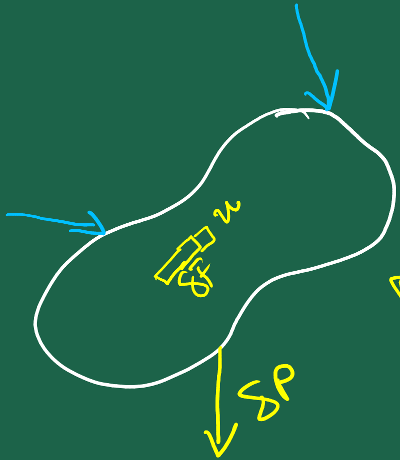


1. Consider the unloaded body and apply a virtual force of magnitude $\delta P = 1$ along the desired dir^n





2. Due to δP, internal forces generated all throughout the body. If the body is statically determinate, we can determine them.



3. With δP still applied bring back the actual loading.

Actual loading → real deformations → Calc ϵ

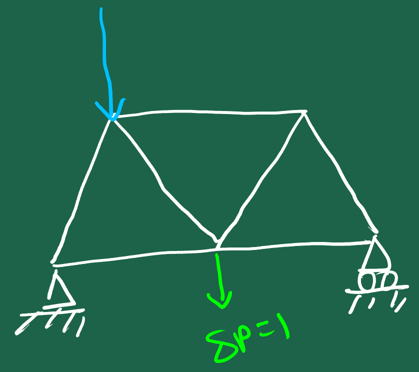
4. Write the ext. virtual work by multiplying δP with Δ, i.e. the deformation along the desired dir^n.

Write the int. virtual work due to all the internal virtual forces mulipl:ed with the deformations at those points.

$$\delta P \times \Delta = \sum \delta F \times u$$

Truss

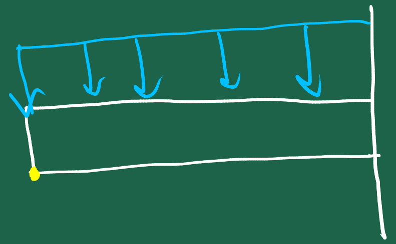
$$1 \times \Delta = \sum_{i=1}^{n} \delta F_i \frac{N_i L_i}{A_i E_i}$$

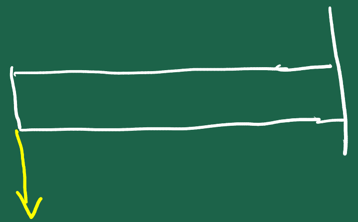


Beam

$$1 \times \Delta = \int_0^L m \, d\theta$$

$$= \int_0^L m \frac{M}{EI} \, dx$$



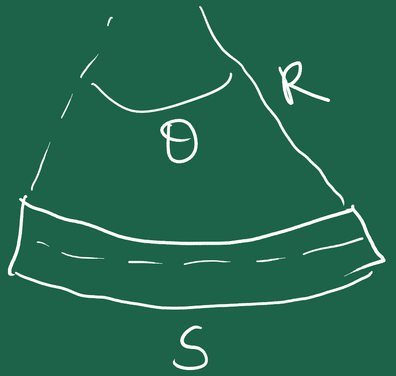


$$MR = EI$$

$$\Rightarrow R = \frac{EI}{M}$$

$$\Rightarrow \frac{1}{R} = \frac{M}{EI}$$

$$dx \leftarrow S = R\theta \rightarrow d\theta$$



$$d\theta = \frac{S}{R} \Leftrightarrow \frac{M}{EI} dx$$

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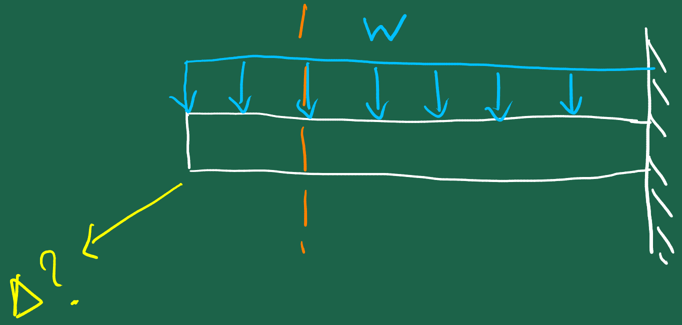
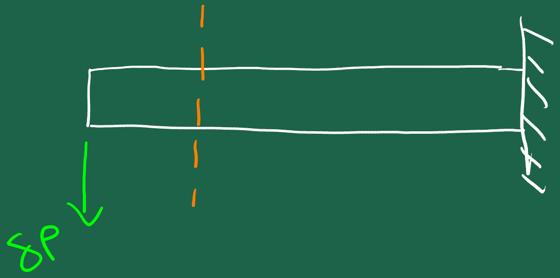
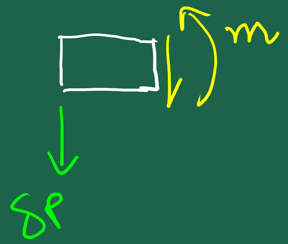
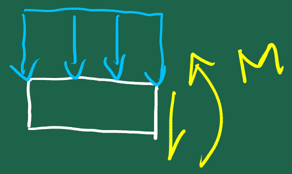






$$m = -\cancel{\delta P}^{\,1}\, x$$



$$M = -\frac{wx^2}{2}$$

$$\cancel{\delta P}^{\,1} \times \Delta = \int_0^L m \frac{M\,dx}{EI}$$

$$= \int_0^L (-x)\frac{\left(-wx^2/2\right)}{EI}\,dx$$

$$= \frac{w}{2EI}\int_0^L x^3\,dx$$

$$= \frac{wL^4}{8EI}$$

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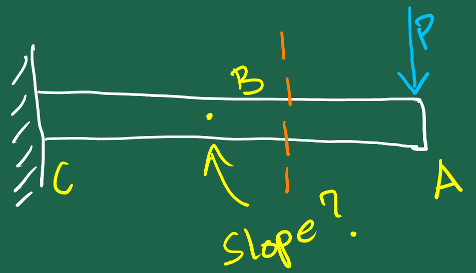
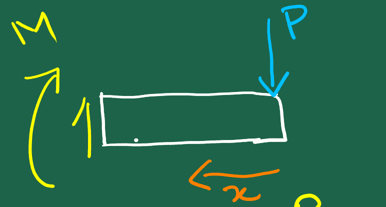
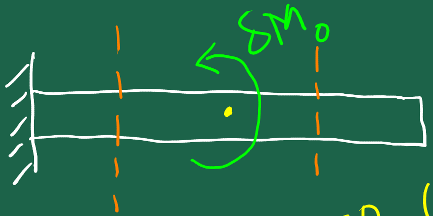




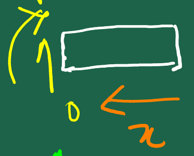
$$M = -Px$$

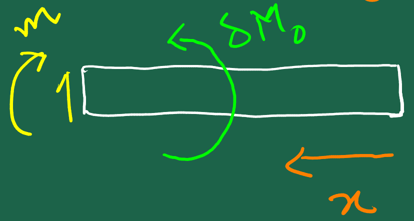
F Δ

M θ



$m=0 \quad (0<x<L/2)$





$$m - \delta M_0 = 0 \Rightarrow m = \cancel{\delta M_0}^{\,1}$$

$(\frac{L}{2}<x<L)$

$$\cancel{\delta M_0}^{\,1} \times \theta = \int_0^L m\,\frac{M\,dx}{EI}$$

$$= \int_0^{L/2} (0)\frac{(-Px)}{EI}dx + \int_{L/2}^{L} (1)\frac{(-Px)}{EI}dx$$

$$= 0 + \frac{(-P)}{EI}\int_{L/2}^{L} x\,dx$$

$$= \frac{(-P)}{EI}\,\frac{3L^2}{8}$$

$$= \frac{-3PL^2}{8EI}$$

CASTIGLIANO'S THEOREMS

$$\Delta = \frac{\partial U}{\partial P}$$

$$\theta = \frac{\partial U}{\partial M}$$

Castigliano's 2nd theorem
OR
Castigliano's theorem on deflection

Applicable <u>only</u> to linear elastic materials

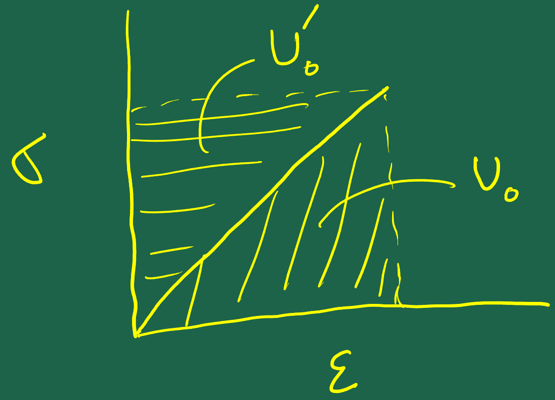
Generalization:

$$\Delta = \frac{\partial U'}{\partial P}$$

$$\theta = \frac{\partial U'}{\partial M}$$

$\rightarrow U'$: Complementary strain energy

$U_0 = U_0'$



$U_0 = U_0'$ for linear elastic materials

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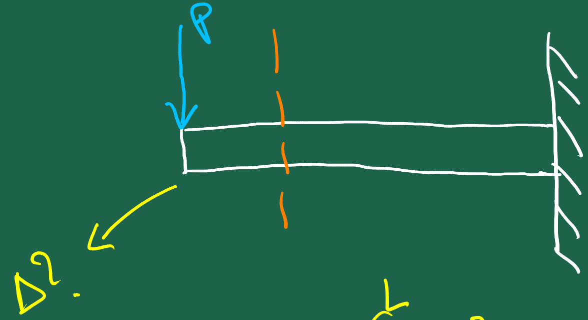

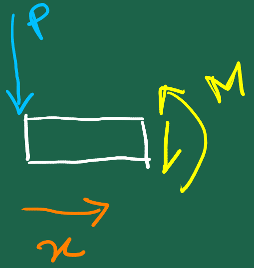

$$M = -Px$$

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\Delta = \frac{\partial U}{\partial P}$$

$$= \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx$$

$$= \int_0^L \frac{2M\frac{\partial M}{\partial P}}{2EI}\,dx$$

$$= \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,dx$$

$$= \int_0^L \frac{(-Px)}{EI}(-x)\,dx$$

$$= \frac{P}{EI}\int_0^L x^2\,dx$$

$$= \frac{PL^3}{3EI}$$

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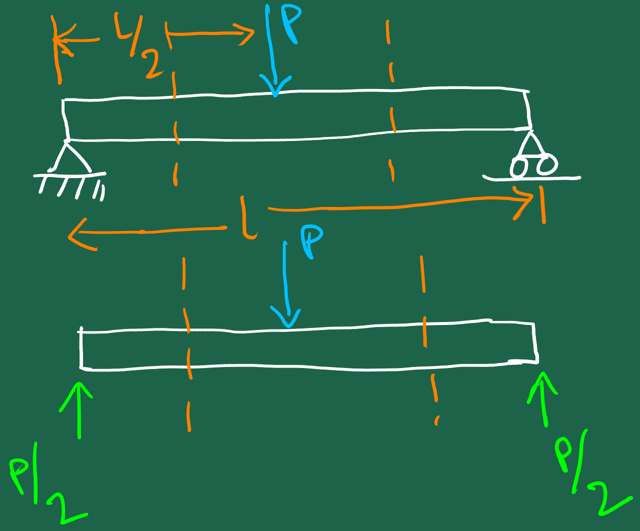

$$\Delta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,dx$$

Because of symmetry

$$\Delta = 2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial P}\,dx$$

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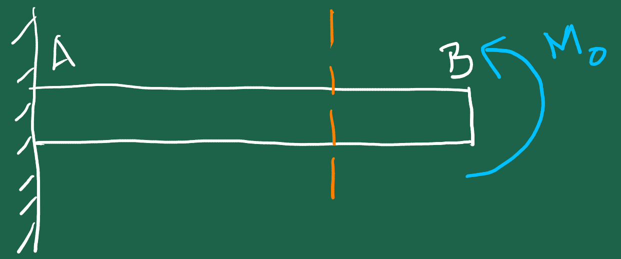

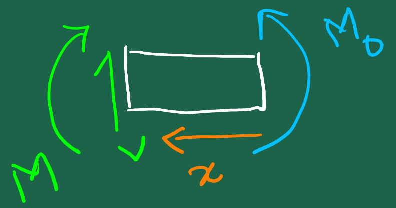

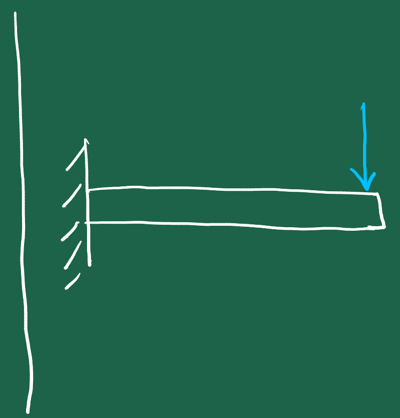

$$M = M_0$$

$$\theta = \frac{\partial U}{\partial M_0} = \frac{\partial}{\partial M_0} \int_0^L \frac{M^2}{2EI}\, dx$$

$$= \int_0^L \frac{M}{EI} \frac{\partial M}{\partial M_0}\, dx$$

$$= \int_0^L \frac{M_0}{EI} (1)\, dx = \frac{M_0 L}{EI}$$

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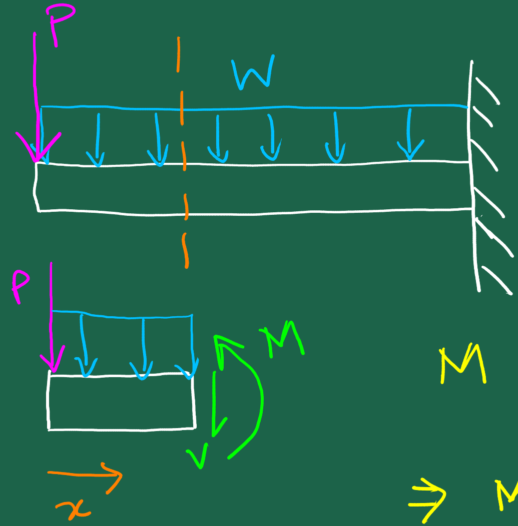


$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

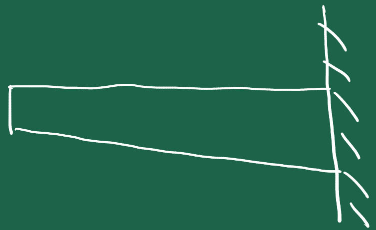
$$M + Px + \frac{wx^2}{2} = 0$$

$$\Rightarrow M = -Px - \frac{wx^2}{2}$$

$$\Delta\Big|_{P=0} = \left[\int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,dx\right]_{P=0}$$

$$= \left[\int_0^L \frac{\left(-Px - wx^2/2\right)}{EI}(-x)\,dx\right]_{P=0} = \int_0^L \frac{\left(-wx^2/2\right)(-x)}{EI}\,dx$$

$$= \frac{wL^4}{8EI}$$



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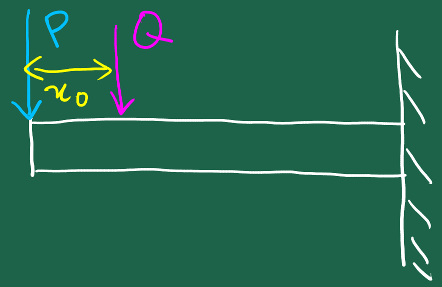

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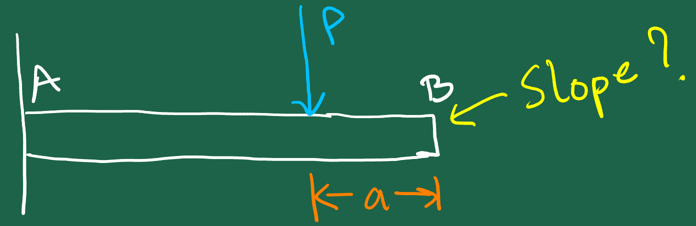
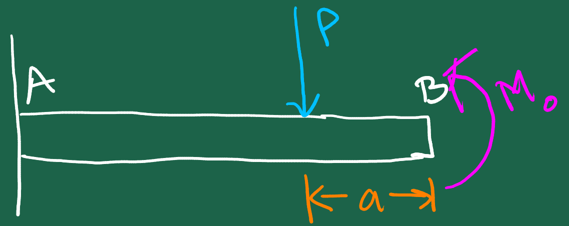


Introduce a dummy or fictitious moment at B.



$$\theta_B\Big|_{M_0=0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx$$

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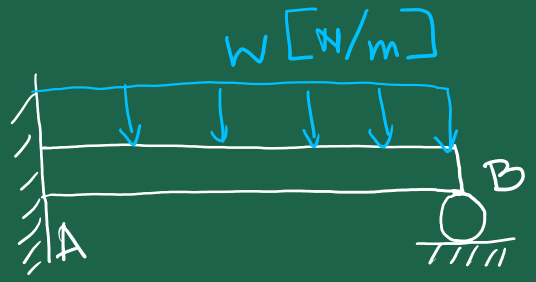
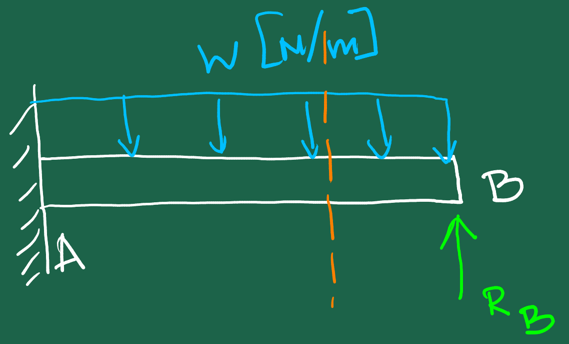
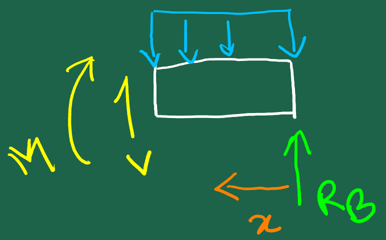






$$M - R_B x + \frac{w x^2}{2} = 0$$

$$\Rightarrow M = R_B x - \frac{w x^2}{2}$$

$$\underset{0}{\cancel{\Delta}} = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial R_B} \, dx$$

$$= \int_0^L \frac{\left(R_B x - \frac{w x^2}{2}\right)}{EI} (x) \, dx$$

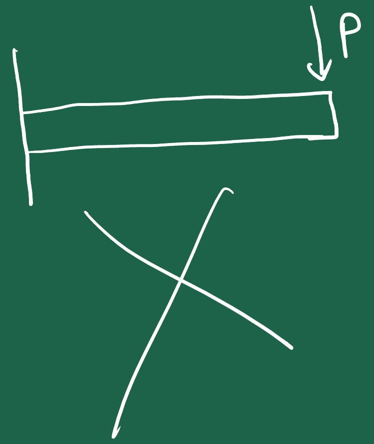
$$= \frac{R_B}{EI} \int_0^L x^2 \, dx - \frac{w}{2EI} \int_0^L x^3 \, dx$$

$$= \frac{R_B L^3}{3EI} - \frac{w L^4}{8EI}$$

$$\therefore \frac{R_B L^3}{3EI} = \frac{w L^4}{8EI} \Rightarrow R_B = \frac{3wL}{8}$$

Very Important:

Do NOT use this $\Delta = 0$ trick to find reaction forces in statically determinate structures



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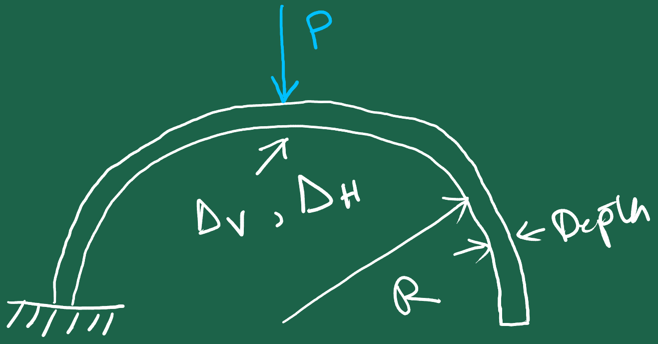
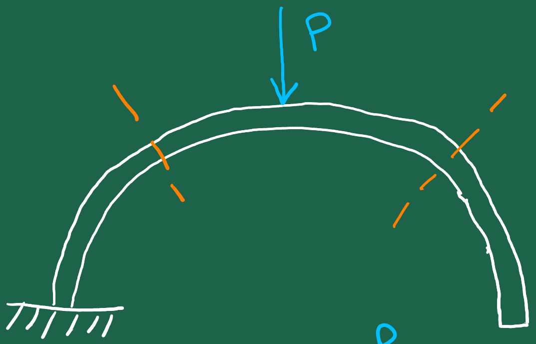
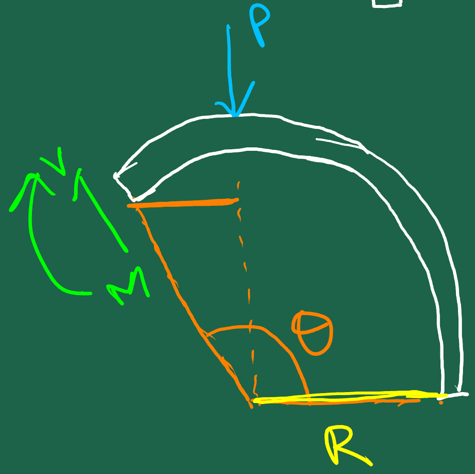


R >> Depth

(Thin)

Δ_V :





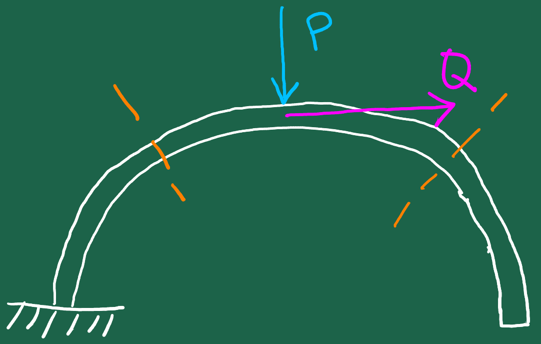
$$M + PR\sin\left(\theta - \frac{\pi}{2}\right) = 0, \quad \frac{\pi}{2} < \theta < \pi$$

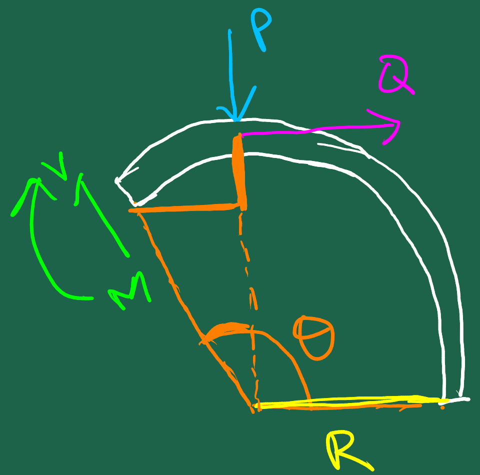
$$\Delta_V = \int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta$$

$$\Delta_V = \int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta$$

$$= \int_0^{\pi/2} 0 \; R\,d\theta \quad + \int_{\pi/2}^{\pi} \frac{\left\{-PR\sin\left(\theta-\pi/2\right)\right\}}{EI}\left\{-R\sin\left(\theta-\pi/2\right)\right\} R\,d\theta$$

Δ_H :





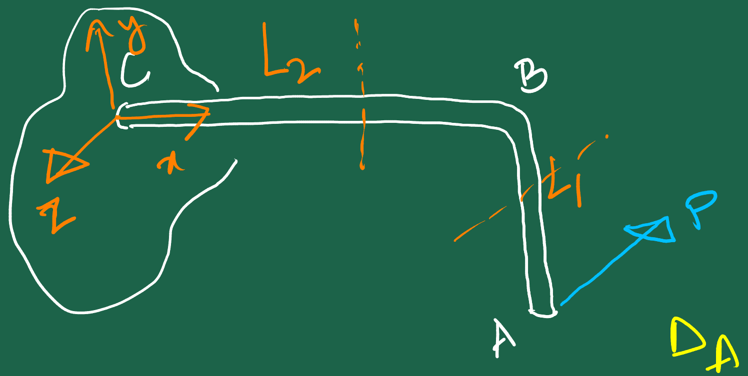
$$M + PR\sin\left(\theta-\pi/2\right) + Q\left\{R - R\cos\left(\theta-\pi/2\right)\right\} = 0, \quad \frac{\pi}{2} < \theta < \pi$$

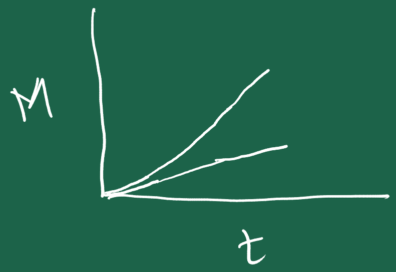
$$M + PR\sin\left(\theta - \frac{\pi}{2}\right) + Q\left\{R - R\cos\left(\theta - \frac{\pi}{2}\right)\right\} = 0, \quad \frac{\pi}{2} < \theta < \pi$$

$$\Delta_H\Big|_{Q=0} = \left[\int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial Q} R\,d\theta\right]_{Q=0}$$

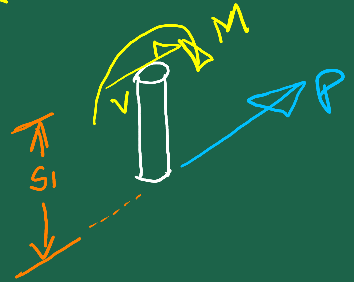
$$= \int_0^{\pi/2} 0\,R\,d\theta + \left[\int_{\pi/2}^{\pi} \frac{M}{EI}\left\{-R\left(1 - \cos\left(\theta - \frac{\pi}{2}\right)\right)\right\} R\,d\theta\right]_{Q=0}$$

#





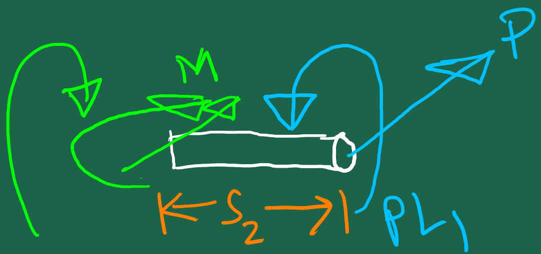
AB :

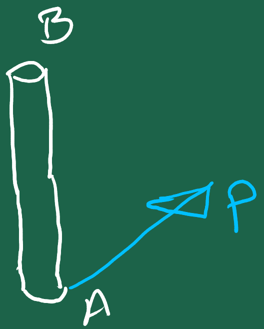


$$M - P s_1 = 0$$

$$\Rightarrow M = P s_1 \quad , \; 0 < s_1 < L_1$$

BC :





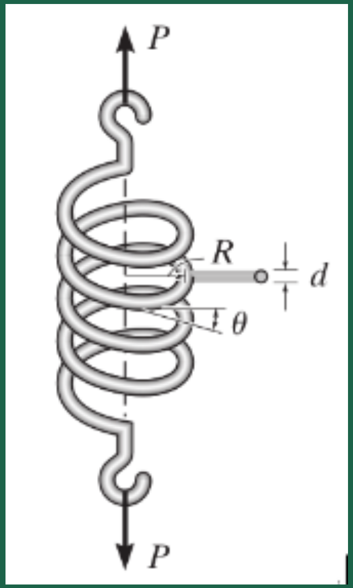
Total strain energy

$$U = \underbrace{\int_0^{L_1} \frac{M^2}{2EI} ds_1}_{AB} + \underbrace{\int_0^{L_2} \frac{M^2}{2EI} ds_2 + \int_0^{L_2} \frac{T^2}{2GJ} ds_2}_{BC}$$

$$\Delta_A = \frac{\partial U}{\partial P} = \int_0^{L_1} \frac{M}{EI}\frac{\partial M}{\partial P}\, ds_1 + \int_0^{L_2} \frac{M}{EI}\frac{\partial M}{\partial P}\, ds_2 + \int_0^{L_2} \frac{T}{GJ}\frac{\partial T}{\partial P}\, ds_2$$

$$= \frac{PL_1^3}{3EI} + \frac{PL_2^3}{3EI} + \frac{PL_1^2 L_2}{GJ}$$

2. The coiled spring has n coils and is made from a material having a shear modulus G. Determine the stretch of the spring when it is subject to the load P. Assume that the coils are close to each other so that $\theta \approx 0°$ and the deflection is caused entirely by the torsional stress in the coil. $[\dfrac{64nPR^3}{d^4G}]$



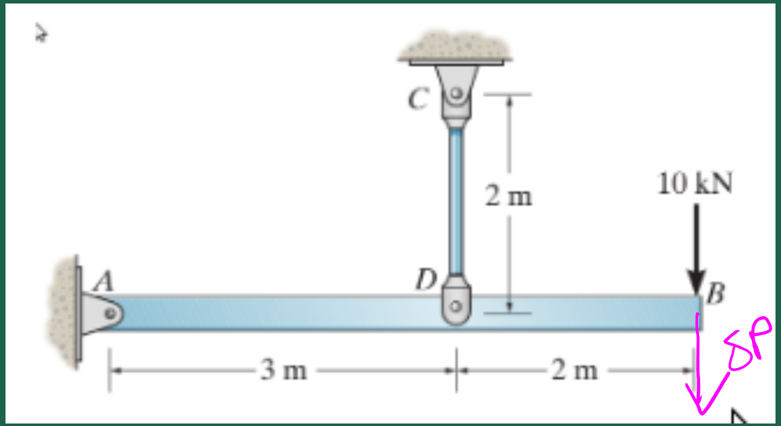
$$T = PR$$

$$U = \int_0^L \frac{T^2}{2GJ}\,dx = \frac{T^2 L}{2GJ} = \frac{T^2\, 2\pi R n}{2GJ} = \frac{P^2R^2\, 2\pi R n}{2GJ}$$

$$J = \frac{\pi d^4}{32} \quad \left(\frac{1}{2}\pi r^4\right)$$

$$\Delta = \frac{\partial U}{\partial P}$$

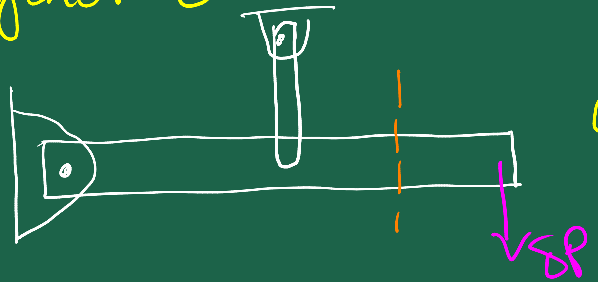
6. Beam AB has a square cross section of 100 mm by 100 mm. Bar CD has a diameter of 10 mm. If both members are made of steel ($E = 200$ GPa), determine the vertical displacement of point B and the slope at A. [43.5 mm, 0.00530 rad]



PVW

Remove the ext. loading and apply δP

$\delta P \to$ generate some internal forces



solve!

$$\delta P \times \Delta = \sum \delta F \times u$$

$$\frac{FL}{AE}$$

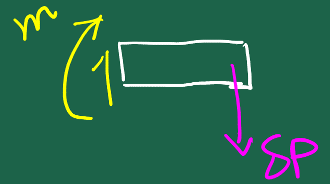
Bring back the external loading and consider all the internal deformations

$$1 \times \Delta_B = \underbrace{}_{AB} + \underbrace{}_{CD}$$

$$= \int_B^D + \int_D^A + \delta F_{CD}\left(\frac{FL}{AE}\right)$$

$$\int_B^D \quad = \int_B^D m\, d\theta \quad = \int_0^2 m\, \frac{M\,dx}{EI}$$

$$= \int_0^2 (-x)\,\frac{(-10x)}{EI}\,dx$$



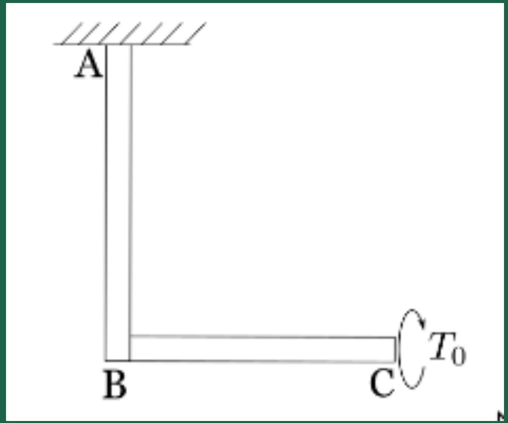
$$m = -\delta P\, x$$

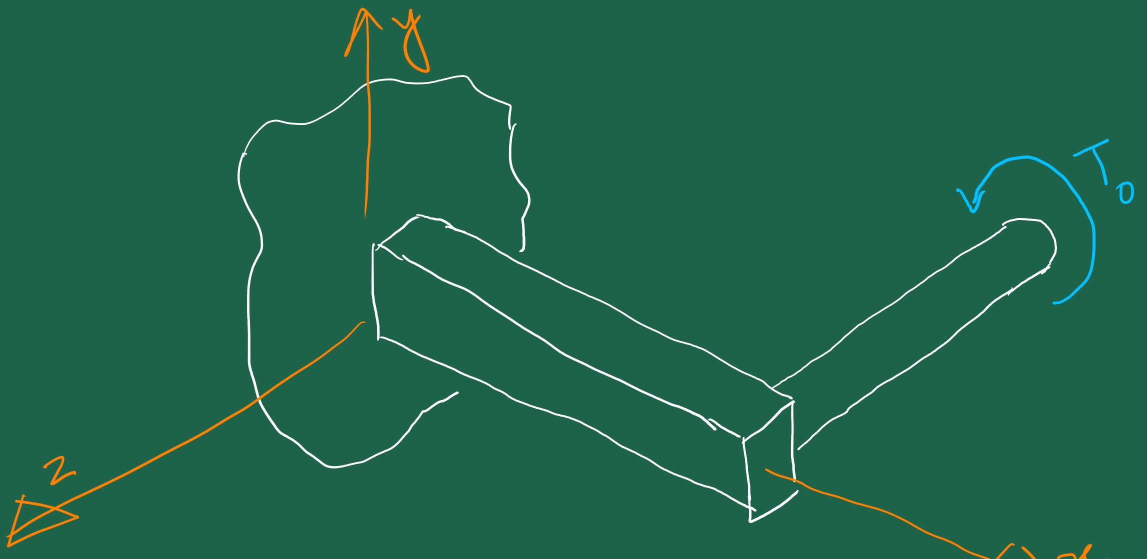
$$= -x$$

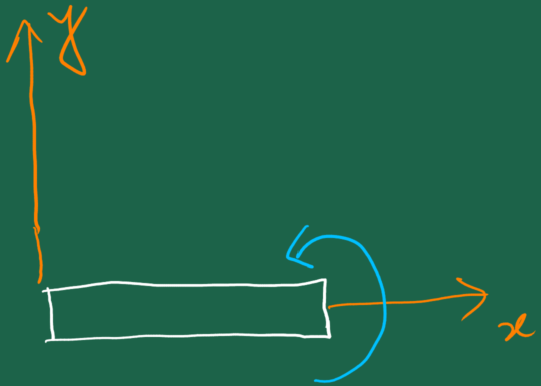
Complete the rest of the steps yourself!

Compare the above sol^n based on PVW with the solution based on Castigliano's theorem

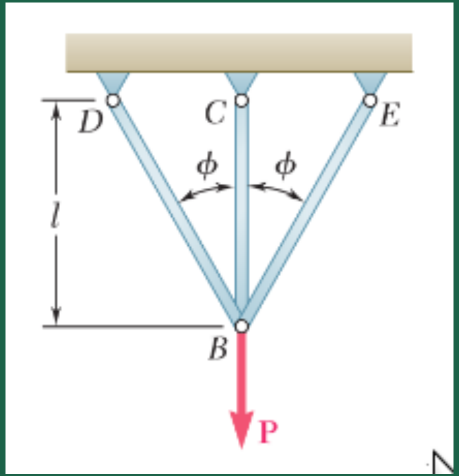
10. A shaft BC (length: 1.2 m) of circular cross-section (diameter 60 mm) is welded to a beam AB (length 1.5 m) of rectangular cross-section (70 mm × 50 mm). A torque $T_0 = 2.50$ kN·mis applied at C as shown. Determine the rotation of the end C. Both the shaft and the beam are made of steel ($E = 200$ GPa; $G = 77.5$ GPa). [0.0523 rad]

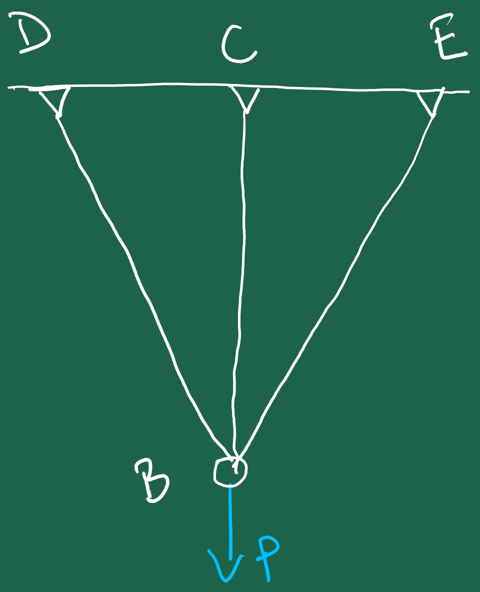


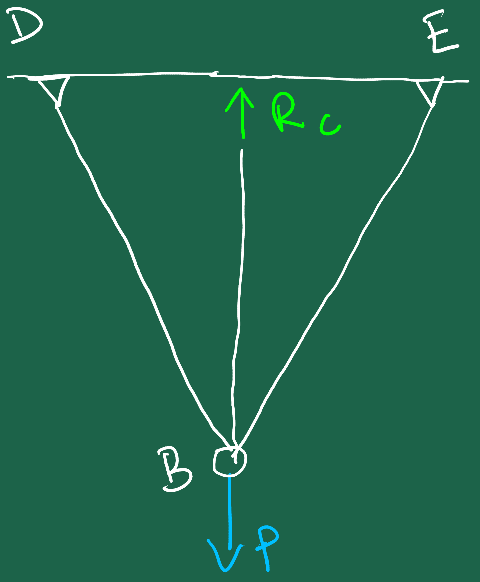




11. Three members of the same material and same cross-sectional area are used to support the load P. Determine the force in the member BC. $\left[\dfrac{P}{1+2\cos^3\phi}\right]$







$$2F_0\cos\phi + R_c = P$$

$$\Rightarrow F_0 = \frac{P-R_c}{2\cos\phi}$$

$$L_{BD} = L_{BE} = \frac{l}{\cos\phi}$$

$$\Delta_c = \frac{\partial U}{\partial R_c} = \frac{\partial U_{BD}}{\partial R_c} + \frac{\partial U_{BE}}{\partial R_c} + \frac{\partial U_{BC}}{\partial R_c}$$

$$= \frac{\partial}{\partial R_c}\left(\frac{F_0^2 L_{BD}}{2AE}\right) + \frac{\partial}{\partial R_c}\left(\frac{F_0^2 L_{BE}}{2AE}\right) + \frac{\partial}{\partial R_c}\left(\frac{R_c^2\, l}{2AE}\right)$$

$$= 2\frac{\partial}{\partial R_C}\left(\frac{F_0^2 L_{BD}}{2AE}\right) + \frac{\partial}{\partial R_C}\left(\frac{R_C^2 l}{2AE}\right)$$

$$F_0 = \frac{P - R_C}{2\cos\phi}$$

$$= 2\,\frac{2F_0}{2AE}\,\frac{l}{\cos\phi}\,\frac{\partial F_0}{\partial R_C} + \frac{2R_C l}{2AE}$$

$$\Rightarrow \frac{\partial F_0}{\partial R_C} = \frac{-1}{2\cos\phi}$$

$$= \frac{2F_0}{AE}\,\frac{l}{\cos\phi}\left(\frac{-1}{2\cos\phi}\right) + \frac{R_C l}{AE}$$

But we know that $\Delta_C = 0$

$$\therefore \frac{2F_0 l}{AE\cos\phi}\left(\frac{+1}{2\cos\phi}\right) = \frac{R_C l}{AE}$$

$$\Rightarrow \frac{P - R_C}{2\cos^3\phi} = R_C \Rightarrow R_C = \frac{P}{1 + 2\cos^3\phi}$$