Kinematics*

1 Deformation and displacement

1.1 Deformation map

Consider a body in a intial reference state denoted by $\mathcal{B}$. Under some given loading, let this body go from $\mathcal{B}$ to a deformed state $\mathcal{B}'$. In this process, various material points that make up the body move in such a way that in general the distance between any two points in the state $\mathcal{B}'$ will be different from that in the state $\mathcal{B}$. We want to have a mathematical framework to quantify this deformation.

We set up a set of coordinate axes and consider the motion of a particular point in the body. Let the position vector of this point referred to our coordinate axes be $\boldsymbol{X}$. (Note that from now on we will use bold symbols like $\boldsymbol{X}$ to refer to a vector instead of $\vec{X}$.) After deformation, let the position vector of this same point be $\boldsymbol{x}$. In terms of components, we have $\boldsymbol{X} \equiv (X_1, X_2, X_3)$ and $\boldsymbol{x} \equiv (x_1, x_2, x_3)$. Using the indicial notation, we can write these as $\boldsymbol{X} \equiv X_i$ and $\boldsymbol{x} \equiv x_i$.

We note that $\boldsymbol{x}$, in general, will be a function of the initial position vector $\boldsymbol{X}$ as well as time, t, i.e. $\boldsymbol{x} \equiv \boldsymbol{x}(\boldsymbol{X}, t)$. This means that we have taken up a Lagrangian description of our system.† In terms of components we have

$$
\begin{aligned}
x_1 &\equiv x_1(X_1, X_2, X_3, t),\\
x_2 &\equiv x_2(X_1, X_2, X_3, t),\\
x_3 &\equiv x_3(X_1, X_2, X_3, t).
\end{aligned}
$$

*Notes prepared by Jeevanjyoti Chakraborty. Contact: jeevan@mech.iitkgp.ac.in

†It is very much possible to use an Eulerian description because there is a one-to-one correspondence between a point in the intial state and the point in the final state. That is, just as we can find out the position $\boldsymbol{x}$ at any time, t, given an initial position $\boldsymbol{X}$, so also we can find out the initial position $\boldsymbol{X}$ by tracking back from the position $\boldsymbol{x}$ at time t.

Using the indical notation, we can write

$$x_i = x_i(X_j, t).$$

DANGER: In the above indical notation, why do we need to write X_j for the dependence of x_i? To answer this, we must note that each of x_1, x_2, and x_3 is a function of all three of X_1, X_2, and X_3. If, suppose, we were to write $x_i(X_i)$ then that would be quite confusing because then two interpretations would be possible. First interpretation could be that x_1 is dependent only on X_1, x_2 only on X_3, and x_3 only on X_3. Second interpretation could be that some kind of summation is involved because there is a repeated index. However, both these interpretations would be totally wrong. To avoid these confusions or wrong interpretations, we write $x_i(X_j)$ and NOT $x_i(X_i)$.

We say that any one of the following:

$$\boldsymbol{x} \equiv \boldsymbol{x}(\boldsymbol{X}, t), \tag{1}$$

$$\text{or,} \quad x_i \equiv x_i(X_j, t) \tag{2}$$

represents the deformation map.

1.2 Displacement

We define the displacement as the difference between the final and the initial position vectors of a point. Since the final position vector $\boldsymbol{x}$ is a function of the initial position vector $\boldsymbol{X}$, therefore the displacement, $\boldsymbol{u}$ of the point is also a function of the initial position vector $\boldsymbol{X}$ and time t

$$\boldsymbol{u}(\boldsymbol{X}, t) := \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X}.$$

From now on, we consider the dependence on time to be implicilty included and stop mentioning t as an independent variable. Thus

$$\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{x}(\boldsymbol{X}) - \boldsymbol{X} \tag{3}$$

In terms of components we have

$$\begin{aligned} u_1(X_1, X_2, X_3) &= x_1(X_1, X_2, X_3) - X_1, \\ u_2(X_1, X_2, X_3) &= x_2(X_1, X_2, X_3) - X_2, \\ u_3(X_1, X_2, X_3) &= x_3(X_1, X_2, X_3) - X_3. \end{aligned}$$

Or, using indical notation, we can write in a very short form:

$$u_i(X_j) = x_i(X_j) - X_i.$$

Now although we can track the motion of an individual particle, just by following this motion, we cannot quantify the deformation. To do that, we need to look at the distance between two material points.

1.3 Quantification of deformation

We consider another neighbouring point Q (neighbour of P) whose position vector in the initial reference state is given by $\boldsymbol{X} + \mathrm{d}\boldsymbol{X}$. Let the position vector of this neighbouring point in the deformed state be $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$. The displacement of this neighbouring point Q will then be

$$\boldsymbol{u}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}) = (\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - (\boldsymbol{X} + \mathrm{d}\boldsymbol{X}). \tag{4}$$

However, from (3) we have $\boldsymbol{u}(\boldsymbol{X}) = \boldsymbol{x} - \boldsymbol{X}$. Therefore, from (4) we obtain

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}) &= \boldsymbol{u}(\boldsymbol{X}) + \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}, \\ \text{or,} \quad \mathrm{d}\boldsymbol{x} &= \mathrm{d}\boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X} + \mathrm{d}\boldsymbol{X}) - \boldsymbol{u}(\boldsymbol{X}). \end{aligned} \tag{5}$$

In terms of components we have

$$\mathrm{d}x_1 = \mathrm{d}X_1 + u_1(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) - u_1(X_1, X_2, X_3), \tag{6a}$$
$$\mathrm{d}x_2 = \mathrm{d}X_2 + u_2(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) - u_2(X_1, X_2, X_3), \tag{6b}$$
$$\mathrm{d}x_3 = \mathrm{d}X_3 + u_3(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) - u_3(X_1, X_2, X_3). \tag{6c}$$

Again, using indical notation, we can write in a very short form

$$\mathrm{d}x_i = \mathrm{d}X_i + u_i(X_j + \mathrm{d}X_j) - u_i(X_j). \tag{7}$$

To proceed, we use Taylor series expansion to put the aforementioned equation in a form involving gradients (instead of differences). However, this step will involve the Taylor series expansion of a function of more than one variable. We first focus on how to go about doing that.

Brief refresher on Taylor series expansion:

To refresh our memory about Taylor series expansion, we first consider a function of one variable. If $f(x)$ denotes the function value corresponding to the point x, then for a neighbouring point $x + \mathrm{d}x$ we have $f(x + \mathrm{d}x)$, which can be expanded about the point x as

$$f(x + \mathrm{d}x) = f(x) + \frac{\mathrm{d}x}{1!}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{(\mathrm{d}x)^2}{2!}\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \cdots$$

Now, instead of one variable, consider a function of two variables. If $f(x, y)$ denotes the function value corresponding to the point (x, y), then for a neighbouring point $x + \mathrm{d}x, y + \mathrm{d}y$, we have $f(x + \mathrm{d}x, y + \mathrm{d}y)$, which can be expanded about the point (x, y) as

$$\begin{aligned} f(x + \mathrm{d}x, y + \mathrm{d}y) &= f(x, y) + \frac{\mathrm{d}x}{1!}\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{1!}\frac{\partial f}{\partial y} + \frac{(\mathrm{d}x)^2}{2!}\frac{\partial^2 f}{\partial x^2} + 2\frac{\mathrm{d}x\mathrm{d}y}{2!}\frac{\partial^2 f}{\partial x \partial y} + \frac{(\mathrm{d}y)^2}{2!}\frac{\partial^2 f}{\partial y^2} + \cdots, \\ &= f(x, y) + \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right) f + \frac{1}{2}\left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right)^2 f + \cdots \end{aligned}$$

Similarly, if we consider a function of three variables we have

$$\begin{aligned} f(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) =& f(x, y, z) + \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}z\frac{\partial}{\partial z}\right) f \\ &+ \frac{1}{2}\left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}z\frac{\partial}{\partial z}\right)^2 f + \cdots \end{aligned}$$

For our purpose, the Taylor series expansion upto only the linear term will be sufficient. So we may write

$$f(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) \approx f(x, y, z) + \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}z\frac{\partial}{\partial z}\right) f$$

The function, f, of three variables evaluated at $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ and represented by $f(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$ may be compared with the second term on the r.h.s of each of Eqs (6a), (6b), (6c), i.e. with the terms representing the displacement components $u_1(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3)$, $u_2(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3)$, and $u_3(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3)$. These displacement components may also be approximated using upto the linear terms of the Taylor series expansion as

$$u_1(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) \approx u_1(X_1, X_2, X_3) + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_1, \tag{8a}$$

$$u_2(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) \approx u_2(X_1, X_2, X_3) + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_2, \tag{8b}$$

$$u_3(X_1 + \mathrm{d}X_1, X_2 + \mathrm{d}X_2, X_3 + \mathrm{d}X_3) \approx u_3(X_1, X_2, X_3) + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_3. \tag{8c}$$

Substituting Eqs (8a), (8b), and (8c) in Eqs (6a), (6b), and (6c), respectively, we obtain

$$\mathrm{d}x_1 = \mathrm{d}X_1 + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_1, \tag{9a}$$

$$\mathrm{d}x_2 = \mathrm{d}X_2 + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_2, \tag{9b}$$

$$\mathrm{d}x_3 = \mathrm{d}X_3 + \left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) u_3. \tag{9c}$$

Now, the operator $\left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right)$ may be interpreted as the dot product of the vector $\mathrm{d}\boldsymbol{X}$ (whose components are $\mathrm{d}X_1$, $\mathrm{d}X_2$, and $\mathrm{d}X_3$) and the gradient vector ∇ (whose components are $\frac{\partial}{\partial X_1}$, $\frac{\partial}{\partial X_2}$, and $\frac{\partial}{\partial X_3}$). Thus, we have

$$\left(\mathrm{d}X_1 \frac{\partial}{\partial X_1} + \mathrm{d}X_2 \frac{\partial}{\partial X_2} + \mathrm{d}X_3 \frac{\partial}{\partial X_3}\right) \equiv \mathrm{d}\boldsymbol{X} \cdot \nabla,$$

which can be used in Eqs (9a), (9b), and (9c) to give

$$\mathrm{d}x_1 = \mathrm{d}X_1 + \mathrm{d}\boldsymbol{X}\cdot\nabla u_1, \tag{10a}$$
$$\mathrm{d}x_2 = \mathrm{d}X_2 + \mathrm{d}\boldsymbol{X}\cdot\nabla u_2, \tag{10b}$$
$$\mathrm{d}x_3 = \mathrm{d}X_3 + \mathrm{d}\boldsymbol{X}\cdot\nabla u_3. \tag{10c}$$

These three equations (10a), (10b), and (10c) can be written in a compact vector form as

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{X} + \mathrm{d}\boldsymbol{X}\cdot\nabla\boldsymbol{u}. \tag{11}$$

Alternatively, Eqs (10a), (10b), and (10c) can be written in a very short form using indical notation as

$$\mathrm{d}x_i = \mathrm{d}X_i + \mathrm{d}X_j\frac{\partial u_i}{\partial X_j} \tag{12}$$

The term $\nabla\boldsymbol{u}$, or, equivalently, $\partial u_i/\partial X_j$ is called the displacement gradient. In the "Mathematical Preliminaries" notes, we mentioned that a term like this is the gradient of a vector. Remember that we have encountered something similar in Fluid Mechanics: the velocity gradient.

2 Strain

Although Eq. (11) or, equivalently, Eq. (12) contains all the information about the deformation, we would like to see explicitly how much deformation has taken place. We do so by comparing the two lengths: $|\mathrm{d}\boldsymbol{x}|$ (after deformation) and $|\mathrm{d}\boldsymbol{X}|$ (before deformation). A similar approach had been used in introducing strain (in Class XI) or in the discussion of normal strain (in first year mechanics): consider a length L and let it get stretched to a length $l = L + \Delta L$; then the normal strain is defined by comparing the two lengths as $(l - L)/L$. In fact, we are going to do something similar in order to obtain a general three-dimensional version of strain that will cover both normal strains and shear strains.

So, we want to compare $|\mathrm{d}\boldsymbol{x}|$ and $|\mathrm{d}\boldsymbol{X}|$. We define the engineering strain as

$$\varepsilon_{\mathrm{E}} := \frac{|\mathrm{d}\boldsymbol{x}| - |\mathrm{d}\boldsymbol{X}|}{|\mathrm{d}\boldsymbol{X}|} \tag{13}$$

However, both $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{X}$ are vectors and their magnitudes, i.e. $|\mathrm{d}\boldsymbol{x}|$ and $|\mathrm{d}\boldsymbol{X}|$ will involve square roots. So, actually it is easier to compare $|\mathrm{d}\boldsymbol{x}|^2$ with $|\mathrm{d}\boldsymbol{X}|^2$ because then we can simply compare the dot products:

$$|\mathrm{d}\boldsymbol{x}|^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x},$$
$$|\mathrm{d}\boldsymbol{X}|^2 = \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}.$$

Now, substituting the expressions for $\mathrm{d}x_1$, $\mathrm{d}x_2$, and $\mathrm{d}x_3$ from Eqs (9a), (9b), and (9c), respectively in the expression for $|\mathrm{d}\boldsymbol{x}|^2$, we have

$$\begin{aligned}|\mathrm{d}\boldsymbol{x}|^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} = &\left(\mathrm{d}X_1 + \mathrm{d}X_1\frac{\partial u_1}{\partial X_1} + \mathrm{d}X_2\frac{\partial u_1}{\partial X_2} + \mathrm{d}X_3\frac{\partial u_1}{\partial X_3}\right)^2\\ &+\left(\mathrm{d}X_2 + \mathrm{d}X_1\frac{\partial u_2}{\partial X_1} + \mathrm{d}X_2\frac{\partial u_2}{\partial X_2} + \mathrm{d}X_3\frac{\partial u_2}{\partial X_3}\right)^2\\ &+\left(\mathrm{d}X_3 + \mathrm{d}X_1\frac{\partial u_3}{\partial X_1} + \mathrm{d}X_2\frac{\partial u_3}{\partial X_2} + \mathrm{d}X_3\frac{\partial u_3}{\partial X_3}\right)^2.\end{aligned}$$

Expanding and collecting terms we obtain the following

$$\begin{aligned}|\mathrm{d}\boldsymbol{x}|^2 = &(\mathrm{d}X_1)^2 + (\mathrm{d}X_2)^2 + (\mathrm{d}X_3)^2\\ &+2(\mathrm{d}X_1)^2\left[\frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right\}\right]\\ &+2(\mathrm{d}X_2)^2\left[\frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right\}\right]\\ &+2(\mathrm{d}X_3)^2\left[\frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right\}\right]\\ &+2\mathrm{d}X_1\mathrm{d}X_2\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right)\\ &+2\mathrm{d}X_2\mathrm{d}X_3\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right)\\ &+2\mathrm{d}X_3\mathrm{d}X_1\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} + \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_3}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_3}{\partial X_3}\frac{\partial u_3}{\partial X_1}\right).\end{aligned} \tag{14}$$

Noting that $|\mathrm{d}\boldsymbol{X}|^2 = (\mathrm{d}X_1)^2 + (\mathrm{d}X_2)^2 + (\mathrm{d}X_3)^2$, we recast Eq. (14) as

$$|\mathrm{d}\boldsymbol{x}|^2 - |\mathrm{d}\boldsymbol{X}|^2 = 2\left[(\mathrm{d}X_1)^2E_{11} + (\mathrm{d}X_2)^2E_{22} + (\mathrm{d}X_3)^2E_{33} + 2\mathrm{d}X_1\mathrm{d}X_2E_{12} + 2\mathrm{d}X_2\mathrm{d}X_3E_{23} + 2\mathrm{d}X_3\mathrm{d}X_1E_{31}\right], \tag{15}$$

where

$$E_{11} = \frac{\partial u_1}{\partial X_1} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_1}\right)^2 + \left(\frac{\partial u_2}{\partial X_1}\right)^2 + \left(\frac{\partial u_3}{\partial X_1}\right)^2\right\}, \tag{16a}$$

$$E_{22} = \frac{\partial u_2}{\partial X_2} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_2}\right)^2 + \left(\frac{\partial u_2}{\partial X_2}\right)^2 + \left(\frac{\partial u_3}{\partial X_2}\right)^2\right\}, \tag{16b}$$

$$E_{33} = \frac{\partial u_3}{\partial X_3} + \frac{1}{2}\left\{\left(\frac{\partial u_1}{\partial X_3}\right)^2 + \left(\frac{\partial u_2}{\partial X_3}\right)^2 + \left(\frac{\partial u_3}{\partial X_3}\right)^2\right\}, \tag{16c}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2}\right), \tag{16d}$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3}\right), \tag{16e}$$

$$E_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} + \frac{\partial u_1}{\partial X_3}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_3}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_3}{\partial X_3}\frac{\partial u_3}{\partial X_1}\right). \tag{16f}$$

Alternatively, Eq. (14) can be written in a very short form using indical notation as

$$|\mathrm{d}\boldsymbol{x}|^2 = |\mathrm{d}\boldsymbol{X}|^2 + 2\mathrm{d}X_i\mathrm{d}X_j\left\{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)\right\}. \tag{17}$$

The terms inside the curly brackets in Eq. (17) represent the six terms, E_{11}, E_{22}, E_{33}, E_{12}, E_{21}, E_{23}, E_{32}, E_{31} and E_{13}; thus we have

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right), \tag{18}$$

so that Eq. (17) becomes

$$|\mathrm{d}\boldsymbol{x}|^2 = |\mathrm{d}\boldsymbol{X}|^2 + 2\mathrm{d}X_i\mathrm{d}X_jE_{ij}. \tag{19}$$

The entity, E_{ij} is what is referred to as the finite strain tensor. It is a second-order tensor (there are two free indices i and j), and it can be represented by a 3×3 matrix. It is also a symmetric tensor. This fact can be readily verified by interchanging i and j in Eq. (18). Thus, there are only 6 unique elements in the 3×3 matrix representation of the finite strain tensor. These 6 elements are of course E_{11}, E_{22}, E_{33}, E_{12}, E_{23}, and E_{31}. And, you can verify for yourself that we indeed have $E_{12} = E_{21}$, $E_{23} = E_{32}$, and $E_{31} = E_{13}$.

VERY IMPORTANT: If each of $\left|\dfrac{\partial u_i}{\partial X_j}\right| \ll 1$, then the quadratic terms in the expressions of each component of E_{ij} can be dropped. Under this consideration, the resulting strain tensor is referred to as the infinitesimal strain tensor and is denoted by $\boldsymbol{\varepsilon}$ or ε_{ij}. Thus, we have

$$E_{ij} \approx \varepsilon_{ij} \tag{20}$$

In indical notation, ε_{ij} is thus given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right), \tag{21}$$

which can be equivalently represented in matrix form as

$$[\boldsymbol{\varepsilon}] \equiv \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2} + \dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3} + \dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1} + \dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3} + \dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1} + \dfrac{\partial u_1}{\partial X_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2} + \dfrac{\partial u_2}{\partial X_3}\right) & \dfrac{\partial u_3}{\partial X_3} \end{bmatrix}. \tag{22}$$

Often we use the rectangular Cartesian coordinate system where X_1 corresponds to x, X_2 corresponds to y, and X_3 corresponds to z. Then u_1 corresponds to u_x or u, u_2 to v or u_y, and u_3 to w or u_z. Additionally, ε_{11} corresponds to ε_{xx}, ε_{22} to ε_{yy}, ε_{33} to ε_{zz}, $\varepsilon_{12} = \varepsilon_{21}$ to $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{23} = \varepsilon_{32}$ to

$\varepsilon_{yz} = \varepsilon_{zy}$, $\varepsilon_{31} = \varepsilon_{13}$ to $\varepsilon_{zx} = \varepsilon_{xz}$, and we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \tag{23}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \tag{24}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}, \tag{25}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{26}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \tag{27}$$

$$\varepsilon_{zx} = \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right). \tag{28}$$

VERY IMPORTANT: These six relations are referred to as the **strain-displacement relations**.

If the displacement components u, v, and w are known as functions of x, y, and z, then the strain components can be found out using these strain-displacement relations. Thus, from a knowledge of the displacement field, the strain field can be determined.

Finally, we reconsider Eq. (19) with $\left|\frac{\partial u_i}{\partial X_j}\right| \ll 1$, so that $E_{ij} \approx \varepsilon_{ij}$

$$|\mathrm{d}\boldsymbol{x}|^2 = |\mathrm{d}\boldsymbol{X}|^2 + 2\mathrm{d}X_i\mathrm{d}X_j\varepsilon_{ij}. \tag{29}$$

We write $\mathrm{d}X_i$ as $|\mathrm{d}\boldsymbol{X}|N_i$ where $N_i \equiv \hat{\boldsymbol{N}}$ is the unit vector along $\mathrm{d}X_i \equiv \mathrm{d}\boldsymbol{X}$. Similarly, we write $\mathrm{d}X_j = |\mathrm{d}\boldsymbol{X}|N_j$. Note that $\mathrm{d}X_i$ and $\mathrm{d}X_j$ represent the same vector $\mathrm{d}\boldsymbol{X}$. Similary, N_i and N_j represent the same unit vector $\hat{\boldsymbol{N}}$. Then from Eq. (29), we have

$$\begin{aligned}|\mathrm{d}\boldsymbol{x}|^2 &= |\mathrm{d}\boldsymbol{X}|^2 + 2|\mathrm{d}\boldsymbol{X}|^2 N_i\varepsilon_{ij}N_j \\ &= |\mathrm{d}\boldsymbol{X}|^2\left(1 + 2N_i\varepsilon_{ij}N_j\right)\end{aligned}$$

Note that the combination $N_i\varepsilon_{ij}N_j$ is a scalar quantity (no free indices) and in compact representation it is the quadratic form $\hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}$,. Thus, we have

$$|\mathrm{d}\boldsymbol{x}|^2 = |\mathrm{d}\boldsymbol{X}|^2 \left(1 + 2\hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}}\right).$$

We can now use the preceding relation to find an expression of the engineering strain in terms of the unit normals and the infinitesimal strain tensor. Thus, we have the following:

$$\begin{aligned}
&\frac{|\mathrm{d}\boldsymbol{x}|^2}{|\mathrm{d}\boldsymbol{X}|^2} = 1 + 2\hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \\
\text{or,}\quad & |1 + \varepsilon_{\mathrm{E}}|^2 = 1 + 2\hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \quad \text{(using Eq. (13))} \\
\text{or,}\quad & 1 + 2\varepsilon_{\mathrm{E}} = 1 + 2\hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \quad \text{(under small deformations } \varepsilon_{\mathrm{E}} \ll 1\text{)} \\
\text{or,}\quad & \varepsilon_{\mathrm{E}} = \hat{\boldsymbol{N}}^T \cdot \boldsymbol{\varepsilon} \cdot \hat{\boldsymbol{N}} \equiv N_i \varepsilon_{ij} N_j \qquad (30)
\end{aligned}$$

Given the strain at a point, this expression allows us to determine the engineering strain along the direction of $\hat{\boldsymbol{N}}$.

3 Rigid-body rotation

Let us look once more at (12)

$$\begin{aligned}
\mathrm{d}x_i &= \mathrm{d}X_i + \frac{\partial u_i}{\partial X_j}\mathrm{d}X_j \\
&= \mathrm{d}X_i + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)\mathrm{d}X_j}_{①} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)\mathrm{d}X_j}_{②} \qquad (31)
\end{aligned}$$

① We have already interpreted this term as the strain. So this term can be interpreted as the total deformation associated purely with the strain.

② This term represents the rigid-body rotation – devoid of any strains.

Let us consider rigid-body rotation in more detail. First consider a situation where there are no strains, so that

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) = 0. \qquad (32)$$

Then we have

$$dx_i = dX_i + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) dX_j. \tag{33}$$

We denote the rotation tensor by Ω:

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right). \tag{34}$$

Since Ω_{ij} is anti-symmetric, the diagonal terms are zero. So the matrix representation can be done using only three independent components:

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \equiv \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} - \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} - \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial X_1} - \frac{\partial u_1}{\partial X_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} - \frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial X_1} - \frac{\partial u_1}{\partial X_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial X_2} - \frac{\partial u_2}{\partial X_3}\right) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}. \tag{35}$$

Here the subscripts of ω have been chosen so that $\Omega_{ij}dX_j$ can be written as $\boldsymbol{\omega} \times d\boldsymbol{X}$ with

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}. \tag{36}$$

Thus from (33), we have

$$d\boldsymbol{x} = d\boldsymbol{X} + \boldsymbol{\omega} \times d\boldsymbol{X}. \tag{37}$$

Note that $\boldsymbol{\omega} \times d\boldsymbol{X}$ must be perpendicular to $d\boldsymbol{X}$. Further, if $|\boldsymbol{\omega}| \ll 1$ and $|d\boldsymbol{X}| \ll 1$ then $|\boldsymbol{\omega} \times d\boldsymbol{X}| \ll 1$, and we have $|d\boldsymbol{x}| \approx |d\boldsymbol{X}|$ and the motion is approximately rigid-body motion.

4 Principal strains and principal axes of strain

We have seen from Eq. (30) that the engineering strain at a point in the direction of the unit normal vector $\hat{\boldsymbol{N}}$ is

$$\varepsilon_{\mathrm{E}} = \varepsilon_{ij} N_i N_j.$$

As we keep on changing $\hat{\boldsymbol{N}}$, ε_{E} will change. An important question to answer is: what is the maximum possible value of ε_{E} and which $\hat{\boldsymbol{N}}$ does it correspond to? This is a special type of extremization problem because we simultaneously need to satisfy the constraint that $|\hat{\boldsymbol{N}}| = 1$. So this is a constrained optimization problem and we need to use the method of Lagrange multipliers. Thus, we set

$$L = \varepsilon_{ij} N_i N_j + \lambda \left(1 - N_k N_k\right), \tag{38}$$

where λ is the Lagrange multiplier. For the extremization

$$\begin{aligned} \frac{\partial L}{\partial N_k} &= 0, \\ \text{or,} \quad \frac{\partial}{\partial n_k} \left(\varepsilon_{ij} N_i N_j + \lambda \left(1 - N_k N_k\right)\right) &= 0. \end{aligned} \tag{39}$$

The apparently difficult part in the above is the partial derivative of the term $\varepsilon_{ij} N_i N_j$. We isolate out

$$\frac{\partial}{\partial N_k}(N_i N_j) = \frac{\partial N_i}{\partial N_k} N_j + N_i \frac{\partial N_j}{\partial N_k} = \delta_{ik} N_j + N_i \delta_{jk}, \tag{40}$$

where the appearance of the Kronecker deltas can be verified through expanding $\partial N_i / \partial N_k$ and $\partial N_j / \partial N_k$ as 3×3 matrices. Going back to (39), we now have

$$\begin{aligned} & \varepsilon_{ij} \delta_{ik} N_j + \varepsilon_{ij} N_i \delta_{jk} - 2\lambda N_k = 0, \\ \text{or,} \quad & \varepsilon_{kj} N_j + \varepsilon_{ik} N_i - 2\lambda N_k = 0, \\ \text{or,} \quad & \varepsilon_{jk} N_j + \varepsilon_{ik} N_i - 2\lambda N_k = 0, \quad \text{(using symmetry property: } \varepsilon_{kj} = \varepsilon_{jk}) \\ \text{or,} \quad & \varepsilon_{ik} N_i + \varepsilon_{ik} N_i - 2\lambda N_k = 0, \quad \text{(since } j \text{ and } i \text{ are repeated indices, using either one is fine)} \\ \text{or,} \quad & 2\varepsilon_{ik} N_i - 2\lambda N_k = 0, \\ \text{or,} \quad & \varepsilon_{ik} N_i - \lambda \delta_{ik} N_i = 0 \quad \text{(using substitution property of Kronecker delta),} \\ \text{or,} \quad & \left(\varepsilon_{ik} - \lambda \delta_{ik}\right) N_i = 0, \end{aligned} \tag{41}$$

which in matrix form is

$$([\boldsymbol{\varepsilon}] - \lambda\mathbb{I})\,[\hat{\boldsymbol{N}}] = 0, \tag{42}$$

where $\mathbb{I}$ is the identity matrix, the matrix equivalent of Kronecker delta.

This is the familiar form of an eigenvalue problem. The direction $\hat{\boldsymbol{N}}$ in which the engineering strain ε_{E} is extremised is an eigenvector of the strain tensor ε_{ij} while λ is the corresponding eigenvalue or the desired "extreme" value of the engineering strain. It is these extreme values that are referred to as the principal strains and the directions are referred to as the principal directions.

5 Strain compatibility

Given a displacement field u_i we can find the strain tensor ε_{ij}. Now what about the reverse situation: given ε_{ij}, can we find u_i?

Since ε_{ij} is symmetric, there are only six independent strain components. Each of these strain components are expressed in terms of the displacement field components. These expressions can be viewed as partial differential equations (PDEs) for u_i components (because ε_{ij} are given). But there is a problem: there are six PDEs but only three variables to find. So there must be additional relationships among the components of ε_{ij}. In other words, any arbitrary set of six numbers cannot be the components of ε_{ij} because then a *compatible* displacement field may not be found. Let's consider just such a situation (in an easier 2D framework). It is given:

$$\varepsilon_{11} = x_2^2, \quad \varepsilon_{12} = 0, \varepsilon_{22} = 0.$$

Then

$$\begin{aligned}\frac{\partial u_1}{\partial x_1} &= x_2^2 \implies u_1 = x_1 x_2^2 + f(x_2),\\ \frac{\partial u_2}{\partial x_2} &= 0 \implies u_2 = g(x_1)\end{aligned}$$

So

$$\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2x_1x_2 + f'(x_2) + g'(x_1).$$

According to the given strain field, $\varepsilon_{12} = 0$. But the above expression can never be identically 0. The problem is with the presence of the $x_1 x_2$ term. Had it been just $2x_1 + f'(x_2) + g'(x_1)$ then we could have had a situation like

$$\begin{aligned} g'(x_1) &= -2x_1 + c, \\ \text{and} \quad f'(x_2) &= -c, \end{aligned}$$

to ensure that $\varepsilon_{12} = 0$ identically. In that case

$$\begin{aligned} f(x_2) &= -cx_2 + k_1, \\ \text{and} \quad g(x_1) &= -x_1^2 + cx_1 + k_2, \end{aligned}$$

where k_1 and k_2 are arbitrary constants.