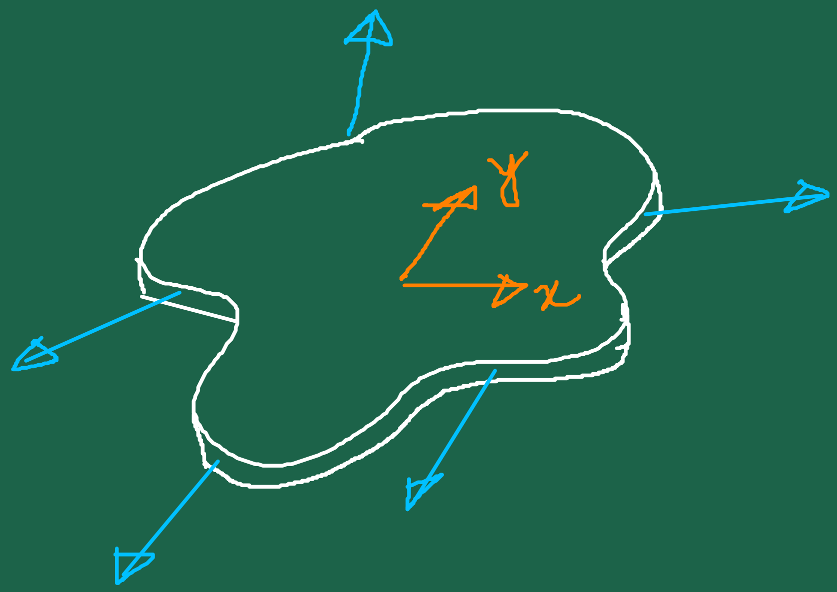
2D Elasticity

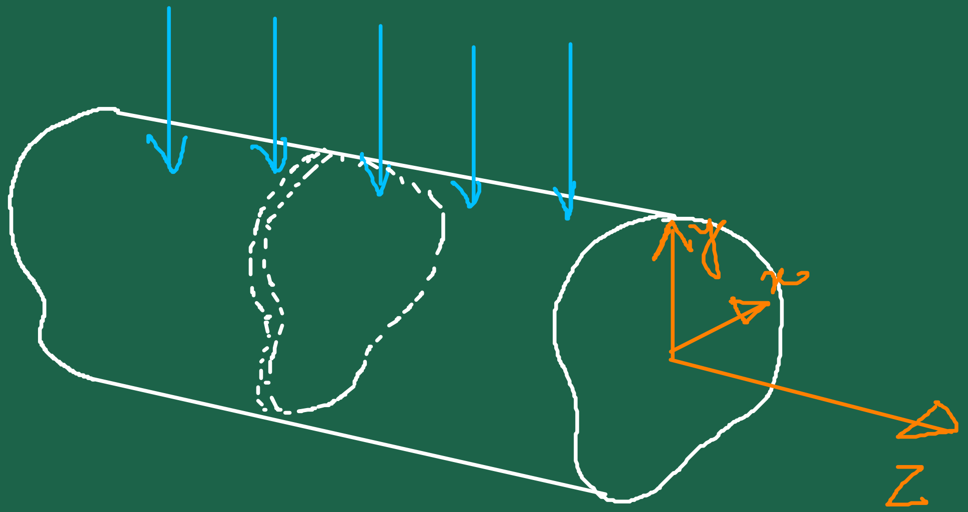
Plane Stress



$$\sigma_{zz} = \sigma_{zx} = \sigma_{yz} = 0$$

$$\sigma_{xy} \equiv \sigma_{xy}(x,y), \quad \sigma_{xx} \equiv \sigma_{xx}(x,y), \quad \sigma_{yy} \equiv \sigma_{yy}(x,y)$$

Plane Strain



$$u \equiv u(x,y) \qquad w \equiv 0$$

$$v \equiv v(x,y)$$

Plane Stress

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}})\right] \rightarrow \varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \cancel{\sigma_{zz}})\right] \qquad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancel{\sigma_{zz}} - \nu(\sigma_{xx} + \sigma_{yy})\right] \qquad \varepsilon_{zz} = -\frac{1}{E}\nu(\sigma_{xx} + \sigma_{yy})$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy} \qquad \varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$

$$\varepsilon_{yz} = \frac{1}{2G}\cancelto{0}{\sigma_{yz}} = 0 \qquad \varepsilon_{yz} = 0$$

$$\varepsilon_{zx} = \frac{1}{2G}\cancelto{0}{\sigma_{zx}} = 0 \qquad \varepsilon_{zx} = 0$$

Plane Strain

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \Rightarrow \varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy} - \nu^2\sigma_{xx} - \nu^2\sigma_{yy}\right)$$

$$= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \Rightarrow \varepsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$0 \leftarrow \cancel{\varepsilon_{zz}} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right]$$

$$\Rightarrow \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\sigma_{xx} \neq 0, \quad \sigma_{yy} \neq 0$$

$$\sigma_{zz} \neq 0$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy}$$

$$\sigma_{xy} \neq 0$$

$$0 \leftarrow \cancel{\varepsilon_{yz}} = \frac{1}{2G}\sigma_{yz} \Rightarrow \sigma_{yz} = 0$$

$$0 \leftarrow \cancel{\varepsilon_{zz}} = \frac{1}{2G}\sigma_{zx} \Rightarrow \sigma_{zx} = 0$$

$$\sigma_{zx} = \sigma_{yz} = 0$$

2D Elasticity ... contd.

Airy Stress Function

For Plane Stress problems

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$$

Stress eqb. eqns

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \end{aligned}\right\} \rightarrow$$

Always satisfied if

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

ϕ : Airy stress function

$$\nabla \cdot \vec{v} = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$v_x = \frac{\partial \psi}{\partial y}$$

$$v_y = -\frac{\partial \psi}{\partial x}$$

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Does this mean that our problem is magically solved?

Boundary conditions must be properly considered

σ_{xx}, σ_{xy}, σ_{yy} $\rightarrow$ ε_{xx}, ε_{xy}, ε_{yy} $\rightarrow$ Dangers that these strains may <u>NOT</u> be compatible

$\rightarrow$ Arbitrary choices of the Airy stress fn are <u>NOT</u> allowed!

Information from the compatibility eqns <u>must</u> be embedded within the mathematical formulation of the stress fn. approach.

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Compatibility eqns

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \leftarrow \text{Constitutive Reln} \quad \leftarrow \text{Stress eqb. eqns}$$

$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\nabla^2}\left(\sigma_{xx} + \sigma_{yy}\right) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2}\right) = 0$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \Rightarrow \left(\nabla^2\right)^2 \phi = 0$$

$$\Rightarrow \boxed{\nabla^4 \phi = 0}$$

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$$\nabla^4 \phi = 0$$

$$\Rightarrow \boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0}$$

∇^4 : Biharmonic operator

Operational Steps :

Assume some general sol^n forms for ϕ
Conditions that must be satisfied for the general sol^n form to satisfy $\nabla^4\phi = 0$
Satisfy the boundary conditions

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Usual to consider polynomial forms for ϕ

\# $\phi = \text{const.} \rightarrow \nabla^4 \phi = 0$ ✓

$\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0$

\# $\phi = Ax, \; Ay, \; Ax + By \rightarrow \nabla^4 \phi = 0$ ✓

$\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = 0$

\# $\phi = Ay^2 \rightarrow \nabla^4 \phi = 0$ ✓

$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = A, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0$

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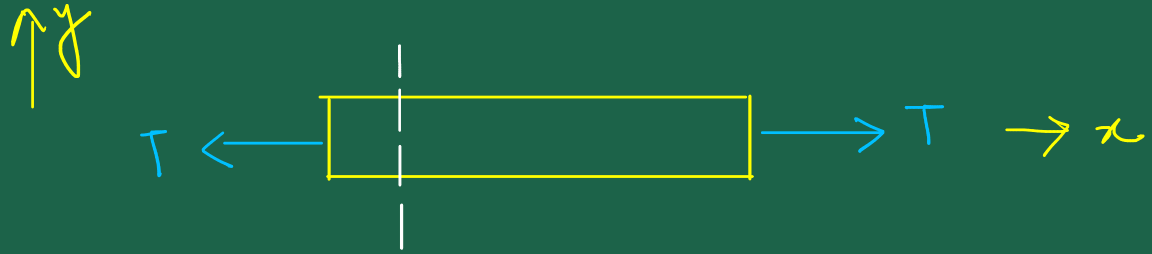

$\sigma_{xx} = \text{const.} = \frac{T}{a} \leftarrow$ c/s area

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2A = \frac{T}{a}$$

$$\phi = Ay^2 = \frac{T}{a} y^2 \checkmark$$

$$\sigma_{yy} = 0, \quad \sigma_{xy} = 0$$

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] = \frac{1}{E} \times \frac{T}{a} \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] = -\frac{\nu}{E} \times \frac{T}{a} \\ \varepsilon_{xy} &= 0 \end{aligned}\right\} \rightarrow u, v \checkmark$$

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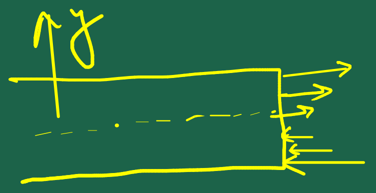
$\phi = Ay^3 \longrightarrow \nabla^4\phi = 0 \ \forall$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6Ay$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 0$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y} = 0$$

$\leftarrow$ Pure bending problem



$\leftarrow \sigma_{xx} \propto y$

$\phi = A_{40}x^4 + A_{31}x^3y + A_{22}x^2y^2 + A_{13}xy^3 + A_{04}y^4$

$\hookrightarrow \nabla^4\phi = 0$ (not satisfied identically)