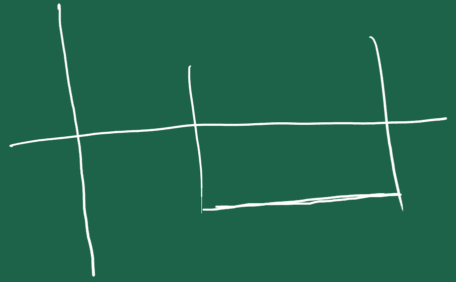
Bending of Beams

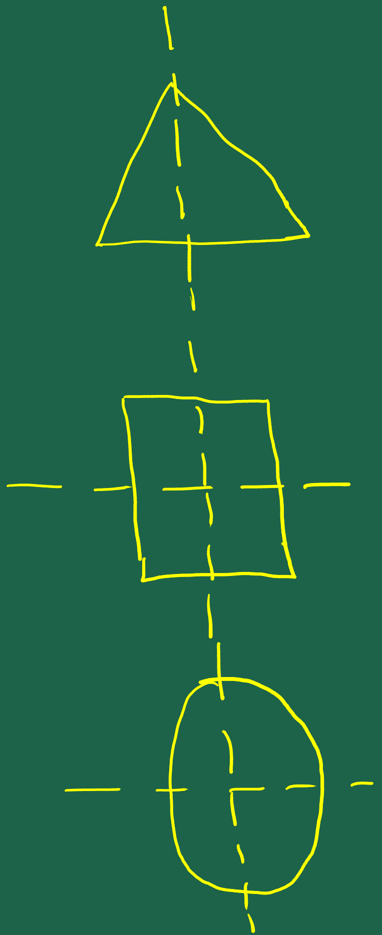
Bending deformation and Flexure Formula

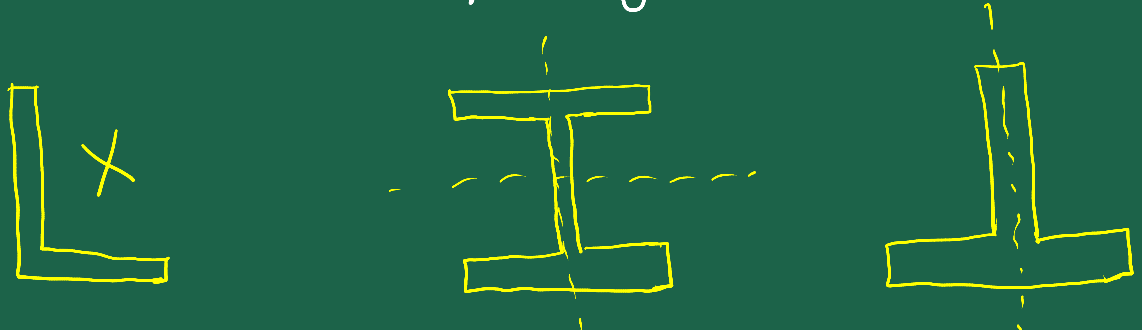
Pure!

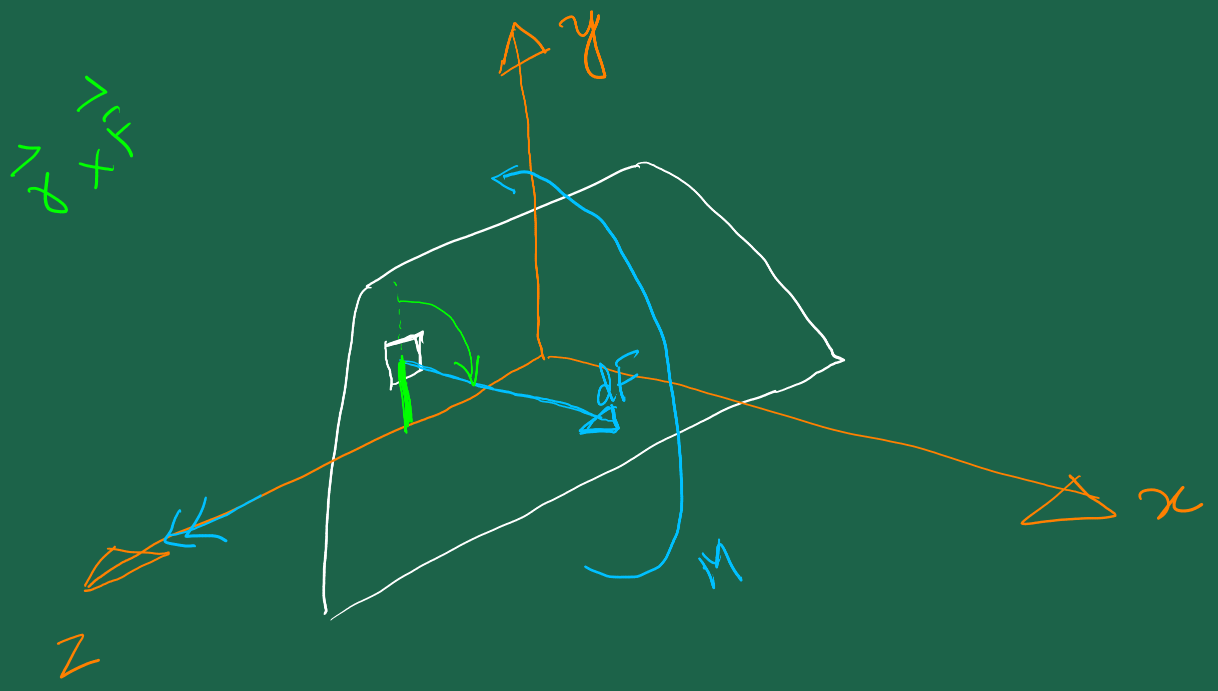


prismatic

c/s does not vary along length







$$M = -\int y \, dF$$

$$\boxed{M = -\int y \, \sigma_{xx} \, dA}$$

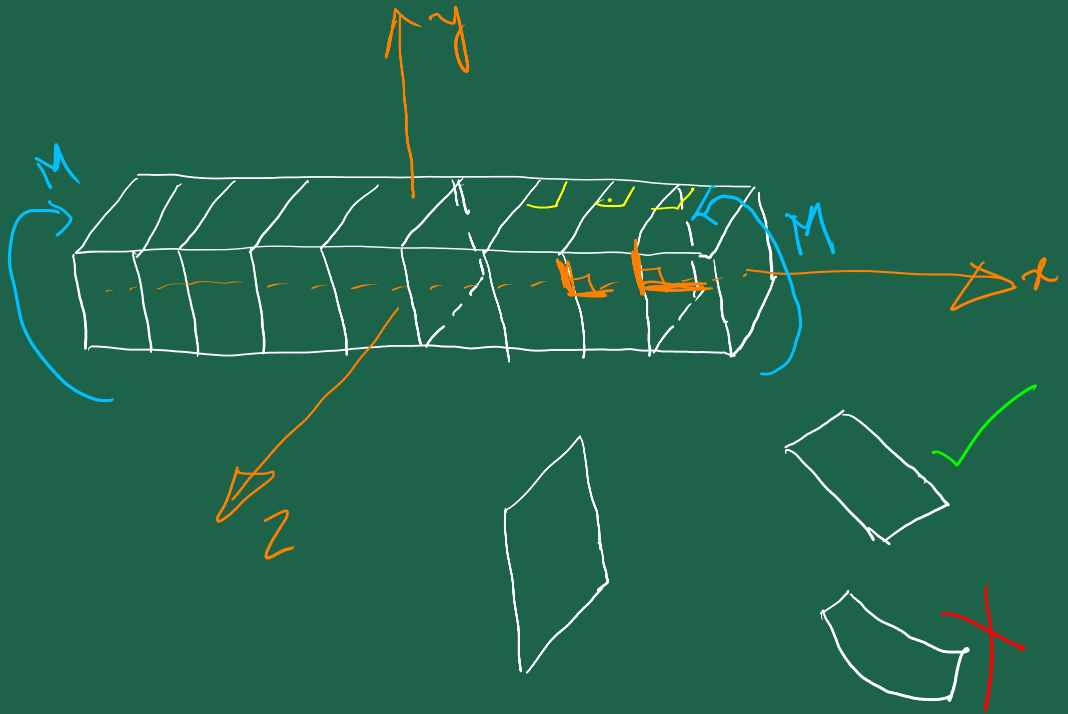
$$\boxed{0 = \int \sigma_{xx} \, dA}$$

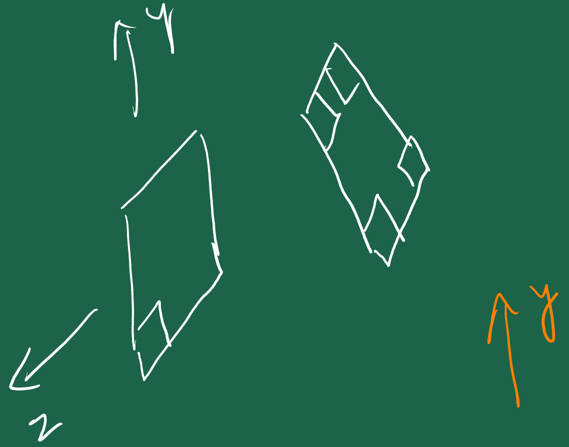
Euler – Bernoulli

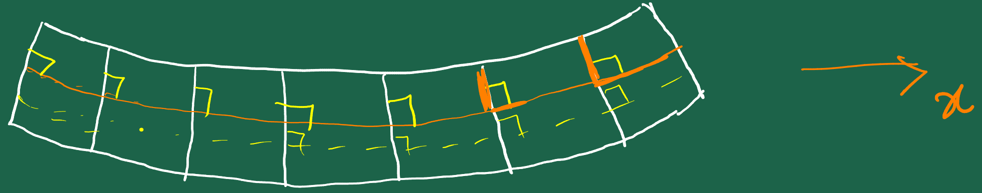
Plane sections remain plane

Plane sections remain perpendicular to the longitudinal axis

Plane sections do not change dimensions

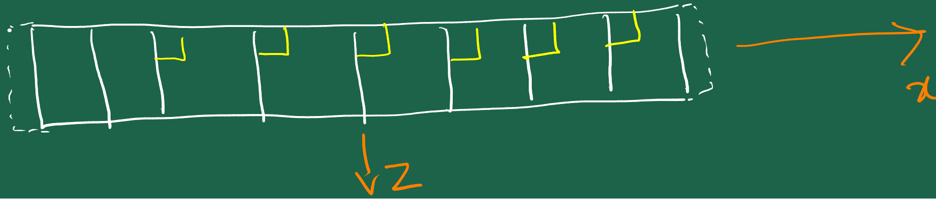






$\varepsilon_{yy} = 0, \quad \varepsilon_{zz} = 0, \quad \boxed{\varepsilon_{xx} \neq 0}$

$\gamma_{xy} = 0 \quad \gamma_{xz} = 0 \quad \gamma_{yz} = 0$



$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \Rightarrow \tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \qquad \left[\tau_{xy} = G\gamma_{xy} \text{ etc.}\right]$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \rightarrow \sigma_{xx} = f_1(\varepsilon_{xx}, \underbrace{\cancel{\varepsilon_{yy}}}_{0}, \underbrace{\cancel{\varepsilon_{zz}}}_{0})$$

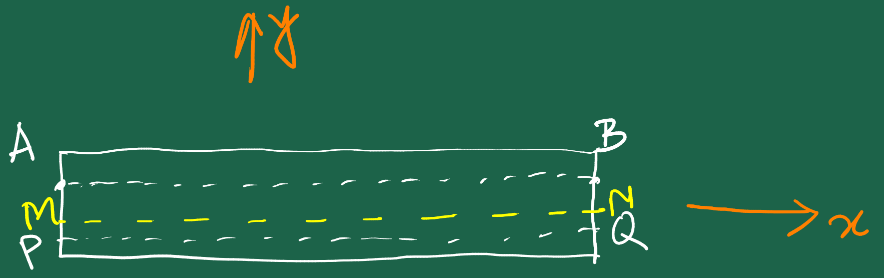
$$\underbrace{\cancel{\varepsilon_{yy}}}_{0} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \qquad \sigma_{yy} = f_2(\varepsilon_{xx}, \underbrace{\cancel{\varepsilon_{yy}}}_{0}, \underbrace{\cancel{\varepsilon_{zz}}}_{0})$$

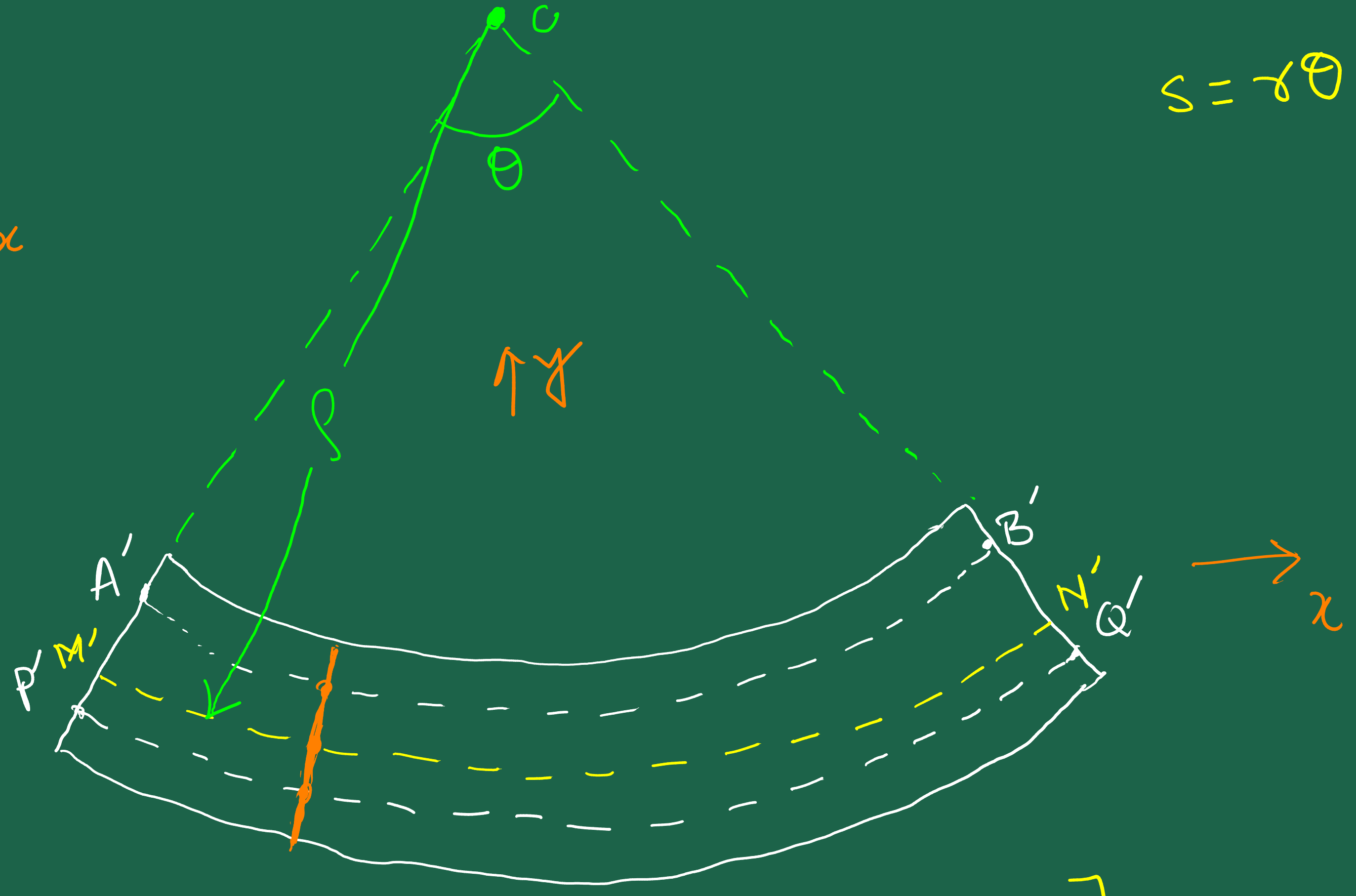
$$\underbrace{\cancel{\varepsilon_{zz}}}_{0} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \sigma_{zz} = f_3(\varepsilon_{xx}, \underbrace{\cancel{\varepsilon_{yy}}}_{0}, \underbrace{\cancel{\varepsilon_{zz}}}_{0})$$

$\sigma = E\varepsilon$

$\rightarrow$ Assume to be 0 ! (strictly not true)

$$\varepsilon_{xx} \neq 0 \; ; \quad \sigma_{xx} \neq 0 \;, \quad \text{Rest are } 0$$





$s = r\theta$

$A'B' < AB$

$P'Q' > PQ$

$M'N' = MN$

→ Neutral Surface

$$\varepsilon_{xx} = \frac{A'B' - AB}{AB} \qquad [AB = MN = M'N']$$

$$= \frac{(\rho - y)\theta - \rho\theta}{\rho\theta} = \frac{-y}{\rho}$$

$$\varepsilon_{xx} = -\frac{y}{\rho} \Rightarrow \sigma_{xx} = E\,\varepsilon_{xx} = -E\frac{y}{\rho}$$

Now,

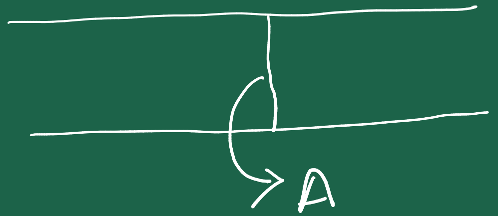
$$M = -\int_A y\,\sigma_{xx}\,dA$$

$$= -\int_A y\left(-E\frac{y}{\rho}\right)dA$$

$$= \int_A \frac{E}{\rho}\,y^2\,dA$$

$$= \frac{E}{\rho}\underbrace{\int_A y^2\,dA}_{\substack{\text{Area moment of inertia} \\ \text{OR} \\ \text{Second moment of area}}} = \frac{EI}{\rho}$$

Area moment of inertia
OR
Second moment of area } I

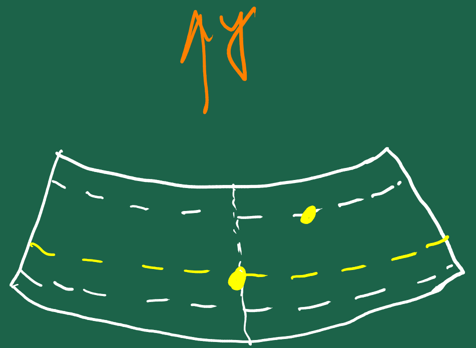


$$\boxed{M = \frac{EI}{\rho}}$$

$$\underbrace{\int y\,dA}_{\text{First moment of area}}$$

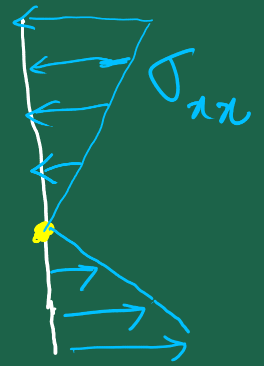
First moment of area

$$\sigma_{xx} = -E\frac{y}{\rho} \qquad \left[\rho = \frac{EI}{M}\right]$$



$$= -E\frac{y}{EI/M}$$

$$\boxed{\sigma_{xx} = -M\frac{y}{I}} \longrightarrow \text{Bending stress}$$



Flexure formula

For points lying above the neutral surface : $y>0 \Rightarrow \sigma_{xx}<0$

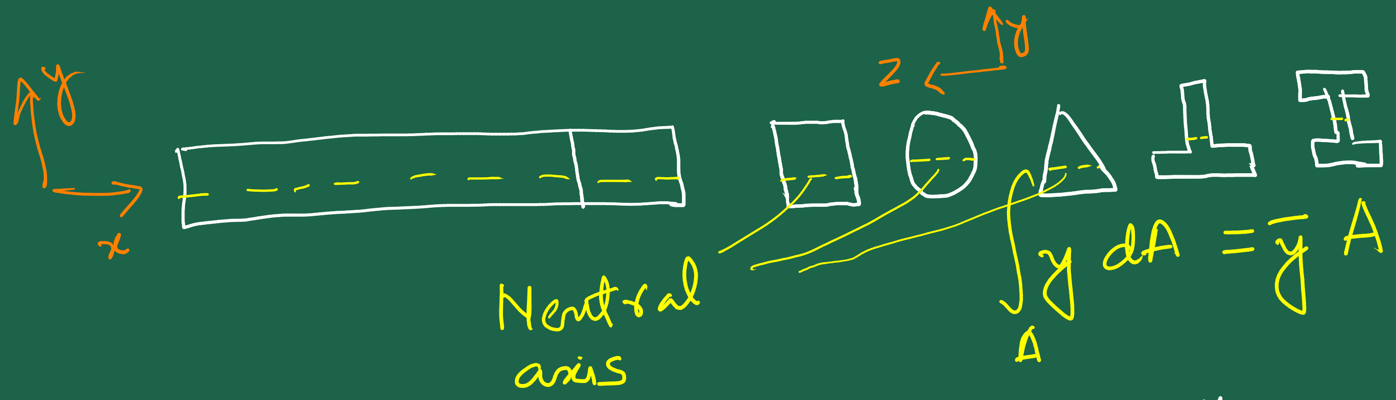
" " " below " " " : $y<0 \Rightarrow \sigma_{xx}>0$

$$0 = \int_A \sigma_{xx}\, dA$$

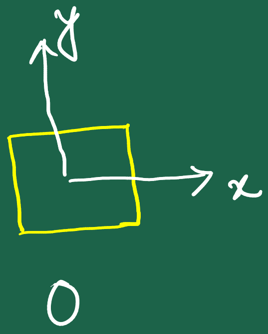
$$\Rightarrow 0 = \int_A \left(-\frac{My}{I}\right) dA$$

$$\Rightarrow \boxed{\int_A y\, dA = 0}$$

$\Rightarrow$ x-axis passes through the centroid



$$\int_A y\, dA = \bar{y} A$$



But we had chosen our x-axis to lie along the neutral surface

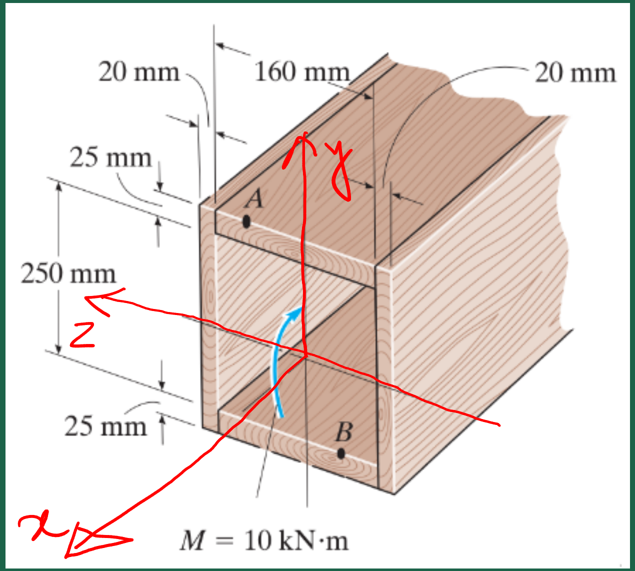
$\therefore$ The implication is that the neutral surface passes through the centroid.

$$\sigma_{xx} = -\frac{My}{I} \rightarrow \sigma_{max} = -\frac{Mc}{I}$$ [c: extremities of y]

$$= -\frac{M}{I/c}$$

$$= -\frac{M}{S} \rightarrow \text{Section Modulus}$$

1. A box beam is constructed from four pieces of wood, glued together as shown. If the moment acting on the cross section is 10 kN·m, determine the stress at points A and B. [$\sigma_A = -6.21$ MPa, $\sigma_B = 5.17$ MPa]

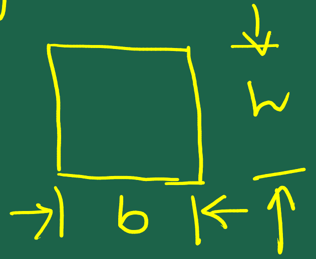


$I = ?$

$I = I_1 - I_2$

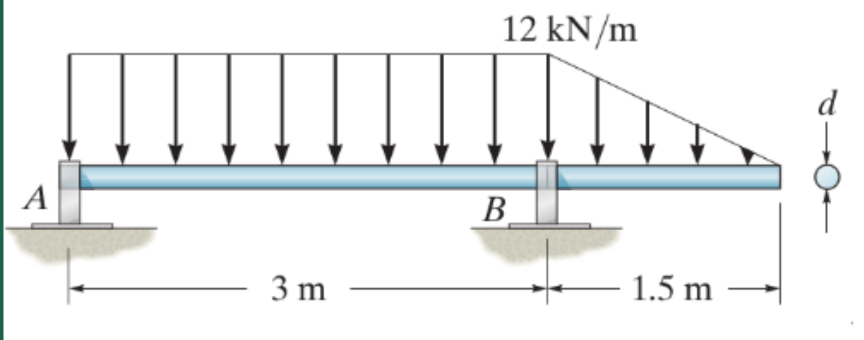
$\sigma_{xx} = -\frac{My}{I}$

Rectangular c/s



$I = \frac{1}{12} bh^3$

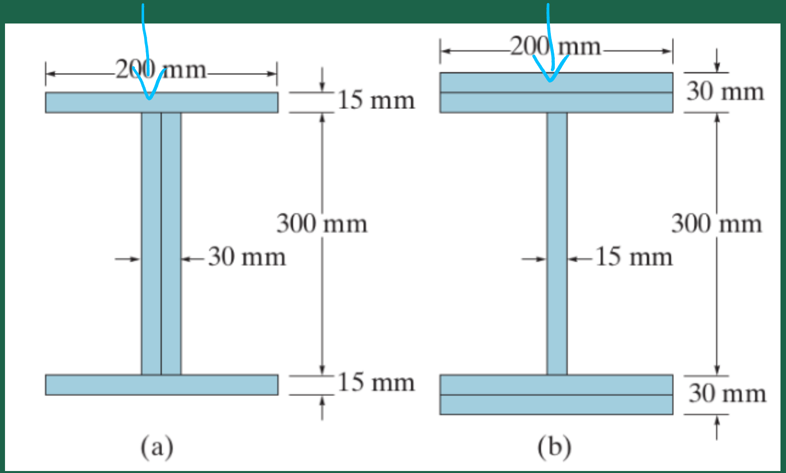
2. The shaft is supported by smooth journal bearings at A and B that only exert vertical reactions on the shaft. Determine its smallest diameter d if the allowable bending stress is $\sigma_{\text{allow}} = 180$ MPa. [86.3 mm]



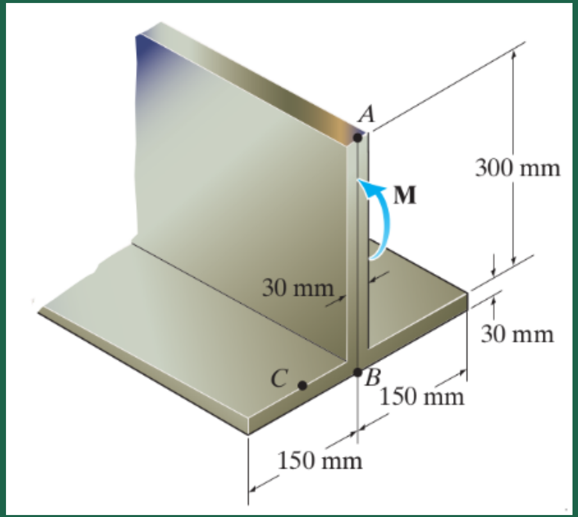
$$\left|\sigma_{xx}\right| \leq \sigma_{allow}$$

$$\sigma_{xx}\Big|_{\text{at a c/s} \to \max} = -\frac{M_{\to \max}\, c}{I}$$

3. Two designs for a beam are to be considered. Determine which one will suport a moment of $M = 150$ kN·m with the least amount of bending stress. What is that stress? [Design (b); $\sigma_{\text{min}} = 74.7$ MPa]



5. If the beam is subjected to an internal moment of $M = 100$ kN·m, determine the bending stress developed at points A, B, and C. [$\sigma_A = 122$ MPa (C), $\sigma_B = 51.1$ MPa (T), $\sigma_C = 35.4$ MPa (T)]



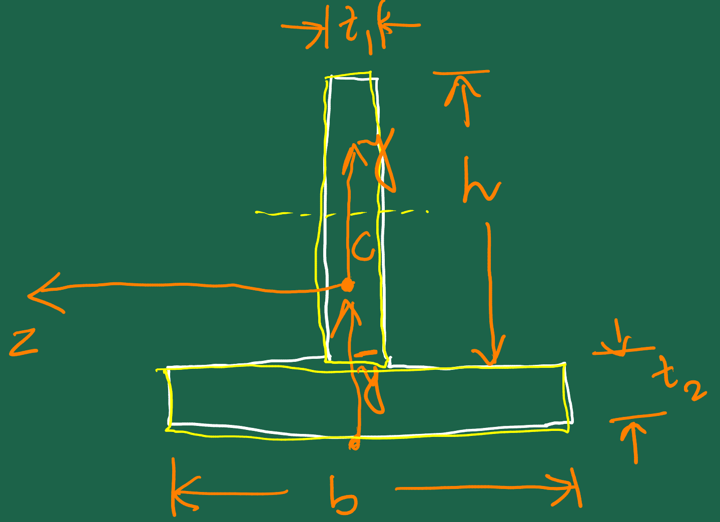
First locate the centroid

z-axis pass through the centroid

I

$\sigma_{xx} = -\frac{My}{I}$

w.r.t. z-axis



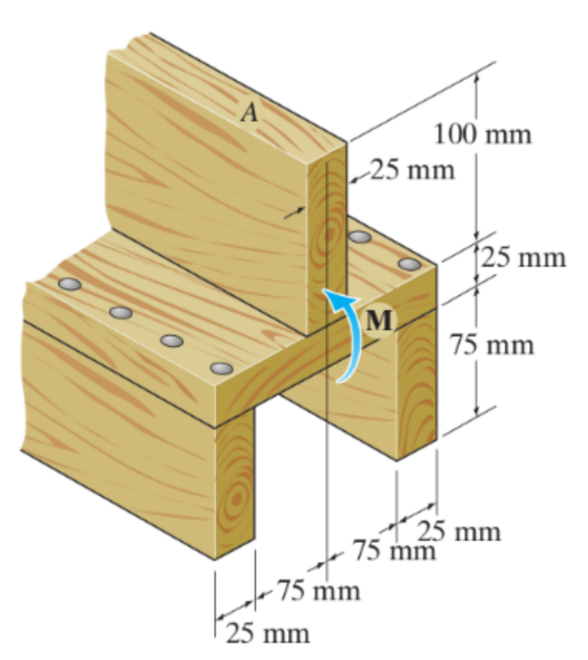
Centroid (c)

$$ht_1\left(t_2+\frac{h}{2}\right) + bt_2\frac{t_2}{2} = \bar{y}\left(ht_1+bt_2\right) \rightarrow \bar{y} ✓$$

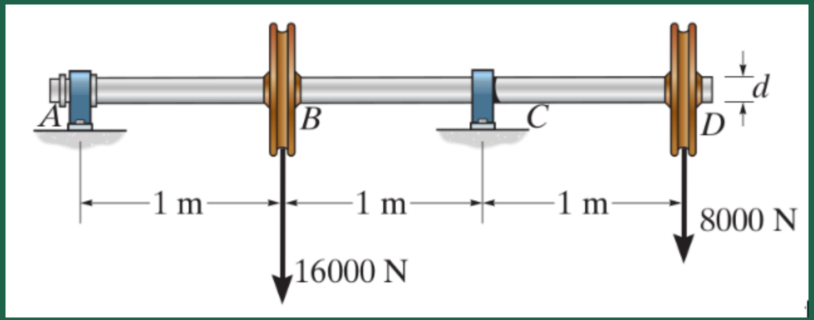
Second moment of area (I):

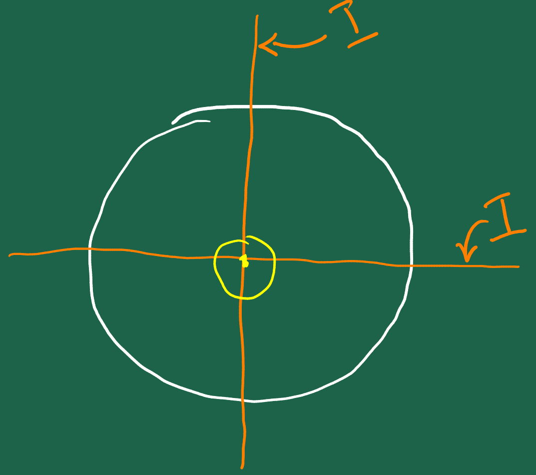
$$I = \frac{1}{12}t_1h^3 + ht_1\left(t_2+\frac{h}{2}-\bar{y}\right)^2 + \frac{1}{12}bt_2^3 + \left(\bar{y}-\frac{t_2}{2}\right)^2 bt_2$$

4. If the beam is subjected to an internal moment of $M = 3$ kN·m, determine the maximum tensile and compressive stress in the beam. Also, sketch the bending stress distribution on the cross section. $[\sigma_{\text{max,c}} = 13 \text{ MPa}, \ \sigma_{\text{max,t}} = 9.5 \text{ MPa}]$



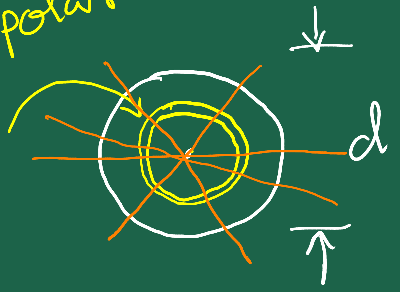
6. The shaft is supported by a smooth thrust bearing at A and a smooth journal bearing at C. If the material has an allowable bending stress of $\sigma_{\text{allow}} = 168$ MPa, determine the required minimum diameter d of the shaft to the nearest mm. [75 mm]

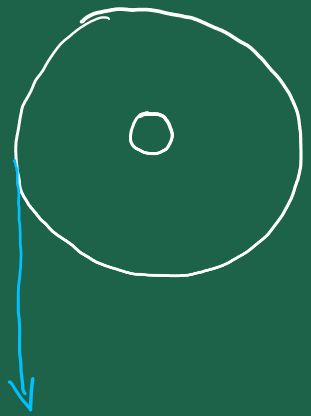




polar moment of inertia

$$\frac{\pi}{2} r^4 = J$$



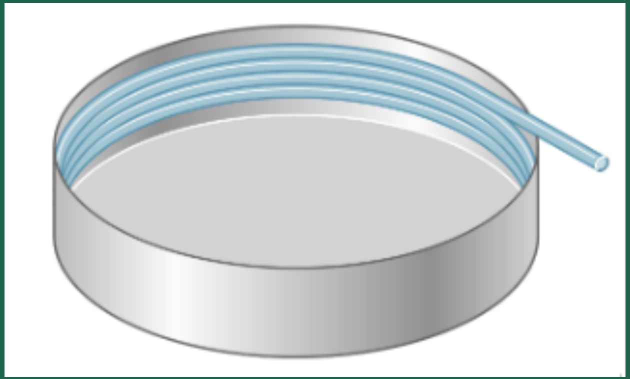


$I =$

$$I + I = J \Rightarrow 2I = J = \frac{\pi}{2} r^4$$

$$\therefore I = \frac{\pi}{4} r^4$$

8. Straight rods of 6 mm diameter and 30 m length are stored by coiling the rods inside a drum of 1.25 m inside diameter. Assuming that the yield strength is not exceeded, determine the maximum stress in a coiled rod and the corresponding bending moment in the rod. Use $E = 200$ GPa. [965 MPa, 20.5 N·m]

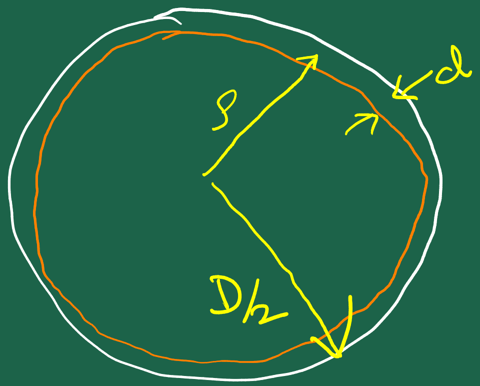


X

$$\sigma_{xx} = -\frac{My}{I}$$

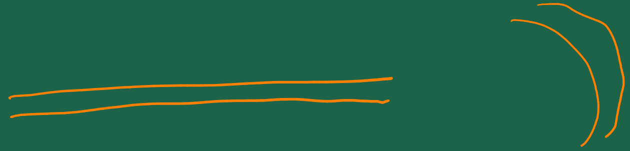
$$\frac{1}{\rho} = \frac{M}{EI}$$

$$\sigma_{xx} = -E\frac{y}{\rho}$$

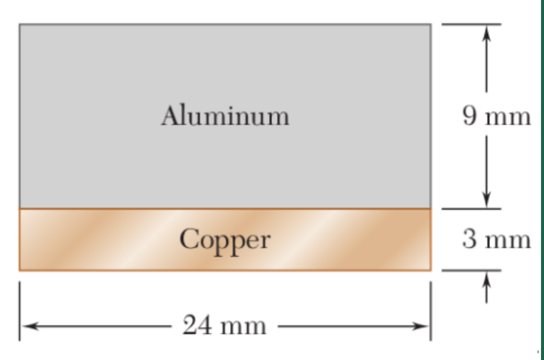


$$\rho = \frac{D}{2} - \frac{d}{2}$$

$$M = \frac{EI}{\rho}$$



9. A copper strip and an aluminium strip are bonded together to form the composite beam shown. Knowing that the beam is bent about a horizontal axis by a couple of moment $M = 35$ N·m, determine the maximum stress in the aluminium strip and the copper strip. The Young's modulus of copper is 105 GPa and that of aluminium is 75 GPa. [−56.0 MPa, 68.4 MPa]



$$\int \sigma_{xx}\, dA = 0$$

$$\int y\, \sigma_{xx}\, dA = -M$$

$$\varepsilon_{xx} = -\frac{y}{\rho} \leftarrow \text{(composite)}$$

$$\sigma_1 = E_1 \varepsilon_{xx} = -E_1 \frac{y}{\rho}$$

$$\sigma_2 = E_2 \varepsilon_{xx} = -E_2 \frac{y}{\rho}$$

$$\int \sigma_{xx}\, dA = 0$$

$$\Rightarrow \int_{A_1} \sigma_1\, dA + \int_{A_2} \sigma_2\, dA = 0$$

$$\Rightarrow -\int_{A_1} \frac{E_1}{\rho} y\, dA - \int_{A_2} \frac{E_2}{\rho} y\, dA = 0$$

$$\Rightarrow E_1/\rho \int_{A_1} y\, dA + \frac{E_2}{\rho} \int_{A_2} y\, dA \Rightarrow E_1 \int_{A_1} y\, dA + E_2 \int_{A_2} y\, dA = 0$$

$$\int y\,\sigma_{xx}\,dA = -M$$

$$\Rightarrow \int_{A_1} y\,\sigma_1\,dA + \int_{A_2} y\,\sigma_2\,dA = -M$$

$$\Rightarrow -\int_{A_1} \frac{E_1}{\rho} y^2\,dA - \int_{A_2} \frac{E_2}{\rho} y^2\,dA = -M$$

$$\Rightarrow \frac{E_1}{\rho}\int_{A_1} y^2\,dA + \frac{E_2}{\rho}\int_{A_2} y^2\,dA = M$$

$$\Rightarrow \frac{E_1 I_1}{\rho} + \frac{E_2 I_2}{\rho} = M \Rightarrow \frac{1}{\rho} = \frac{M}{E_1 I_1 + E_2 I_2}$$

$$\sigma_1 = -E_1\frac{y}{\rho}$$

$$\sigma_2 = -E_2\frac{y}{\rho}$$

$$\sigma_1 = -E_1 \frac{y}{\rho}$$

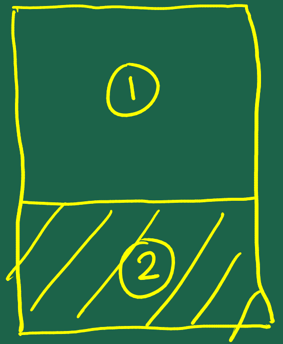
$$\sigma_2 = -E_2 \frac{y}{\rho}$$

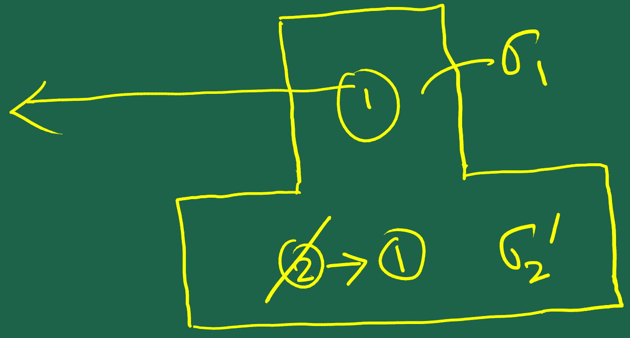
$$E_2 = nE_1$$

$$\sigma_2 = -E_1 n \frac{y}{\rho}$$

$$dF_1 = \sigma_1\, dA = -E_1 \frac{y}{\rho}\, dA$$

$$dF_2 = \sigma_2\, dA = -E_1 n \frac{y}{\rho}\, dA = -E_1 \frac{y}{\rho}(n\, dA)$$





Transformed c/s

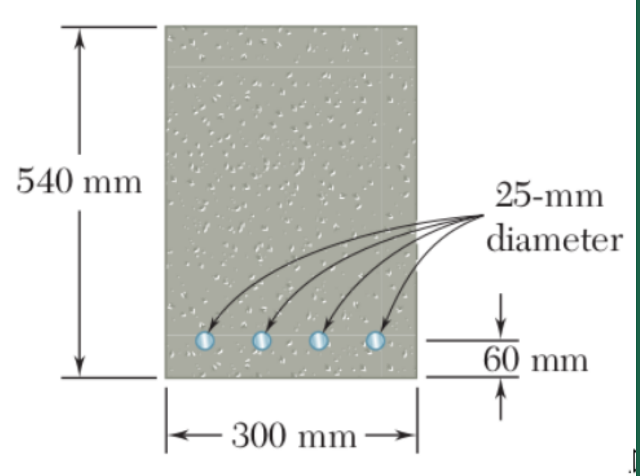
$$\sigma_2 = \frac{dF_2}{dA} \qquad \sigma_2' \frac{dF_2}{n\, dA}$$

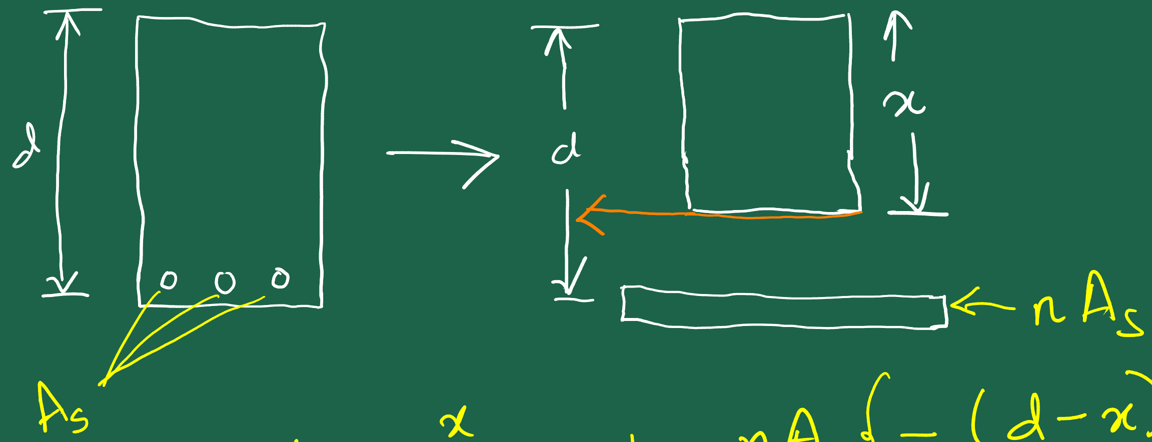
$$\sigma_2 = n\sigma_2'$$

σ_1 ✓

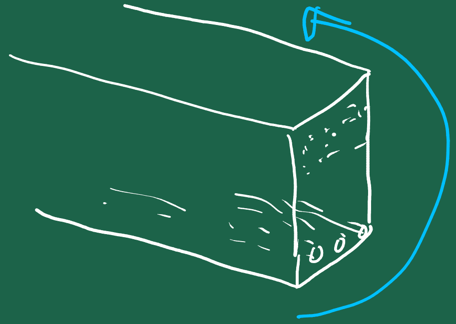
$\sigma_2 = n\sigma_2'$ ✓

10. The reinforced concrete beam shown is subjected to a positive bending moment of 175 kN·m. Knowing that the modulus of elasticity if 25 GPa for the concrete and 200 GPa for the steel, determine the stress in the steel and the maximum stress in the concrete. [212 MPa, −15.59 MPa]





$$n = \frac{E_s}{E_c}$$



$$b x \frac{x}{2} + nA_s\{-(d-x)\} = 0$$

$$\Rightarrow \frac{1}{2} b x^2 + nA_s x - nA_s d = 0$$