Assignment Sheet in Fluid Mechanics*
Mathematical Preliminaries

Note: The purpose of the following exercises and problems is to prepare you for some of the very important derivations, especially in the context of the Navier-Stokes equations. Working through these exercises and problems will also impart a certain clarity and depth of thought when tackling the problems in the chapter on kinematics.

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1. Expand $\nabla^2\mathbf{v}$ in terms of the components of the velocity vector and appropriate derivaties in the rectangular Cartesian coordinate system. Interpret ∇^2 as

$$\nabla^2 \equiv \nabla \cdot \nabla \equiv \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right).$$

   (In this assignment sheet, we use $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ to represent unit vectors along the x, y, and z directions, respectively; thus $\mathbf{e}_x \equiv \mathbf{i}$, $\mathbf{e}_y \equiv \mathbf{j}$, and $\mathbf{e}_z \equiv \mathbf{k}$.)

2. Expand $\mathbf{v} \cdot \nabla\mathbf{v}$ in terms of the components of the velocity and appropriate derivatives in the rectangular Cartesian coordinate system. Interpret $\mathbf{v} \cdot \nabla\mathbf{v}$ as the operator $\mathbf{v} \cdot \nabla$ operating on $\mathbf{v}$.

3. What are the index notation representations of $\nabla^2\mathbf{v}$ and $\mathbf{v} \cdot \nabla\mathbf{v}$? (*Hint:* First think what is the order of the tensor denoted by each of the expressions. The number of free indices in the index notation representation should be equal to the order of the tensor.)

4. Expand $\nabla \times \mathbf{v}$ in terms of the components of the velocity vector and appropriate derivatives in the rectangular Cartesian coordinate system. While it is possible to do this rather algorithmically using

$$\nabla \times \mathbf{v} \equiv \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & vz \end{vmatrix},$$

   but this algorithmic approach will lead you into trouble in cylindrical coordinates. Instead, evaluate the expression using

$$\nabla \times \mathbf{v} \equiv \left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \times \left(v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z\right),$$

   and noting that $\mathbf{e}_x \times \mathbf{e}_y = \mathbf{e}_z$, $\mathbf{e}_y \times \mathbf{e}_z = \mathbf{e}_x$, $\ldots \mathbf{e}_y \times \mathbf{e}_x = -\mathbf{e}_z$, and so on.

5. In terms of the components of the velocity vector and appropriate derivatives in the cylindrical coordinate system, expand (a) $\mathbf{v} \cdot \nabla\mathbf{v}$, (b) $\nabla^2\mathbf{v}$, and (c) $\nabla \times \mathbf{v}$. The following points will help you in carrying out the expansions:

   (i) In cylindrical coordinates, the gradient operator is given by

$$\nabla \equiv \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \mathbf{e}_z \frac{\partial}{\partial z}$$

*Compiled and prepared by Jeevanjyoti Chakraborty. For queries, email: jeevan@mech.iitkgp.ac.in

(ii) Interpret the term $\mathbf{v}\cdot\nabla\mathbf{v}$ as the operator $\mathbf{v}\cdot\nabla$ operating on $\mathbf{v}$. Note that

$$\mathbf{v}\cdot\nabla \equiv v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}.$$

When making $\mathbf{v}\cdot\nabla$ operate on $\mathbf{v}$, be careful to note that $\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta$ and $\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r$.

(iii) We already know that $\nabla^2 \equiv \nabla\cdot\nabla$. So

$$\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)$$

Again when expanding the above, be careful to note that $\frac{\partial\mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta$ and $\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r$.

(iv) Similarly, when evaluating $\nabla\times\mathbf{v}$, be careful about the derivatives of the unit vectors.

6. In the chapter on kinematics as well as that on Navier-Stokes equation, we come across the term $\nabla\mathbf{v}$, the gradient of the velocity vector. We want to find the expansion of $\nabla\mathbf{v}$ in a systematic fashion. It might seem to be doable through a natural extension of the ideas used in expanding $\mathbf{v}\cdot\nabla$ and ∇^2, but doing that results in something which in matrix representation is the transpose of $\nabla\mathbf{v}$.

   Start by writing $\nabla\mathbf{v}$ as $\mathbf{v}\otimes\nabla$ (and, no, that is *not* an operator like $\mathbf{v}\cdot\nabla$). The symbol $\otimes$ represents a tensor product. But contrary to expectation, note that ∇ and $\mathbf{v}$ have changed order from the original $\nabla\mathbf{v}$.

   (a) First, work through the following in rectangular Cartesian coordinate system:

$$\begin{aligned}\nabla\mathbf{v} \equiv \mathbf{v}\otimes\nabla &= (v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z)\otimes\left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\\ &= \frac{\partial v_x}{\partial x}\mathbf{e}_x\otimes\mathbf{e}_x + \frac{\partial v_x}{\partial y}\mathbf{e}_x\otimes\mathbf{e}_y + \frac{\partial v_x}{\partial z}\mathbf{e}_x\otimes\mathbf{e}_z \ldots\\ &\quad + \frac{\partial v_y}{\partial x}\mathbf{e}_y\otimes\mathbf{e}_x + \frac{\partial v_y}{\partial y}\mathbf{e}_y\otimes\mathbf{e}_y + \frac{\partial v_y}{\partial z}\mathbf{e}_y\otimes\mathbf{e}_z \ldots\\ &\quad + \frac{\partial v_z}{\partial x}\mathbf{e}_z\otimes\mathbf{e}_x + \frac{\partial v_z}{\partial y}\mathbf{e}_z\otimes\mathbf{e}_y + \frac{\partial v_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z,\end{aligned}$$

   which in matrix notation can be represented as a 3×3 matrix. Which term will go into which place is determined by the tensor products of the unit vectors. For instance, the term with $\mathbf{e}_x\otimes\mathbf{e}_x$ goes into 1st row, 1st column, the one with $\mathbf{e}_x\otimes\mathbf{e}_y$ goes into 1st row, 2nd column, the one with $\mathbf{e}_z\otimes\mathbf{e}_y$ goes into 3rd row, 2nd column, and so on. Write the equivalent matrix representation.

   (b) Now, following the steps of the rectangular Cartesian coordinate system, work out the expansion in cylindrical coordinates to show that the equivalent matrix representation is

$$\nabla\mathbf{v} \equiv \mathbf{v}\otimes\nabla \equiv \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} & \frac{\partial v_r}{\partial z}\\ \frac{\partial v_\theta}{\partial r} & \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} & \frac{\partial v_\theta}{\partial z}\\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial\theta} & \frac{\partial v_z}{\partial z}\end{bmatrix}$$

   (c) Now, find the strain-rate tensor $\mathbf{E} := \frac{1}{2}\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right\}$ in both rectangular and cylindrical coordinates.