Assignment Sheet in Fluid Mechanics*
Navier-Stokes Equations and Exact Solutions

1. Rewrite the Navier-Stokes equations in terms of the mechanical pressure defined as $\bar{p} := -\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$.

   (a) What happens when the flow is solenoidal, i.e. when $\nabla \cdot \mathbf{v} = 0$ without ρ necessarily being a constant? Is there any difference from the case when ρ is a constant?

   (b) What happens when the Stokes hypothesis is valid, i.e. when the bulk viscosity, $\mu_v = \lambda + \frac{2}{3}\mu$ is zero?

2. Consider a horizontal parallel plate channel of height H through which there is a steady, incompressible, viscous flow of a Newtonian fluid. Set the origin of a Cartesian coordinate system at the lower plate with x directed parallel to the plates. The fluid flow is due to the combined effect of an applied pressure gradient (parallel to x) and the shearing motion of the upper plate moving with velocity U in the positive x–direction. The density and the dynamic viscosity of the fluid are ρ and μ respectively.

   (a) Start from the Navier-Stokes equations, make the simplifying assumptions that the flow is unidirectional and that there is no velocity gradient in the x–direction, and obtain the velocity profile of the flow, i.e. find the velocity component, u, along x as a function of the transverse coordinate y. Express your answer in dimensionless form using $u^* = u/U$ and $y^* = y/H$. This answer should involve a non-dimensional group, $V_\mathrm{r} = -\frac{H^2}{2\mu U}\frac{\mathrm{d}P}{\mathrm{d}x}$, where $\frac{\mathrm{d}P}{\mathrm{d}x}$ is the applied pressure gradient.

   (b) Find the location, $y^* = y^*_{\mathrm{extremum}}$, of the turning points where u^* reaches a local extremum value. Also find this extremum value of u^*. When is this extremum a local maximum or turning point? When is it a local minimum or turning point? Justify mathematically in terms of the value of V_r and interpret physically in terms of the sign of the applied pressure gradient. (Remember: $U > 0$, always)

   (c) Considering the two separate cases: $V_\mathrm{r} = 1$ and $V_\mathrm{r} = -1$, find the position and the value of the extremum velocity corresponding to each case. Additionally, find the shear stress at the extremum positions for each case.

   (d) For what values of V_r is it possible to have flow reversal in the channel, i.e. a situation where u^* is positive for some values of y^* and negative for other values of y^*? Justify your answer in terms of the location of the extremum position.

   [(a) $u^* = V_\mathrm{r}\left(y^* - y^{2*}\right) + y^*$; (b) $y^*_{\mathrm{extremum}} = \dfrac{V_\mathrm{r}+1}{2V_\mathrm{r}}$, $u^*_{\mathrm{extremum}} = \dfrac{(V_\mathrm{r}+1)^2}{4V_\mathrm{r}}$, For the nature of the extrema, look into the sign of $\dfrac{\mathrm{d}^2u^*}{\mathrm{d}y^{*2}}$; (c) When $V_\mathrm{r} = 1$, $y^*_{\mathrm{extremum}} = 1$ and $u^*_{\mathrm{extremum}} = 1$; when $V_\mathrm{r} = -1$, $y^*_{\mathrm{extremum}} = 0$ and $u^*_{\mathrm{extremum}} = 0$; shear stress is 0 in both cases; (d) When $V_\mathrm{r} < 1$.]

3. Consider steady laminar axisymmetric fully developed incompressible flow of a constant-property Newtonian fluid through a horizontal concentric cylindrical annulus of inner radius R_1 and outer radius R_2, driven by a constant externally applied axial pressure gradient.

*Compiled and prepared by Jeevanjyoti Chakraborty; for queries email: jeevan@mech.iitkgp.ac.in

(a) Determine the radial distribution of the axial velocity, $v_z(r)$.

(b) Determine the average velocity through the annulus.

Note the following: The divergence of the velocity field, $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_z\mathbf{e}_z$, in cylindrical coordinates is given by

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}.$$

The Navier-Stokes equations for a constant-property fluid may be expressed in cylindrical coordinates as

$$\begin{aligned}
\rho\left(\frac{\mathrm{D}v_r}{\mathrm{D}t} - \frac{v_\theta^2}{r}\right) &= -\frac{\partial P}{\partial r} + \mu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right), \\
\rho\left(\frac{\mathrm{D}v_\theta}{\mathrm{D}t} + \frac{v_r v_\theta}{r}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right), \\
\rho\frac{\mathrm{D}v_z}{\mathrm{D}t} &= -\frac{\partial P}{\partial z} + \mu\nabla^2 v_z,
\end{aligned}$$

where

$$\begin{aligned}
P &= p - \rho\mathbf{g}\cdot\mathbf{r}, \\
\frac{\mathrm{D}}{\mathrm{D}t} &\equiv \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}, \\
\nabla^2 &\equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

[(a) $v_z = -\dfrac{1}{4\mu}\dfrac{\mathrm{d}P}{\mathrm{d}z}\left[(R_2^2 - R_1^2)\dfrac{\ln(r/R_1)}{\ln(R_2/R_1)} - (r^2 - R_1^2)\right]$; (b) $\overline{v}_z = -\dfrac{1}{4\mu}\dfrac{\mathrm{d}P}{\mathrm{d}z}\left[\dfrac{R_2^2 + R_1^2}{2} - \dfrac{R_2^2 - R_1^2}{2\ln(R_2/R_1)}\right]$]

4. *Circular Couette Flow:* Consider the flow of an incompressible, Newtonian fluid between two concentric circular cylinders. Let the radius and angular velocity of the inner cylinder be R_1 and Ω_1 and those for the outer cylinder be R_2 and Ω_2. Use cylindrical coordinates and appropriate simplifying assumptions.

   (a) Show that the equations of motion in the radial and the tangential directions are

   $$\begin{aligned}
   \frac{v_\theta^2}{r} &= \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}, \\
   0 &= \mu\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}}(rv_\theta)\right].
   \end{aligned}$$

   (b) Using the θ–momentum equation and the appropriate boundary conditions, show that

   $$v_\theta = \frac{1}{1 - (R_1/R_2)^2}\left[\left\{\Omega_2 - \Omega_1\left(\frac{R_1}{R_2}\right)^2\right\}r + \frac{R_1^2}{r}(\Omega_1 - \Omega_2)\right].$$

   (c) Consider the special case of $\Omega_1 = 0$. Show that the torque exerted on either cylinder, per unit length, equals $4\pi\mu\Omega a^2 b^2/(b^2 - a^2)$. Note that the general expression for the shear stress $\tau_{r\theta}$ at any point is given by

   $$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right).$$

(d) For the special case of flow outside a cylinder rotating in an infinite fluid, set $\Omega_2 = 0$ as $R_2 \to \infty$ and $\Omega_1 = \Omega$ at $R_1 = R$ to show that

$$v_\theta = \frac{\Omega R^2}{r}.$$

Is the flow irrotational?

(e) For the special case of flow inside a rotating cylinder, set $\Omega_1 = 0$ at $R_1 = 0$, and $\Omega_2 = \Omega$ at $R_2 = R$ to show that

$$v_\theta = \Omega r.$$

Compare this motion with that of a solid circular disk of radius R rotating at an angular velocity Ω.

5. Consider a Couette-flow like situation where instead of one fluid, we now have two immiscible fluids flowing between two parallel plates. The two plates are separated by a distance H, the lower plate is moved with a velocity U to the right, and the upper plate is stationary. Use rectangular Cartesian coordinate system and set the origin at the lower plate. Fluid 1 is present in $y \in [d, H]$ while Fluid 2 is in $y \in [0, d]$. The dynamic viscosity coefficients for Fluid 1 and 2 are μ_1 and μ_2, respectively. Make suitable simplifying assumptions; additionally assume that the interface between Fluid 1 and 2 remains horizontal.

   (a) Solve for the velocity distributions in the two fluids. In addition to the two boundary conditions at the two walls you will need two more conditions. These will be found at the interface of Fluids 1 and 2: the velocity in the x–direction should be continuous and the shear stress should be equal.

   (b) For the special case $\mu_1/\mu_2 \to 0$, what happens to the velocity distributions? Show that in this case the shear stress at the interface tends to 0.

   (c) The special case in part (b) may be thought of as a model for the flow of a single layer of a liquid (eg. water) with the top surface free (no wall, exposed to air) because the dynamic viscosity of air is very small compared to that of water. For this model, the appropriate boundary conditions would be no slip at the bottom wall and no shear at the top surface. Confirm that this model indeed recovers the velocity distribution for Fluid 2 found in part (b).

$$\text{[(a) For Fluid 1: } u = \frac{U}{H + \left(\frac{\mu_1}{\mu_2} - 1\right) d}(H - y), \text{ for Fluid 2: } u = U - \frac{\left(\frac{\mu_1}{\mu_2}\right) U}{H + \left(\frac{\mu_1}{\mu_2} - 1\right) d} y]$$

6. Consider a Poiseuille flow-like situation where instead of one fluid, we have two immiscible fluids flowing through a circular cylindrical pipe of radius R. The origin is set at the centre of the circular cross-section so that the z-axis points along the cylindrical axis. The regions occupied by the two fluids are concentric. Fluid 1 in the region $0 \leq r \leq d$ is referred to as being in the core region, and Fluid 2 in the region $d \leq r \leq R$ is referred to as being in the annular region. Both fluids are subjected to the same linear pressure gradient in the axial direction. Assume that both fluids are incompressible. Further, assume that the flow is steady, fully developed along the axial direction and axisymmetric, and that the angular component of velocity is zero.

   (a) From a consideration of the continuity equation and other physical requirement(s), show that the radial component of velocity (v_r) must be zero in both the fluids.

   (b) If the dynamic viscosity of Fluid 1 in the core region is μ_C and that of Fluid 2 in the annular region is μ_A, find the velocity profiles in the axial direction (v_z) in both the regions.

(c) If the two fluids were to be replaced by a single fluid such that the volumetric flow rate of this single fluid is the same as that of the two fluids combined, express the dynamic viscosity (μ) of this single fluid in terms of μ_C, μ_A, d, and R. Note that the volumetric flow rate according to the Hagen-Poiseuille law is $Q = -\dfrac{\pi R^4}{8\mu}\dfrac{\mathrm{d}P}{\mathrm{d}z}$ (here $-\dfrac{\mathrm{d}P}{\mathrm{d}z} < 0$).

(d) If the ratio $\varepsilon = \frac{R-d}{R} \ll 1$, then show that $\mu = \mu_\mathrm{C}\left[1 + 4\varepsilon\left(\dfrac{\mu_\mathrm{C}}{\mu_\mathrm{A}} - 1\right)\right]$.

[(b) For Fluid 1: $v_z = -\dfrac{1}{4\mu_\mathrm{C}}\dfrac{\mathrm{d}P}{\mathrm{d}z}(\delta^2 - r^2) - \dfrac{1}{4\mu_\mathrm{A}}\dfrac{\mathrm{d}P}{\mathrm{d}z}(R^2 - \delta^2)$, for Fluid 2: $v_z = -\dfrac{1}{4\mu_\mathrm{A}}\dfrac{\mathrm{d}P}{\mathrm{d}z}(R^2 - r^2)$;

(c) $\dfrac{1}{\mu} = \dfrac{1}{\mu_\mathrm{A}}\left[\left(\dfrac{\delta}{R}\right)^4\left(\dfrac{\mu_\mathrm{A}}{\mu_\mathrm{C}} - 1\right) + 1\right]$]

7. Consider the laminar flow of a fluid layer flowing down a plane inclined at an angle θ with the horizontal. If h is the thickness of the fluid layer in the fully developed region, show that the velocity distribution is

$$v = \frac{\rho g \sin\theta}{2\mu}\left(h^2 - y^2\right),$$

where the x–axis points along the free surface, and the $y - axis$ points toward the plane. Show that volume flow rate per unit width is

$$Q = \frac{\rho g h^3 \sin\theta}{3\mu},$$

and the shear stress on the wall is $\tau = \rho g h \sin\theta$.