Kinematics

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1 Introduction

In kinematics, we are concerned with how the fluid motion looks, i.e. we care only about displacement, velocity, acceleration and so on without concerning ourselves with the agents (forces and moments) that bring about this motion in the first place.

2 Lagrangian vs Eulerian description

In the Lagrangian description, we focus our attention on an individual fluid particle and follow its motion. In fact, this way of following the motion is what you are accustomed to in all your experiences with mechanics (at the pre-JEE level) as well in your first year. When we follow the motion of a block moving down an inclined plane or a ball rolling on a table, we are implicitly using this Lagrangian description. Here, the independent variables are taken as time (t) and a label for the particle. For the sake of convenience, this label can be taken as the position vector $\boldsymbol{X}$ of the particle at the initial time $t = 0$. Thus, a generic flow variable $\mathcal{F}$ may be expressed as $\mathcal{F}(\boldsymbol{X}, t)$ in this description. For instance, the position vector itself may be considered as a flow variable. The position vector at any time, t is written as $\boldsymbol{x} \equiv \boldsymbol{x}(\boldsymbol{X}, t)$ which is the position at time, t of a particle which was at $\boldsymbol{X}$ at $t = 0$. Very important: $\boldsymbol{x}(\boldsymbol{X}, 0) = \boldsymbol{X}$.

In the Eulerian description, we focus our attention on a particular point in space (referred to as a spatial point). Through this point in space, various particles pass through at various times. Here, the independent variables are the position vectors of the various fixed spatial points $\boldsymbol{x}$ and time, t. Thus, a generic flow variable $\mathcal{F}$ may be expressed as $\mathcal{F}(\boldsymbol{x}, t)$ in this description.

Note: $\boldsymbol{x}$ is a *dependent* variable in Lagrangian description while $\boldsymbol{x}$ is an *independent* variable in Eulerian description.

Which is a better or more natural way of descrition? The answer depends on what we are trying to describe. For the case of a block sliding down an incline, the task of tracking the motion of the block naturally calls for the use of the Lagrangian description. On the other hand, if we are interested in tracking the fluid motion that takes place in the region around the foot of a dam, we do not need to follow what is happening to individual particles of water; rather we need to investigate what is happening in those spatial points. So in this case, the Eulerian description is more natural.

*For the discussions in this chapter, I mostly follow the chapter on "Kinematics" in the book "Fluid Mechanics" (4th edition) by Kundu and Cohen

3 Connection between Lagrangian and Eulerian description

The Lagrangian and Eulerian descriptions are two different ways of looking at the same thing. So it is possible to establish some connection between the two. Consider the generic flow variable $\mathcal{F}$ again. In the Eulerian description, the value of $\mathcal{F}$ at a particular point $\boldsymbol{x}$ at some time t may be equivalently be thought of as the value of $\mathcal{F}$ that is shown by the particle that happens to occupy this position $\boldsymbol{x}$ at that time. This way of thinking connects the Eulerian description to the Lagrangian description. We can also think of it the other way round. Thus, there is a one-to-one correspondence between the two descriptions:

$$\mathcal{F}(\boldsymbol{X}, t) \iff \mathcal{F}(\boldsymbol{x}(\boldsymbol{X}, t), t).$$

A very useful mathematical connection can be established between the two descriptions on the basis of time derivatives. Indeed, there are two kinds of time derivatives:

- $\left.\dfrac{\partial}{\partial t}\right|_{\boldsymbol{X}} \equiv \dfrac{\mathrm{D}}{\mathrm{D}t}$: This time derivative is taken keeping our focus fixed on a particular particle, the one that has the label $\boldsymbol{X}$; it is referred to as the material derivative or substantial derivative or particle derivative
- $\left.\dfrac{\partial}{\partial t}\right|_{\boldsymbol{x}} \equiv \dfrac{\partial}{\partial t}$: This time derivative is taken keeping our focus fixed on a particular spatial point, denoted by $\boldsymbol{x}$

Before establishing the connection between these two time derivatives, note the definition of velocity:

$$\mathbf{v} := \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{X}} \equiv \frac{\mathrm{D}\boldsymbol{x}}{\mathrm{D}t}. \tag{1}$$

Now, to establish the connection we use the chain rule of differentiation:

$$\begin{aligned}
\frac{\mathrm{D}\mathcal{F}}{\mathrm{D}t} &\equiv \left.\frac{\partial \mathcal{F}(\boldsymbol{x}(\boldsymbol{X}, t), t)}{\partial t}\right|_{\boldsymbol{X}} \\
&= \left.\frac{\partial \mathcal{F}}{\partial t}\right|_{\boldsymbol{x}} \left.\frac{\partial t}{\partial t}\right|_{\boldsymbol{X}} + \left.\frac{\partial \mathcal{F}}{\partial \boldsymbol{x}}\right|_{t} \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{X}} \\
&= \frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \boldsymbol{x}} \\
&\equiv \frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{F},
\end{aligned}$$

where we note the following:

- We have used the definition of velocity from Eq. (1)
- We have dropped the subscript t in the partial derivative with respect to $\boldsymbol{x}$
- The partial derivative with respect to $\boldsymbol{x}$ is nothing but the gradient denoted by ∇
- The dot product between $\mathbf{v}$ and $\nabla \mathcal{F}$ ensures that the order of the tensor denoted by the second term is the same as that of the first term

We can now write

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{2}$$

4 Streamline, Pathline, Streakline

There are three common ways to visualize the flow: using streamlines, pathlines, and streaklines.

Streamlines are those curves which are everywhere tangent to the velocity field. If, in a rectangular Cartesian coordinate system, we consider an infinitesimal element $\mathrm{d}\boldsymbol{s} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ of a streamline with $\mathbf{v} = (v_x, v_y, v_z)$ being the local velocity vector then by the definition of streamline, we have

$$\mathrm{d}\boldsymbol{s} \times \mathbf{v} = 0,$$
$$\text{or,} \quad \mathbf{i}(v_y\mathrm{d}z - v_z\mathrm{d}y) - \mathbf{j}(v_x\mathrm{d}z - v_z\mathrm{d}x) + \mathbf{k}(v_x\mathrm{d}y - v_y\mathrm{d}x) = 0,$$

from which we obtain after equating each of the components to zero:

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}z}{v_z}, \tag{3}$$

which is the equation that is used to determine the equation of the streamline.

A *pathline* is the locus traced out by a particle as it passes through different positions in space at different times. Therefore, it is just the position vector of a particle in the Lagrangian description, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$.

A *streakline* is the curve that joins at a certain time, t the end-points of the pathlines of the various particles that passes through a specified point from some initial time to the time, t.

Streamlines, pathlines, streaklines overlap each other when the flow is steady.

5 Relative motion

We consider the relative motion between two neighbouring fluid particles located at $\boldsymbol{x}$ and $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$. The relative velocity between the two points is given by

$$\begin{aligned}\mathbf{v}(\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - \mathbf{v}(\boldsymbol{x}) &= \mathbf{v}(\boldsymbol{x}) + \mathrm{d}\boldsymbol{x} \cdot \frac{\partial \mathbf{v}}{\partial \boldsymbol{x}} + \cdots - \mathbf{v}(\boldsymbol{x}) \qquad \text{(Taylor series expansion)} \\ &\approx \mathrm{d}\boldsymbol{x} \cdot \frac{\partial \mathbf{v}}{\partial \boldsymbol{x}} \equiv \mathrm{d}\boldsymbol{x} \cdot \nabla \mathbf{v}\end{aligned}$$

You have already come across the grouping $\nabla \mathbf{v}$ earlier in the assignment sheet on Mathematical Preliminaries. It is the gradient of the velocity and is a second-order tensor, equivalently represented in index notation as $\dfrac{\partial v_i}{\partial v_j}$. It can be equivalently represented as a 3×3 matrix. Like all matrices, this one can be decomposed into a sum of symmetric and anti-symmetric parts; thus

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{anti-symmetric}}, \tag{4}$$

where we observe that the first part is indeed symmetric because there is no effective change in the expression if we interchange the indices i and j; on the other hand, in the second part if we interchange i and j the resulting expression is negative of the original one which is why it is anti-symmetric. We may equivalently write the above decomposition as

$$\nabla \mathbf{v} = \underbrace{\frac{1}{2}\left\{\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathsf{T}}\right\}}_{\text{symmetric}} + \underbrace{\frac{1}{2}\left\{\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathsf{T}}\right\}}_{\text{anti-symmetric}}, \tag{5}$$

where the superscript 'T' denotes the transpose. Again, you have already come across this symmetric part in the assignment sheet on Mathematical Preliminaries. Indeed, this symmetric part is used to define what is referred to as the strain-rate tensor.[1]

[1] We'll see later that extensive use of this strain-rate tensor will be made in the context of the Navier-Stokes equations.