# Problem Sheet 3: Constitutive Relations

1. The constitutive law for a linear, elastic, isotropic solid is given by

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G\varepsilon_{ij}.$$

From this relation find the inverse relation, i.e. $\varepsilon_{ij}$ in terms of $\sigma_{ij}$. To do this, first set $i = j = p$, utilize the fact that $\varepsilon_{pp} \equiv \varepsilon_{kk}$, and then obtain the expression for $\varepsilon_{kk}$; next substitute this expression back in the above equation to find $\varepsilon_{ij}$.

2. Compare the relation found in the previous question with the following

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

to show that $E = \dfrac{G\left(3\lambda + 2G\right)}{\lambda + G}$ and $\nu = \dfrac{\lambda}{2\left(\lambda + G\right)}$.

3. From the expressions of $E$ and $\nu$ found in the previous question, show that

$$\lambda = \frac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)} \qquad \text{and} \qquad G = \frac{E}{2\left(1+\nu\right)}.$$

4. For a hydrostatic state of compression $\sigma_{ij} = -p\delta_{ij}$ ($p$ being a positive constant), show that the dilatation (refer the problem sheet on kinematics) is given by $-p/K$, where $K$ is referred to as the bulk modulus. Also show

$$K = \frac{E}{3\left(1-2\nu\right)} \equiv \frac{3\lambda + 2G}{3}.$$

If $\nu \to 1/2$, what happens to $K$? Also, what happens to the volumetric deformation? What can you say about the compressibility of the material? How does the first invariant of strain feature in all this?

5. If the Young's modulus $E$, the bulk modulus $K$, and the shear modulus $\mu$ are required to be positive, show that the Poisson's ratio $\nu$ must satisfy the inequality

$$-1 < \nu < \frac{1}{2}.$$

For most real materials, however, $0 < \nu < 1/2$. Show that this more restrictive inequality implies $\lambda > 0$. Materials that have negative Poisson's ratio are referred to as *auxetic* materials.

6. Strain gauges are little devices used to measure the strain on the free surface of a body, along a particular direction.* Usually three strain gauges are combined together to form what is called a strain rosette. Following the measurement of strains along three directions, various pieces of information regarding the state of strain and stress may be determined.

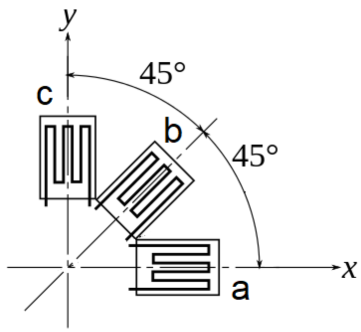


Figure 1: Rectangular strain rosette

Consider a rectangular rosette, as shown in Figure 1 that is cemented to the *free surface* of an airplane wing. The material of the airplane wing is an aluminium alloy with Young's modulus, $E = 72$ GPa and Poisson's ratio, $\nu = 0.33$. Under load, the strain readings are $\varepsilon_a = 0.00250$, $\varepsilon_b = 0.00140$, $\varepsilon_c = -0.00125$.

(a) Determine the state of strain, i.e. find the various strain components.

(b) Determine the principal stresses.

(c) Determine the orientation of the principal planes.

(d) Determine the maximum shear stress.

(e) Determine the orientation of the plane on which the maximum shear stress acts.

[(a) $\varepsilon_{xx} = 0.00250$, $\varepsilon_{yy} = -0.00125$, $\varepsilon_{xy} = 0.000775$; (b) $p_1 = 176.99$ MPa, $p_2 = -42.67$ MPa; (c) At 11.23° to the $x$-axis or $y$-axis; (d) $\tau_{\max} = 109.83$ MPa; (e) At 56.23° to the $x$-axis. ]

7. Using the constitutive relation for a linear, elastic, isotropic solid in the mechanical equilibrium equations ($\nabla \cdot \boldsymbol{\sigma} = 0$ OR $\dfrac{\partial \sigma_{ij}}{\partial X_j} = 0$) where body forces are absent, show that the following equation is obtained:

$$(\lambda + G)\nabla(\nabla \cdot \boldsymbol{u}) + G\nabla^2 \boldsymbol{u} = 0 \quad \text{OR} \quad (\lambda + G)\frac{\partial}{\partial X_i}\left(\frac{\partial u_k}{\partial X_k}\right) + G\frac{\partial^2 u_i}{\partial X_j^2} = 0.$$

*They work on the principle that the electrical resistance of a wire changes with change in length.