## Problem Sheet 1: Kinematics

1. The components of a displacement field are

$$
\begin{aligned}
& u_{x}=\left(x^{2}+20\right) \times 10^{-4}, \\
& u_{y}=2 y z \times 10^{-3} \\
& u_{z}=\left(z^{2}-x y\right) \times 10^{-3} .
\end{aligned}
$$

(a) Consider two points in the undeformed system $(2,5,7)$ and $(3,8,9)$. Find the change in distance between these points.
(b) What are the components of the strain tensor?
(c) The rotation tensor is given by $\boldsymbol{\Omega}=\nabla \boldsymbol{u}-\boldsymbol{\varepsilon}$ or $\Omega_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial X_{j}}-\frac{\partial u_{j}}{\partial X_{i}}\right)$. What are the components of the rotation tensor?
(d) Compute the strain at $(2,-1,3)$.

$$
\begin{array}{r}
{\left[(\mathrm{a}) 0.07 \text {; (d) } \varepsilon_{11}=4 \times 10^{-4}, \varepsilon_{22}=0.006, \varepsilon_{33}=0.006, \varepsilon_{12}=0, \varepsilon_{31}=5 \times 10^{-4},\right.} \\
\left.\varepsilon_{23}=-0.002\right]
\end{array}
$$

2. Consider a cubical volume element with three of its edges at any one chosen vertex oriented along the principal directions. The original length of each side is $d X^{(k)}$ and the final length is $d x^{(k)}$, such that

$$
d x^{(k)}=d X^{(k)}\left(1+\varepsilon^{(k)}\right), \quad k=1,2,3 \quad \text { (no summation implied) }
$$

where $\varepsilon^{(k)}$ is the principal strain in the $k$-direction.
(a) Dilatation is defined as the relative change in volume:

$$
\text { Dilatation }=\frac{(\text { final vol. })-(\text { initial vol. })}{(\text { initial vol. })} .
$$

Find an expression for dilatation assuming that $\left|\varepsilon^{(k)}\right| \ll 1$. Rewrite this expression in terms of the components of displacement gradient $u_{i, j}$.
(b) If the initial and final volumes are $V_{0}$ and $V_{\mathrm{f}}$ respectively with corresponding densities $\rho_{0}$ and $\rho_{\mathrm{f}}$ (assumed uniform throughout the volumes), then mass conservation gives the relation $\rho_{\mathrm{f}} V_{\mathrm{f}}=\rho_{0} V_{0}$. Using this relation and the result of part (a), relate $\rho_{0}, \rho_{\mathrm{f}}$, and the divergence of the displacement, $u_{i, i}$. What happens when $\rho_{0}=\rho_{\mathrm{f}}$, i.e. when the density is constant?
3. Since dilatation can be expressed solely in terms of the normal strain components (refer previous problem), these normal strain components are said to be responsible for changes in volume while the shearing strains are responsible for changes in shape. Often, the (infinitesimal) strain tensor is decomposed into two parts: the mean normal strain $\varepsilon_{\mathrm{M}}$, which accounts for volumetric change, and the deviatoric strain $\varepsilon_{\mathrm{D}}$, which accounts for shape change.
(a) Define $\varepsilon_{\mathrm{M}}=\frac{1}{3}(\nabla \cdot \mathbf{u}) \mathbb{I}$, where $\mathbb{I}$ is the identity tensor; or, equivalently, $\left(\varepsilon_{M}\right)_{i j}=$ $\frac{1}{3} u_{i, i} \delta_{i j}$. Find the matrix representation of $\varepsilon_{\mathrm{D}}$.
(b) The definition of $\varepsilon_{\mathrm{M}}$ ensures that the mean normal strain represents a state of equal elongation in all directions. Under this state of strain the elemental volume deforms in such a way that the shape remains similar to the original shape. Since $\varepsilon_{\mathrm{M}}$ accounts for the volumetric strain, the volumetric change associated with $\varepsilon_{\mathrm{D}}$ should be zero. Check if this is so by finding the dilatation of $\varepsilon_{\mathrm{D}}$.
4. The problem of finding the principal strains at a point reduces to the eigenvalue problem:

$$
\left(\varepsilon_{i j}-\lambda_{i j}\right) n_{j}=0, \quad \text { or, equivalently, in matrix form } \quad([\varepsilon]-\lambda[\mathbb{I}])[\hat{\mathbf{n}}]=0
$$

Non-trivial solutions of this problem may be found by using the condition that

$$
\operatorname{det}([\varepsilon]-\lambda[\mathbb{I}])=0
$$

where $\operatorname{det}([A])$ means determinant of $[A]$. The above gives us:

$$
\left|\begin{array}{ccc}
\varepsilon_{11}-\lambda & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22}-\lambda & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}-\lambda
\end{array}\right|=0
$$

that reduces to

$$
\lambda^{3}-J_{1} \lambda^{2}+J_{2} \lambda-J_{3}=0
$$

(a) Find expressions for $J_{1}, J_{2}$, and $J_{3}$ in terms of the components of $\varepsilon_{i j}$.
(b) $J_{1}, J_{2}$, and $J_{3}$ are referred to as the strain invariants. What do you think is the motivation behind calling them invariants? Hint: Principal strains pertain to the actual physical situation while the components of $\varepsilon_{i j}$ are a consequence of the choice of our coordinate axes.
(c) The principal strain tensor is such that in its matrix representation the diagonal elements are the $\lambda$ 's while the off-diagonal elements are zero. Find expressions for $J_{1}, J_{2}$, and $J_{3}$ in terms of the principal strains $\lambda_{1}, \lambda_{2}$, and $\lambda_{3}$.
5. The strain field at a point $P(x, y, z)$ in an elastic body is given by

$$
\varepsilon=\left[\begin{array}{ccc}
20 & 3 & 2 \\
3 & -10 & 5 \\
2 & 5 & -8
\end{array}\right] \times 10^{-6}
$$

Determine the following values:
(a) The strain invariants
(b) The principal strains
(c) The mean normal strain and the deviatoric strain

$$
\begin{array}{r}
{\left[(\mathrm{a}) J_{1}=2 \times 10^{-6}, J_{2}=-318 \times 10^{-12}, J_{3}=1272 \times 10^{-18} ; ~(\mathrm{~b}) \lambda_{1}=20.5 \times 10^{-6},\right.} \\
\left.\lambda_{2}=-14.1 \times 10^{-6}, \lambda_{3}=-4.39 \times 10^{-6}\right]
\end{array}
$$

6. Consider a strain field such that

$$
\varepsilon_{11}=A x_{2}^{2}, \quad \varepsilon_{22}=A x_{1}^{2}, \quad \varepsilon_{12}=B x_{1} x_{2}, \quad \varepsilon_{33}=\varepsilon_{32}=\varepsilon_{31}=0
$$

Find the relationship between $A$ and $B$ such that it is possible to obtain a single-valued continuous displacement field which corresponds to the given strain field. [ $B=2 A]$
7. Consider the strain-displacement relations in a rectangular Cartesian coordinate system and verify that

$$
\begin{align*}
& \frac{\partial^{2} \varepsilon_{x x}}{\partial y^{2}}+\frac{\partial^{2} \varepsilon_{y y}}{\partial y^{2}}=2 \frac{\partial^{2} \varepsilon_{x y}}{\partial x \partial y}  \tag{1}\\
& \frac{\partial^{2} \varepsilon_{x x}}{\partial y \partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{y z}}{\partial x}+\frac{\partial \varepsilon_{z x}}{\partial y}+\frac{\partial \varepsilon_{x y}}{\partial z}\right) \tag{2}
\end{align*}
$$

Extend the ideas of these two equations to obtain

$$
\begin{align*}
& \frac{\partial^{2} \varepsilon_{y y}}{\partial z^{2}}+\frac{\partial^{2} \varepsilon_{z z}}{\partial y^{2}}=2 \frac{\partial^{2} \varepsilon_{y z}}{\partial y \partial z}  \tag{3}\\
& \frac{\partial^{2} \varepsilon_{z z}}{\partial x^{2}}+\frac{\partial^{2} \varepsilon_{x x}}{\partial z^{2}}=2 \frac{\partial^{2} \varepsilon_{z x}}{\partial z \partial x}  \tag{4}\\
& \frac{\partial^{2} \varepsilon_{y y}}{\partial z \partial x}=\frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{z x}}{\partial y}+\frac{\partial \varepsilon_{x y}}{\partial z}+\frac{\partial \varepsilon_{y z}}{\partial x}\right)  \tag{5}\\
& \frac{\partial^{2} \varepsilon_{z z}}{\partial x \partial y}=\frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{x y}}{\partial z}+\frac{\partial \varepsilon_{y z}}{\partial x}+\frac{\partial \varepsilon_{z x}}{\partial y}\right) \tag{6}
\end{align*}
$$

These six equations are referred to as the compatibility equations.
8. The six compatibility equations in the previous question are not actually independent. To see this, first obtain from Eqs. (2), (5), and (6) the following:

$$
\begin{align*}
\frac{\partial^{4} \varepsilon_{x x}}{\partial y^{2} \partial z^{2}} & =\frac{\partial^{3}}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{y z}}{\partial x}+\frac{\partial \varepsilon_{z x}}{\partial y}+\frac{\partial \varepsilon_{x y}}{\partial z}\right)  \tag{7}\\
\frac{\partial^{4} \varepsilon_{y y}}{\partial z^{2} \partial x^{2}} & =\frac{\partial^{3}}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{z x}}{\partial y}+\frac{\partial \varepsilon_{x y}}{\partial z}+\frac{\partial \varepsilon_{y z}}{\partial x}\right)  \tag{8}\\
\frac{\partial^{4} \varepsilon_{z z}}{\partial x^{2} \partial y^{2}} & =\frac{\partial^{3}}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{x y}}{\partial z}+\frac{\partial \varepsilon_{y z}}{\partial x}+\frac{\partial \varepsilon_{z x}}{\partial y}\right) . \tag{9}
\end{align*}
$$

Next, add Eqs. (7) and (8) and compare with what you obtain after differentiating Eq. (1) w.r.t $z$ twice. This comparison shows that Eqs. (7), (8), and (9) are really the three independent equations.
9. A two-dimensional problem of a rectangular bar stretched by uniform end loadings results in the following strain field:

$$
\varepsilon=\left[\begin{array}{ccc}
C_{1} & 0 & 0 \\
0 & -C_{2} & 0 \\
0 & 0 & 0
\end{array}\right],
$$

where $C_{1}$ and $C_{2}$ are constants. Assuming the field depends only on $x$ and $y$, integrate the strain-displacement relations to determine the displacement components and identify any rigid-body motion terms.

$$
\begin{aligned}
& {\left[u_{x}=C_{1} x+K y+D, u_{y}\right.}=-C_{2} y-K x+E ; \text { rigid body rotation about } z \text {-axis: } \\
&\left.\omega_{z}=-K ; \text { translation along } x: D ; \text { translation along } y: E\right]
\end{aligned}
$$

10. Consider the simple tension of a prismatic bar fixed at one end. Let $\varepsilon$ be the unit elongation of the bar in the longitudinal direction and $\nu \varepsilon$ the unit lateral contraction. The components of displacement of a point with coordinates $x, y, z$ are

$$
u=\varepsilon x, \quad v=-\nu \varepsilon y, \quad w=-\nu \varepsilon z .
$$

Verify that a plane in the bar given, before deformation, by the equation

$$
a x+b y+c z+d=0
$$

will remain a plane after deformation.
11. A state of strain is given by

$$
\varepsilon \equiv\left[\begin{array}{ccc}
A z & 0 & 0 \\
0 & A z & 0 \\
0 & 0 & B z
\end{array}\right]
$$

From the strain-displacement relations for normal strains show that the three components of the displacement vector are $u=A x z+f(y, z), v=A y z+g(x, z), w=$ $\frac{1}{2} B z^{2}+h(x, y)$, where $f(y, z), g(x, z)$, and $h(x, y)$ are arbitrary functions. Now, use the strain displacement relations for shear strains to obtain $f(y, z)$ and $g(x, z)$ in terms of some arbitrary constants if it is given that $h(x, y)=-\frac{1}{2} A x^{2}-\frac{1}{2} A y^{2}-C_{1} x+C_{2}+C_{3} y$, where $C_{1}, C_{2}$, and $C_{3}$ are also arbitrary constants.

$$
\left[f(y, z)=K y+C_{1} z+C_{6}, g(x, z)=-K x-C_{3} z+C_{5}\right]
$$

12. Show that if the rotation is zero throughout a body (irrotational deformation), the displacement vector is the gradient of a scalar potential function. Hint: Use the idea from irrotational fluid flow.
