Nondimensionalization and Boundary Layer Equations

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Consider a 2D flow situation:

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{1}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{2}$$

Choose the following nondimensionalization scheme:

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{\delta}, \quad v_x^* = \frac{v_x}{U}, \quad v_y^* = \frac{v_y}{V}, \quad P^* = \frac{P}{\rho U^2}, \quad t^* = \frac{t}{L/U}. \tag{3}$$

Here, L is called the characteristic length scale along x and δ is the characteristic length scale along y. Similarly, U and V are the characteristic scales for the velocities along x and y. The pressure scale is ρU^2 and the scale for time is obtained from the scales for length and velocity. It is very important to note that it is the actual physical situation (often called the "physics of the problem") that determines these scales.

Using this nondimensionalization scheme, we have

$$\frac{\partial v_x}{\partial x} = \frac{\partial(U v_x^*)}{\partial x^*}\frac{\partial x^*}{\partial x} = \frac{U}{L}\frac{\partial v_x^*}{\partial x^*} \tag{4a}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial(U v_x^*)}{\partial y^*}\frac{\partial y^*}{\partial y} = \frac{U}{\delta}\frac{\partial v_x^*}{\partial y^*} \tag{4b}$$

$$\frac{\partial v_x}{\partial t} = \frac{\partial(U v_x^*)}{\partial t^*}\frac{\partial t^*}{\partial t} = \frac{U}{L/U}\frac{\partial v_x^*}{\partial t^*} = \frac{U^2}{L}\frac{\partial v_x^*}{\partial t^*} \tag{4c}$$

$$\frac{\partial^2 v_x}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}\right) = \frac{\partial}{\partial x^*}\left(\frac{U}{L}\frac{\partial v_x^*}{\partial x^*}\right)\frac{\partial x^*}{\partial x} = \frac{U}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} \tag{4d}$$

$$\frac{\partial^2 v_x}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial y}\right) = \frac{\partial}{\partial y^*}\left(\frac{U}{\delta}\frac{\partial v_x^*}{\partial y^*}\right)\frac{\partial y^*}{\partial y} = \frac{U}{\delta^2}\frac{\partial^2 v_x^*}{\partial y^{*2}} \tag{4e}$$

And similarly (4f)

$$\frac{\partial v_y}{\partial x} = \frac{V}{L}\frac{\partial v_y^*}{\partial x^*} \tag{4g}$$

$$\frac{\partial v_y}{\partial y} = \frac{V}{\delta}\frac{\partial v_y^*}{\partial y^*} \tag{4h}$$

$$\frac{\partial v_y}{\partial t} = \frac{VU}{L}\frac{\partial v_y^*}{\partial t^*} \quad \text{(note this carefully)} \tag{4i}$$

$$\frac{\partial^2 v_y}{\partial x^2} = \frac{V}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} \tag{4j}$$

$$\frac{\partial^2 v_y}{\partial y^2} = \frac{V}{\delta^2}\frac{\partial^2 v_y^*}{\partial y^{*2}} \tag{4k}$$

What is the point of rewriting everything in terms of the nondimensional variables (i.e. the starred variables)? What are we trying to achieve? If we have chosen the proper characteristic scales through a correct understanding of the physics of the problem, then rewriting everything in terms of the nondimensional variables helps us to extract the essential physical meaning out of each terms of a governing equation. (Most importantly, it also results in equation forms where the various terms can be easily compared against each other on the basis of their relative strengths ... but we will discuss this important point a bit later)

So how is it that we extract the essential physical meaning? To answer this, we note that rewriting in terms of the nondimensional variables results in an entity that is a product of two parts: one part is a certain combination of the characteristic scales while the other part is a certain combination of the nondimensional (starred) variables. For instance, from (4)(a), we have

$$\frac{\partial v_x}{\partial x} = \frac{U}{L}\frac{\partial v_x^*}{\partial x^*}. \tag{5}$$

On the LHS, we have $\frac{\partial v_x}{\partial x}$ which is a measure of how much the velocity in the x-direction increases for every unit traversal along that direction and it has the dimension of the reciprocal of time. That is the essence, the physical meaning of the term. On the RHS, the part U/L is a combination of the characteristic velocity scale along x and the length scale along x while the part $\frac{\partial v_x^*}{\partial x^*}$ is a combination of the nondimensional variables v_x^* and x^*. The significance of writing $\frac{\partial v_x}{\partial x}$ as a product of these two parts is that the physical meaning is captured by the first part (U/L) while the specific form is captured by the second part $(\frac{\partial v_x^*}{\partial x^*})$. This second part, taken in isolation, is just a mathematical entity devoid of any physical meaning. The responsibility of capturing the physical meaning of the LHS is shouldered entirely by the first part (U/L) while the responsibility of capturing the mathematical form is shouldered entirely by the second part $(\frac{\partial v_x^*}{\partial x^*})$.

This division or separation of physical meaning and mathematical form has a tremendous consequence on how we view a particular governing equation and how we may simplify it based on our physical understanding of the actual problem under consideration. It can also offer beautiful insights into the problem even before solving the problem. The key thing to note is that once this division or separation of physical meaning and mathematical form has been done, we can think about the physics of the problem in terms of the parts (like U/L) that denote the physical meaning without worrying about the mathematical form which is often complicated due to the presence of partial derivatives. An example of how such a physical discussion might be possible is given in the following.

Let us consider that the velocity fields in (1) and (2) correspond to an incompressible fluid so that $\nabla \cdot \mathbf{v} = 0$, which we rewrite using the nondimensionalization scheme as

$$\begin{aligned} \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0, \\ \text{or,} \quad \frac{U}{L}\frac{\partial v_x^*}{\partial x^*} + \frac{V}{\delta}\frac{\partial v_y^*}{\partial y^*} &= 0. \end{aligned}$$

In the second line above, we have done the separation between the physical meaning and the mathematical form in both the terms. Now, let us consider the physics of the problem. It is given that the fluid flow is two-dimensional which means that physically both the $x-$ and $y-$ component of the velocities must have a role to play. Neither one of the terms is relatively more important than the other term. The physical meaning of the first term is captured by U/L while the physical meaning of the second terms is captured by V/δ. In order to ensure that neither outweighs the other, we must have that the magnitude of these two parts must be of the same order. Thus, we can conclude that

$$\begin{aligned} \frac{U}{L} &\sim \frac{V}{\delta}, \\ \text{or,} \quad V &\sim \frac{U\delta}{L}, \end{aligned} \tag{6}$$

and

$$\boxed{\frac{\partial v_x^*}{\partial x^*} + \frac{\partial v_y^*}{\partial y^*} = 0.} \tag{7}$$

We use (6) to rewrite (4)(g)-(k) as

$$\frac{\partial v_y}{\partial x} = \frac{U\delta}{L^2}\frac{\partial v_y^*}{\partial x^*} \tag{8a}$$

$$\frac{\partial v_y}{\partial y} = \frac{U}{L}\frac{\partial v_y^*}{\partial y^*} \tag{8b}$$

$$\frac{\partial v_y}{\partial t} = \frac{U^2\delta}{L}\frac{\partial v_y^*}{\partial t^*} \tag{8c}$$

$$\frac{\partial^2 v_y}{\partial x^2} = \frac{U\delta}{L^3}\frac{\partial^2 v_y^*}{\partial x^{*2}} \tag{8d}$$

$$\frac{\partial^2 v_y}{\partial y^2} = \frac{U}{\delta L}\frac{\partial^2 v_y^*}{\partial y^{*2}} \tag{8e}$$

Using all the above in (1), we obtain

$$\rho\left(\frac{U^2}{L}\frac{\partial v_x^*}{\partial t^*} + \frac{U^2}{L}v_x^*\frac{\partial v_x^*}{\partial x^*} + \frac{U\delta}{L}\frac{U}{\delta}v_y^*\frac{\partial v_x^*}{\partial y^*}\right) = -\frac{\rho U^2}{L}\frac{\partial P^*}{\partial x^*} + \mu\left(\frac{U}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U}{\delta^2}\frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

Dividing throughout by $\rho U^2/L$, we obtain from the above

$$\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{\mu}{\rho U L}\left(\frac{\partial^2 v_x^*}{\partial x^{*2}} + \left(\frac{L}{\delta}\right)^2\frac{\partial^2 v_x^*}{\partial y^{*2}}\right), \tag{9}$$

where the group $\mu/(\rho U L)$ is identified as the inverse of the Reynolds number (Re), i.e. $Re = \rho U L/\mu$.

DANGER!:
What happens when $Re \gg 1$ or $Re \to \infty$, i.e. $\dfrac{1}{Re} \to 0$?

Suppose we do not *yet* know that $L \gg \delta$ so that L/δ is a finite non-large number. Then with the knowledge that all terms involving the starred variables must be of the same order, a product with a small number $\dfrac{1}{Re}$ is guaranteed to be very small.

Then, we *can*[1] drop the second-order terms from Eq. (9). However, doing so leads to a situation where an equation which was originally a second-order differential equation is forcefully reduced to a first-order differential equation. The consequence of such a drastic step is that all the boundary conditions that are originally associated with the boundary conditions can no longer be satisfied.

What is the resolution to this dangerous situation?

We consider the ratio $\varepsilon = \dfrac{\delta}{L}$ to be very small in such a way that the following holds true

$$Re\left(\frac{L}{\delta}\right)^2 \sim 1\,, \tag{10}$$

$$\text{or,}\quad \varepsilon^2 = \left(\frac{\delta}{L}\right)^2 \sim \frac{1}{Re}\,, \tag{11}$$

$$\text{or,}\quad \varepsilon = \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}\,. \tag{12}$$

[1]This is the really dangerous statement!

Now, if we use the smallness of $\dfrac{1}{Re}$ to drop terms from Eq. (9), we have the following reduced form

$$\boxed{\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{\partial^2 v_x^*}{\partial y^{*2}},} \tag{13}$$

where we use Eq. (10), to retain the last term. This retention ensures that the reduced equation is still a second-order differential equation.

Next, we nondimensionalize Eq. (2) using the same nondimensionalization scheme as before to obtain

$$\rho\left(\frac{VU}{L}\frac{\partial v_y^*}{\partial t^*} + \frac{UV}{L} v_x^* \frac{\partial v_y^*}{\partial x^*} + \frac{V^2}{\delta} v_y^* \frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\rho U^2}{\delta}\frac{\partial P^*}{\partial y^*} + \mu\left(\frac{V}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{V}{\delta^2}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right), \tag{14}$$

and then use $V \sim U\dfrac{\delta}{L}$ to obtain

$$\rho\left(\frac{U^2\delta}{L^2}\frac{\partial v_y^*}{\partial t^*} + \frac{U^2\delta}{L^2} v_x^* \frac{\partial v_y^*}{\partial x^*} + \frac{U^2\delta}{L^2} v_y^* \frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\rho U^2}{\delta}\frac{\partial P^*}{\partial y^*} + \mu\left(\frac{U\delta}{L^3}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{U}{\delta L}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right). \tag{15}$$

Dividing throughout by $\dfrac{\rho U^2}{\delta}$, we have

$$\left(\frac{\delta}{L}\right)^2\left(\frac{\partial v_y^*}{\partial t^*} + v_x^* \frac{\partial v_y^*}{\partial x^*} + v_y^* \frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\partial P^*}{\partial y^*} + \frac{\mu}{\rho U L}\left(\left(\frac{\delta}{L}\right)^2\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{\partial^2 v_y^*}{\partial y^{*2}}\right). \tag{16}$$

Now, noting, as before, that $\varepsilon^2 = \left(\dfrac{\delta}{L}\right)^2 \sim \dfrac{1}{Re}$, we obtain

$$\frac{1}{Re}\left(\frac{\partial v_y^*}{\partial t^*} + v_x^* \frac{\partial v_y^*}{\partial x^*} + v_y^* \frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\partial P^*}{\partial y^*} + \frac{1}{Re^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 v_y^*}{\partial y^{*2}}, \tag{17}$$

$$\text{or,}\quad \varepsilon^2\left(\frac{\partial v_y^*}{\partial t^*} + v_x^* \frac{\partial v_y^*}{\partial x^*} + v_y^* \frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\partial P^*}{\partial y^*} + \varepsilon^4\frac{\partial^2 v_y^*}{\partial x^{*2}} + \varepsilon^2\frac{\partial^2 v_y^*}{\partial y^{*2}} \tag{18}$$

Finally, taking the limit $Re \to \infty$, or equivalently $\varepsilon \to 0$, again as before, we obtain

$$\boxed{0 = \frac{\partial P^*}{\partial y^*}.} \tag{19}$$

The three equations in boxes, Eqs (7), (13), and (19) are the non-dimensional boundary layer equations. Because they are so important, we write them again together:

$$\frac{\partial v_x^*}{\partial x^*} + \frac{\partial v_y^*}{\partial y^*} = 0, \tag{20a}$$

$$\frac{\partial v_x^*}{\partial t^*} + v_x^* \frac{\partial v_x^*}{\partial x^*} + v_y^* \frac{\partial v_x^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{\partial^2 v_x^*}{\partial y^{*2}}, \tag{20b}$$

$$\frac{\partial P^*}{\partial y^*} = 0. \tag{20c}$$