Constitutive Relations*

1 The need for constitutive relations

Just like in fluid mechanics, in solid mechanics too, we need to ensure that mass conservation is satisfied. We can proceed exactly as was done in fluid mechanics to obtain the following

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

which is, of course, the continuity equation. In this equation, ρ is the mass density of the material that can, in general, vary with time and position, and $\mathbf{v}$ is the velocity. It is extremely important to note that this equation which was introduced in the context of studying fluids holds true for solids too. *Mass conservation must be satisfied, whether it is fluid or solid.*

Next, again like in fluid mechanics, we need to ensure the balance of linear momentum. This balance has already been discussed in the previous chapter on "Stress", and it led to the Cauchy's equations of motion

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{b}$ is the body force per unit volume. We had also discussed that for a body in equilibrium, this equation reduces to the following form, referred to as the mechanical equilibrium equations:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = 0. \tag{3}$$

Let us take stock of what we *have* till now, and we want we *need* in order to have a closed system of equations.

First, in the continuity equation (1), we have the unknown density ρ and the three unknown components of the velocity vector $\mathbf{v}$. Alternatively, instead of the unknown velocity components, we can say that the

*Notes prepared by Jeevanjyoti Chakraborty. Contact: jeevan@mech.iitkgp.ac.in unknowns are the three components of the displacement vector $\mathbf{u}$. We note that the velocity can be immediately found once the displacement is found using the kinematic definition: $\mathbf{v} := \dfrac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}$. In any case, what we have are $(1+3=)4$ unknowns and 1 equation.

Next, in the mechanical equilibrium equations (3), we have the six unknown components of the stress tensor $\boldsymbol{\sigma}$ and three equations. So, in total, we have $(1+3+6=)10$ unknowns and $(1+3=)4$ equations. There is, therefore, a discrepancy of 6 equations. These are what we *need* to find.

We were in a similar situation back in fluid mechanics. We had resolved the discrepancy by using the constitutive relations, i.e. relations which represented the behaviour of the constitutive material. In particular, we had used the simplest possible fluid mechanical behaviour for Newtonian fluids in the form

$$\boldsymbol{\sigma} = \bar{\lambda}\,\mathrm{tr}\left(\dot{\mathbf{E}}\right)\mathbf{I} + 2\mu\dot{\mathbf{E}}, \tag{4}$$

where $\bar{\lambda}$ and μ were material constants (in fact, μ was the dynamic viscosity), $\dot{\mathbf{E}}$ was the strain-rate tensor[†], tr represented the trace[‡], $\mathbf{I}$ was the identity tensor.

The immediate question to ask ourselves is: Can we resolve the discrepancy in the current situation of mechanics of solids again through something similar? The answer is a resounding YES.

We use the following constitutive relation

$$\boldsymbol{\sigma} = \lambda\,\mathrm{tr}\left(\boldsymbol{\varepsilon}\right)\mathbf{I} + 2G\boldsymbol{\varepsilon}, \tag{5}$$

where λ and G are material constants and referred to as Lamé parameters and $\boldsymbol{\varepsilon}$ is the infinitesimilar strain tensor. We note that just like Eq. (4) was valid only for a Newtonian fluid, Eq. (5) is valid only for linear, elastic, isotropic solids. Remember that the condition for linearity and isotropy was embedded within the characteristics of Newtonian fluid too.

[†]Remember that $\dot{\mathbf{E}} := \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right)$.

[‡]Remember the trace is just the sum of the diagonal elements of the matrix representation of a second-order tensor (like $\dot{\mathbf{E}}$).

Now, although Eq. (5) gives us six equations that apparently covers the 6-equation discrepancy, but note that six equations are actually expressions of the six components of $\boldsymbol{\sigma}$ in terms of the six components of $\boldsymbol{\varepsilon}$. Thus, in introducing the six equations, we have apparently ended up in introducing another six unknowns! But remember that we already have our six strain-displacement relations.