# Differential equations of motion
# Static fluid, Euler's equation, Bernoulli's equation

Jeevanjyoti Chakraborty
jeevan@mech.iitkgp.ernet.in

## 1 Static fluid behaviour

For static fluid:

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = 0. \tag{1}$$

$$\text{Therefore,} \quad \rho \mathbf{b} + \nabla \cdot \boldsymbol{\sigma}^{\mathsf{T}} = \mathbf{0}. \tag{2}$$

Now in the static fluid we know that

$$\mathbf{T}_{(\mathbf{n})} = -p\hat{\mathbf{n}} \quad \text{or,} \quad T_i = -pn_i. \tag{3}$$

But, we had obtained previously that $T_i = \sigma_{ji} n_j$. Therefore

$$\sigma_{ji} n_j = -pn_i. \tag{4}$$

Now, we use the power of Kronecker delta as a substitution operator to write

$$\sigma_{ji} n_j = -pn_j \delta_{ji}, \tag{5}$$

so that we obtain finally

$$\sigma_{ji} = -p\delta_{ji}, \quad \text{or,} \quad \boldsymbol{\sigma} = -p\mathbf{I}, \tag{6}$$

where $\mathbf{I}$ is the identity matrix. It is important to note that we no longer need to write '$\mathsf{T}$' to denote transpose because $\mathbf{I}$ is symmetric.[1]

Now we substitute Eq. (6) in (2) to obtain:

$$\nabla \cdot (-p\mathbf{I}) + \rho \mathbf{b} = 0. \tag{7}$$

We can write the previous using indical notation and again exploit the power of the Kronecker delta as a substitution operator as follows

$$\begin{aligned}
&\nabla \cdot (-p\mathbf{I}) + \rho \mathbf{b} = 0 \\
\text{or,} \quad &\frac{\partial}{\partial x_j} (-p\delta_{ji}) + \rho b_i = 0 \\
\text{or,} \quad &-\frac{\partial p}{\partial x_i} + \rho b_i = 0 \\
\text{or,} \quad &-\nabla p + \rho \mathbf{b} = 0.
\end{aligned} \tag{8}$$

[1] Actually the symmetricity of the stress tensor is not limited to the special case of static fluid. It can be shown that conservation of angular momentum leads to the conclusion that the stress tensor must always be symmetric (even when motion is taking place; the only condition that must be satisfied is that there should not be any body couples acting on the volume.

# 2 Flowing fluid: Inviscid theory

## 2.1 Euler's equation

If the form of the stress tensor that is applicable in the case of static fluid is substituted in the general equation of motion (Cauchy's equation) then what we obtain is the Euler's equation:

$$\begin{aligned} & \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} + \nabla \cdot (-p\mathbf{I}) \, , \\ \text{or,} \quad & \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} - \nabla p. \end{aligned} \tag{9}$$

This equation is the basis of the inviscid theory of flowing fluids.

## 2.2 Towards Bernoulli's equation

We expand the LHS of (9)

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \rho \mathbf{b} - \nabla p, \tag{10}$$

and use the following identity from vector calculus

$$\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times \boldsymbol{\omega}, \tag{11}$$

where $\omega$ is the vorticity, to obtain from (10)

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} (\mathbf{v} \cdot \mathbf{v}) - \mathbf{v} \times \boldsymbol{\omega} = \frac{\rho \mathbf{b} - \nabla p}{\rho}. \tag{12}$$