Viscous Fluid Flow

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1 Motivation to incorporate viscous effects

In the previous chapter (before Mid-Sem) we had seen that mathematicians had tried (way, way back) to model dynamic fluid flow behaviour by incorporating the static fluid flow relation $\boldsymbol{\sigma} = -p\mathbf{I}$ into the equation of motion obtained from the conservation of linear momentum. While this inviscid theory has its uses, but it was found out by mathematicians themselves that some of the phenomena observed in practice by engineers simply could not be predicted by the equations. One of the most famous examples of this discrepancy was the d'Alembert's paradox: a solid sphere or circular cylinder immersed in a flowing fluid experiences drag; this drag just could not be predicted by the mathematics. Another example is that the lift force observed in lubricated systems (originating from lubricating fluid pressure) cannot be predicted by the theory. Another simple example from common experience is that it is not possible to clean the dusty surface of a car just by driving fast. Inviscid theory however predicts that it is possible!

It was found that these discrepancies between theory and practice can be corrected by including viscous effects in the model for flowing fluid.

2 Dynamic fluid behaviour involving viscosity

2.1 What we *have* till now, and what we *need*

We have one equation (the continuity equation or the mass conservation equation) involving the density, ρ (which in the general case need not be a known constant), and the three components of the velocity:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1}$$

We also have three equations from momentum conservation:

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} + \nabla \cdot \boldsymbol{\sigma}, \tag{2}$$

which involve the 9 unknown components of $\boldsymbol{\sigma}$ besides ρ and $\mathbf{v}$ again. Thus we have $(1+3) = 4$ equations involving $(1+3+9) = 13$ unknowns. We can fix this discrepancy (but not completely) by invoking the principle of conservation of angular momentum. Without going into the details we will state that using this principle we obtain that the stress is symmetric as long as there are no body couples acting. Thus

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}, \tag{3}$$

which means that instead of 9 unknown components we now have 6 unknown components of the stress. Thus, taking stock again we have: $1+3=4$ equations and $(1+3+6) = 10$ unknowns. Therefore we need

6 more equations to fix the discrepancy. These equations are provided by constitutive behaviour (or, how the material behaves). Note that all the equations that we have written till now are applicable equally to all materials (as long as we are treating them in the continuum sense). It is at the point where we invoke the material behaviour that equations become different for different materials.

Here, we will present a discussion of the equation describing the simplest possible material behaviour of a flowing fluid.

2.2 Flowing fluid with viscous effects

We now want to obtain an equation that will depict the simplest possible material behaviour of a flowing fluid with viscous effects incorporated. The first thing we want to ensure is that in the special case where the velocity is zero, the static fluid behaviour should be recoverable from this flowing fluid behaviour. That can be done by writing for the stress

$$\boldsymbol{\sigma} = -p\mathbf{I} + f(\mathbf{v}), \tag{4}$$

where $f(\mathbf{v}) = 0$ when $\mathbf{v} = 0$.

It is important to note that transmission of the information about momentum takes place not due to the velocity *per se*; rather it takes place due to the gradients in velocity. Since the stresses are intrinsically connected to this transmission of momentum information, therefore, a more appropriate version of the above relation is

$$\boldsymbol{\sigma} = -p\mathbf{I} + f(\nabla\mathbf{v}). \tag{5}$$

We have already seen in earlier lectures that the gradient in velocity may be decomposed into the symmetric and the antisymmetric parts as:

$$\nabla\mathbf{v} = \underbrace{\frac{1}{2}\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right\}}_{\text{Symmetric}} + \underbrace{\frac{1}{2}\left\{\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathsf{T}}\right\}}_{\text{Antisymmetric}}. \tag{6}$$

Remember that the symmetric part represents the rate of deformation tensor (or, the rate of strain tensor or, the strain-rate tensor), henceforth denoted by $\mathbf{E}$ while the antisymmetric part denotes the average rotation rate.

A very subtle point: The material behaviour cannot involve any sort of dependence on the rotation because material behaviour should come into play only when there is some deformation – change in the internal structure – of the material. If the material behaviour were to depend on rotation then we would be able to "see" different material behaviour of a bucket full of water simply by hopping around it in different ways!

Therefore, the description of the material behaviour must not include a dependence on the rotation rate part of the velocity gradient. And so an even more appropriate version of Eq. (5) is

$$\boldsymbol{\sigma} = -p\mathbf{I} + f(\mathbf{E}). \tag{7}$$

The simplest possible form is for a linear, instantaneous, isotropic fluid for which the above relation becomes (see the Appendix for a more extensive discussion)

$$\boldsymbol{\sigma} = -p\mathbf{I} + \lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{8}$$

where λ and μ are two material-specific parameters. This material behaviour is said to be that of a Newtonian fluid. Water is a typical example of Newtonian fluid.

Here, $\mathrm{tr}(\mathbf{E})$ is referred to as the trace of $\mathbf{E}$, and it denotes the sum of the diagonal elements in the matrix representation of the tensor $\mathbf{E}$. Thus

$$
\begin{aligned}
\mathrm{tr}(\mathbf{E}) &= \mathrm{tr}\left[\frac{1}{2}\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right\}\right] \\
&= \frac{1}{2}\left[\mathrm{tr}\left(\nabla\mathbf{v}\right) + \mathrm{tr}\left\{(\nabla\mathbf{v})^{\mathsf{T}}\right\}\right] \\
&= \frac{1}{2} \times 2\mathrm{tr}\left(\nabla\mathbf{v}\right) \\
&= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \\
&= \nabla \cdot \mathbf{v}
\end{aligned}
$$

Therefore, from (8), we have

$$
\boldsymbol{\sigma} = -p\mathbf{I} + \lambda\nabla\cdot\mathbf{v}\mathbf{I} + 2\mu\mathbf{E}. \tag{9}
$$

Note that p is the thermodynamic pressure. Now, define "mechanical pressure" as $\bar{p} := -\frac{1}{3}\mathrm{tr}\left(\boldsymbol{\sigma}\right)$ so that

$$
\begin{aligned}
\bar{p} &= -\frac{1}{3}\mathrm{tr}\left[-p\mathbf{I} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}\right] \\
&= -\frac{1}{3}\left[-3p + 3\lambda\nabla\cdot\mathbf{v} + 2\mu\nabla\cdot\mathbf{v}\right] \\
&= p - \lambda\nabla\cdot\mathbf{v} - \frac{2}{3}\mu\nabla\cdot\mathbf{v} \\
&= p - \left(\lambda + \frac{2}{3}\mu\right)\nabla\cdot\mathbf{v}.
\end{aligned}
$$

There are two ways in which the thermodynamic pressure may be equal to the mechanical pressure:

1. When the bulk viscosity $\mu_v := \lambda + \frac{2}{3}\mu = 0$ (referred to as the Stokes' hypothesis). This holds in two situations:
   (a) Characteristic time scales are large compared to the molecular relaxation time scales
   (b) When the fluid under consideration is a monoatomic gas
2. When the fluid is incompressible implying $\nabla\cdot\mathbf{v} = 0$

Now, we would like to use the material behaviour equation (9) in the general equation for the conservation of linear momentum (2) (where we also use the symmetry of the stress condition (3) obtained from the conservation of angular momentum). For this substitution, we need to evaluate $\nabla\cdot\boldsymbol{\sigma}$. Thus

$$
\begin{aligned}
\nabla\cdot\boldsymbol{\sigma} &= \nabla\cdot\left[-p\mathbf{I} + \lambda\nabla\cdot\mathbf{v}\mathbf{I} + 2\mu\mathbf{E}\right] \\
&= -\nabla p + \nabla\left(\lambda\nabla\cdot\mathbf{v}\right) + 2\nabla\cdot\left[\mu\frac{1}{2}\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right\}\right]
\end{aligned}
$$

Take λ and μ as constants. Then

$$
\nabla\cdot\boldsymbol{\sigma} = -\nabla p + \lambda\nabla\left(\nabla\cdot\mathbf{v}\right) + \mu\nabla\cdot\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathsf{T}}\right\} \tag{10}
$$

We can rewrite the term involving μ by noting the following

$$\begin{aligned}
\left\{\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathsf{T}}\right\} &\equiv \frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\
&= \frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial^2 v_j}{\partial x_j \partial x_i} \\
&= \frac{\partial^2 v_i}{\partial x_j^2} + \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) \\
&\equiv \nabla^2 \mathbf{v} + \nabla\left(\nabla \cdot \mathbf{v}\right)
\end{aligned}$$

Then from (10), we have

$$\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} &= -\nabla p + \lambda \nabla\left(\nabla \cdot \mathbf{v}\right) + \mu \nabla^2 \mathbf{v} + \mu \nabla\left(\nabla \cdot \mathbf{v}\right) \\
&= -\nabla p + \left(\lambda + \frac{2}{3}\mu\right)\nabla\left(\nabla \cdot \mathbf{v}\right) + \mu \nabla^2 \mathbf{v} + \frac{1}{3}\nabla\left(\nabla \cdot \mathbf{v}\right). \qquad (11)
\end{aligned}$$

Using (11) in (2) (where we additionally use (3)), we obtain

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} - \nabla p + \left(\lambda + \frac{2}{3}\mu\right)\nabla\left(\nabla \cdot \mathbf{v}\right) + \mu \nabla^2 \mathbf{v} + \frac{1}{3}\nabla\left(\nabla \cdot \mathbf{v}\right), \qquad (12)$$

which is the celebrated Navier-Stokes equation. Note that in class, we had obtained the Navier-Stokes equation in the following form:

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} - \nabla p + \left(\lambda + \mu\right)\nabla\left(\nabla \cdot \mathbf{v}\right) + \mu \nabla^2 \mathbf{v}, \qquad (13)$$

where the only difference with (12) is that the combination of terms $\lambda + \frac{2}{3}\mu$ representing the bulk viscosity has not been separated out.

Note the following:

1. Using just the Stokes' hypothesis $\left(\lambda + \frac{2}{3}\mu = 0\right)$, we have

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \mu \nabla^2 \mathbf{v} + \frac{1}{3}\mu \nabla\left(\nabla \cdot \mathbf{v}\right)$$

2. For incompressible fluid, $\nabla \cdot \mathbf{v} = 0$, we have

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \mu \nabla^2 \mathbf{v}$$

3. In terms of the mechanical pressure, we have

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla \bar{p} + \mu \nabla^2 \mathbf{v} + \frac{1}{3}\mu \nabla\left(\nabla \cdot \mathbf{v}\right)$$

Appendix

Consider Eq. (7) where both $\boldsymbol{\sigma}$ and $p\mathbf{I}$ are symmetric. Therefore, $f(\mathbf{E})$ must also be symmetric. Denote it by $\boldsymbol{\tau}$ which can be any general function of $\mathbf{E}$. Now we invoke certain simplifying assumptions:

- **Linearity**: We assume for simplicity that the functional dependence on $\mathbf{E}$ is linear. So $\boldsymbol{\tau}$ will depend only on $\mathbf{E}$ and not on $\mathbf{E}^2$ or $\mathbf{E}^3$ and so on.

- **No history effects**: We assume that the response of the material is instantaneous – the way that $\mathbf{E}$ had evolved does not influence the current behaviour, i.e. there are no history effects. Had there been history effects, the functional dependence would involved some sort of time integral.
- **Isotropy**: We assume that the material behaviour is same in all directions. Note that this assumption is completely separate from the condition of uniformity.

From the first two of the simplifying assumptions (linearity and no history effects), we can conclude that $\boldsymbol{\tau}$ will be "proportional" to $\mathbf{E}$. Since we want to relate a second-order tensor to another second-order tensor, the connection will be through a fourth-order tensor; thus

$$\tau_{ij} = C_{ijkl} E_{kl}, \tag{14}$$

where C_{ijkl} is the fourth-order tensor. Now, we can just interchange the indices i and j to obtain

$$\tau_{ji} = C_{jikl} E_{kl}. \tag{15}$$

But $\boldsymbol{\tau}$ is symmetric, i.e. $\tau_{ij} = \tau_{ji}$. Thus, subtracting (15) from (14) we obtain

$$\begin{aligned} 0 &= \left(C_{ijkl} - C_{jikl}\right) E_{kl}, \\ \text{or,} \quad C_{ijkl} &= C_{jikl}. \end{aligned} \tag{16}$$

Similarly, interchanging k and l in (14) and using $E_{kl} = E_{lk}$ (since $\mathbf{E}$ is symmetric too), we can obtain

$$C_{ijkl} = C_{ijlk}. \tag{17}$$

Now we would like to incorporate the assumption of isotropy to obtain a particular form for C_{ijkl}. For this, we invoke a result (without going into the proof) from tensor analysis that a fourth-order isotropic tensor can be represented in terms of Kronecker deltas:

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{18}$$

where α, β, and γ are arbitrary scalars. We interchange the indices i and j to obtain

$$C_{jikl} = \alpha\delta_{ji}\delta_{kl} + \beta\delta_{jk}\delta_{il} + \gamma\delta_{jl}\delta_{ik}. \tag{19}$$

However, we have already established that $C_{ijkl} = C_{jikl}$. Therefore, subtracting (19) from (18) we obtain

$$0 = \left(\beta - \gamma\right)\delta_{ik}\delta_{jl} + \left(\gamma - \beta\right)\delta_{il}\delta_{jk}. \tag{20}$$

This equation must be identically true for any combination of i, j, k, and l. For instance, if we set $i \neq j \neq k \neq l$, then we end up with $0 = 0$ which even though true does not yield anything useful. Let's set $i = k$ and $j = l$ with $i \neq j$. Then, from (20) we have

$$\beta = \gamma. \tag{21}$$

Similarly, if we interchange the indices k and l in (18) and proceed to use the condition that $C_{ijkl} = C_{ijlk}$, we will again end up with the condition that $\beta = \gamma$. So, the condition of isotropy together with the symmetry conditions give us

$$C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{22}$$

Then we have from (14)

$$\begin{aligned} \tau_{ij} &= \left\{\alpha\delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\} E_{kl} \\ &= \alpha\delta_{ij}E_{kk} + \beta\left(\delta_{ik}E_{kj} + \delta_{il}E_{jl}\right) \\ &= \alpha\delta_{ij}E_{kk} + \beta\left(E_{ij} + E_{ji}\right) \\ &= \alpha\delta_{ij}E_{kk} + 2\beta E_{ij}. \end{aligned} \tag{23}$$

It is conventional to use the two scalars λ and μ instead of α and β to represent the above relation; thus

$$\tau_{ij} = \lambda E_{kk}\delta_{ij} + 2\mu E_{ij}, \tag{24}$$

or, in compact tensor notation

$$\boldsymbol{\tau} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{25}$$

so that from (7), we obtain

$$\boldsymbol{\sigma} = -p\mathbf{I} + \lambda \mathrm{tr}(\mathbf{E})\mathbf{I} + 2\mu\mathbf{E}, \tag{26}$$

which is exactly what is mentioned in (8).