2D Elasticity*

1 Plane stress

The salient features of a plane stress situation are:

- Domain bounded by two parallel planes
- Distance between two planes is small compared to other dimensions
- When referred to a rectangular Cartesian coordinate system, it is usual to take the two planes to be represented by $z = \pm h$ such that the *very small* distance between the two planes is $2h$, and the $x - y$ plane (denoting the mid-surface) is used to describe the problem.
- The two planes $z = h$ and $z = -h$ are stress-free so that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ on each face.
- It is expected that there will be very little variation in σ_{zz}, σ_{xz}, and σ_{yz} through the thickness; hence, they are approximately taken as zero throughout the entire domain.
- Again because of the very small thickness, the non-zero stress components will have little variation with z; they are approximately taken to be *independent* of z. Thus,

$$\sigma_{xx} \equiv \sigma_{xx}(x, y), \sigma_{yy} \equiv \sigma_{yy}(x, y), \sigma_{xy} \equiv \sigma_{xy}(x, y).$$

- There can be no body forces in the z-direction; this follows immediately from the z-component of the mechanical equilibrium equation.

From constitutive equation of a linear, elastic, isotropic solid, we have

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for the plane stress situation:

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right), \tag{1a}$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right), \tag{1b}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right), \tag{1c}$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}, \tag{1d}$$

$$\varepsilon_{yz} = 0, \tag{1e}$$

$$\varepsilon_{zx} = 0. \tag{1f}$$

Adding (1a) and (1b) and substituting in (1c), we have

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}\left(\varepsilon_{xx} + \varepsilon_{yy}\right). \tag{2}$$

NOTE:

- Even though $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, we still have $\varepsilon_{zz} \neq 0$.
- All strains are independent of z.

Now, consider the compatibility equations[†]:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \tag{3a}$$

$$\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z^2}} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\cancel{\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z}}, \tag{3b}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_{xx}}{\partial z^2}} = 2\cancel{\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x}}, \tag{3c}$$

$$\cancel{\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\cancel{\frac{\partial \varepsilon_{yz}}{\partial x}} + \cancel{\frac{\partial \varepsilon_{zx}}{\partial y}} + \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}}\right), \tag{3d}$$

$$\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(-\cancel{\frac{\partial \varepsilon_{zx}}{\partial y}} + \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}} + \cancel{\frac{\partial \varepsilon_{yz}}{\partial x}}\right), \tag{3e}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\cancel{\frac{\partial \varepsilon_{xy}}{\partial z}} + \cancel{\frac{\partial \varepsilon_{yz}}{\partial x}} + \cancel{\frac{\partial \varepsilon_{zx}}{\partial y}}\right). \tag{3f}$$

[†]Refer to Problem Sheet 1 on "Kinematics".

The only compatibility equation that is considered is (3a). The other three compatibility equations involving ε_{zz} are neglected.

Now, using (1a) and (1b) in (3a), we have, after rearranging

$$\left(-\nu\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\sigma_{xx}+\left(\frac{\partial^2}{\partial x^2}-\nu\frac{\partial^2}{\partial y^2}\right)\sigma_{yy}=2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}. \tag{4}$$

Adding and subtracting $\dfrac{\partial^2\sigma_{xx}}{\partial x^2}$ and $\dfrac{\partial^2\sigma_{yy}}{\partial y^2}$, we have

$$-(1+\nu)\left(\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right)+\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}. \tag{5}$$

Consider the equilibrium equations

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+F_x=0, \tag{6a}$$
$$\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+F_y=0, \tag{6b}$$

Differentiating the first of the preceding equations with respect to x and the second with respect to y, we have

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{xy}}{\partial x\partial y}=-\frac{\partial F_x}{\partial x}, \tag{7a}$$
$$\frac{\partial^2\sigma_{xy}}{\partial y\partial x}+\frac{\partial^2\sigma_{yy}}{\partial y^2}=-\frac{\partial F_y}{\partial y}. \tag{7b}$$

Adding these two equations, we obtain

$$\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}+2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}=-\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right). \tag{8}$$

Substituting (8) in (5), we have

$$\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=-(1+\nu)\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}\right). \tag{9}$$

We consider the equilibrium equations agains but assume the body forces to be conservative in nature so that they are expressed as the gradient of a scalar potential, i.e. $F_x = -\dfrac{\partial V}{\partial x}$ and $F_y = -\dfrac{\partial V}{\partial y}$, so that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial V}{\partial x} = 0, \tag{10a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial V}{\partial y} = 0. \tag{10b}$$

We define the following:

$$\sigma_{xx} - V = \frac{\partial^2 \varphi}{\partial y^2}, \tag{11a}$$

$$\sigma_{yy} - V = \frac{\partial^2 \varphi}{\partial x^2}, \tag{11b}$$

$$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \tag{11c}$$

so that (10a) and (10b) are satisfied identically.

Using the expressions of σ_{xx} and σ_{yy} in Eq. (9), we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(V + \frac{\partial^2 \varphi}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(V + \frac{\partial^2 \varphi}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right),$$

or, $\quad 2\nabla^2 V + \nabla^2(\nabla^2 \varphi) = (1+\nu)\nabla^2 V,$

or, $\quad \nabla^4 \varphi = -(1-\nu)\nabla^2 V \qquad (12)$

Note that ∇^4 is referred to as the biharmonic operator.

If the body forces vanish or if $\nabla^2 V = 0$ then we have

$$\nabla^4 \varphi = 0. \tag{13}$$

This equation is referred to as the biharmonic equation, and its solutions are referred to as the biharmonic solutions.

2 Plane Strain

Consider an infinitely long cylindrical (prismatic) body

- If body forces and tractions on the lateral boundaries are independent of the longitudinal axis variable, z, and have no z−component then the deformation field can be taken as

  $$u \equiv u(x, y), \quad v \equiv v(x, y)$$

  This deformation is referred to as a state of plane strain in the $x - y$ plane.

- All cross sections will have identical displacements and thus the 3D problem is reduced to a 2D problem.

Considering the preceding points, we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),$$
$$\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{yz} = 0.$$

The only compatibility equation that does not reduce to a $0 = 0$ form is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}. \tag{14}$$

Considering the constitutive equation for a linear, elastic, isotropic material we have

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \tag{15a}$$
$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right], \tag{15b}$$
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \implies \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right), \tag{15c}$$
$$\varepsilon_{xy} = \frac{1 + \nu}{E}\sigma_{xy}, \tag{15d}$$
$$\sigma_{yz} = \sigma_{zx} = 0 \quad (\text{Since } \varepsilon_{yz} = \varepsilon_{zx} = 0) \tag{15e}$$

Note that unlike the plane stress case, in the plane strain case we have $\sigma_{zz} \neq 0$.

Substituting the expression of σ_{zz} in the expressions for ε_{xx} and ε_{yy}, we have

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right], \tag{16a}$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right]. \tag{16b}$$

We next use these expressions for ε_{xx} and ε_{yy} (from Eq. (16)(a) and (b)) as well as for ε_{xy} (from Eq. (15)(c)) in the compatibility equation, Eq. (14), to obtain

$$\frac{1+\nu}{E}\frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] + \frac{1+\nu}{E}\frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] = 2\frac{1+\nu}{E}\frac{\partial^2\sigma_{xy}}{\partial x\partial y}, \tag{17}$$

$$\text{or,}\quad (1-\nu)\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right) - \frac{\partial^2\sigma_{xx}}{\partial x^2} - \frac{\partial^2\sigma_{yy}}{\partial y^2} = 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y}. \tag{18}$$

Considering the equilibrium equations, we have

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + F_x = 0 \implies \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial f_x}{\partial x} = 0, \tag{19a}$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + F_y = 0 \implies \frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + \frac{\partial f_y}{\partial y} = 0. \tag{19b}$$

Partially differentiating the first of the preceding equations with respect to x and the second with respect to y, and adding them, we obtain

$$\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} + 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right). \tag{20}$$

Using Eq. (20) in Eq. (18), we obtain

$$(1-\nu)\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right) = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right). \tag{21}$$

Proceeding similarly as in the plane stress case we consider a conservative body force per unit volume, we set

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} + V, \tag{22a}$$

$$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} + V, \tag{22b}$$

$$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}, \tag{22c}$$

where V is a scalar potential such that $f_x = -\dfrac{\partial V}{\partial x}$ and $f_y = -\frac{\partial V}{\partial y}$.

Using these expressions in Eq. (21), we have

$$(1-\nu)\nabla^2 \left(\frac{\partial^2 \varphi}{\partial y^2} + V + \frac{\partial^2 \varphi}{\partial x^2} + V \right) = \nabla^2 V,$$

$$\text{or,} \quad \nabla^4 \varphi = -\frac{1-2\nu}{1-\nu} \nabla^2 V. \tag{23}$$

Again, as in the plane stress case, if the body forces vanish or if $\nabla^2 V = 0$ then we have

$$\nabla^4 \varphi = 0. \tag{24}$$

Summary for plane stress and plane strain cases:

$$\text{Plane stress:} \quad \nabla^4 \varphi = -(1-\nu)\nabla^2 V, \tag{25}$$

$$\text{Plane strain:} \quad \nabla^4 \varphi = -\frac{1-2\nu}{1-\nu} \nabla^2 V. \tag{26}$$