Differential equations of motion

From integral to differential form of momentum conservation equation

Jeevanjyoti Chakraborty
jeevan@mech.iitkgp.ernet.in

1 What we need for a differential form

The integral form of the conservation of linear momentum equation, referred to a material volume, was:

$$\int_{V_\mathrm{m}} \left[\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) \right] \mathrm{d}V = \Sigma \mathbf{F}. \tag{1}$$

The integrand in the left hand side can be expanded as

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial \mathbf{t}} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}), \tag{2}$$

where the right hand side can be further simplified by clubbing the second and the fourth terms, and then using the continuity equation $\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \mathbf{0}$ to obtain

$$\begin{aligned} \frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) &= \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \\ &= \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}. \end{aligned} \tag{3}$$

We thus have from Eq. (1) the following

$$\int_{V_\mathrm{m}} \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \,\mathrm{d}V = \Sigma \mathbf{F}, \tag{4}$$

where the force $\mathbf{F}$ can be decomposed into body force contributions (like those due to gravity or electromagnetic forces) and surface force contributions. Thus,

$$\int_{V_\mathrm{m}} \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} \,\mathrm{d}V = \underbrace{\int_{V_\mathrm{m}} \rho \mathbf{b} \,\mathrm{d}V}_{\text{Body force}} + \underbrace{\int_{S_\mathrm{m}} \mathbf{T} \,\mathrm{d}S}_{\text{Surface force}}, \tag{5}$$

where $\mathbf{b}$ is a generic representation of a body force per unit mass and $\mathbf{T}$ is a force vector acting on a small surface patch of the volume considered; $\mathbf{T}$ is referred to as the surface traction (explained a little later).

Now, had the second integral (involving $\mathbf{T}$) been a volume integral, we could have easily taken all the volume integrals on one side and then using the arbitrariness of V_m, we could have obtained the differential form of the momentum equation. But we don't have that. So in order to obtain the differential form, we need to transform the surface integral into a volume integral. The first step towards that end is to write the surface traction vector in terms of the stress. But before we do that let us first look at traction itself in some more detail.

2 Traction and stress

2.1 Traction across a plane at a point

Consider an area A in a given plane and containing a point P within a body as shown in Fig. 1. Suppose the plane divides the body into two regions, Region I and Region II. Consider Region I. Draw a normal $\hat{\mathbf{n}}$ to the plane at P and pointing from Region I towards Region II. Over the area A, Region II exerts forces on Region I. Suppose this system of forces is statically equivalent to a force $\mathbf{F}$ acting at P in a definite direction and a couple $\mathbf{C}$ about a definite axis. Let us make the area A small, ensuring that the point P is always inside it. Then the force $\mathbf{F}$ and the couple $\mathbf{C}$ tend to zero limits and the direction of $\mathbf{F}$ tends to a limiting direction. We assume that as A tends to zero, the number $|\mathbf{F}|/A$ tends to a non-zero limit while $|\mathbf{C}|/A$ tends to 0 (which is sensible because smaller the area, the smaller will be the distance from the definite axis referred to earlier leading to a couple that vanishes).

We define a vector

$$\mathbf{T} = \lim_{A \to 0} \frac{\mathbf{F}}{A} \tag{6}$$

called the *stress vector* or the *traction vector*.

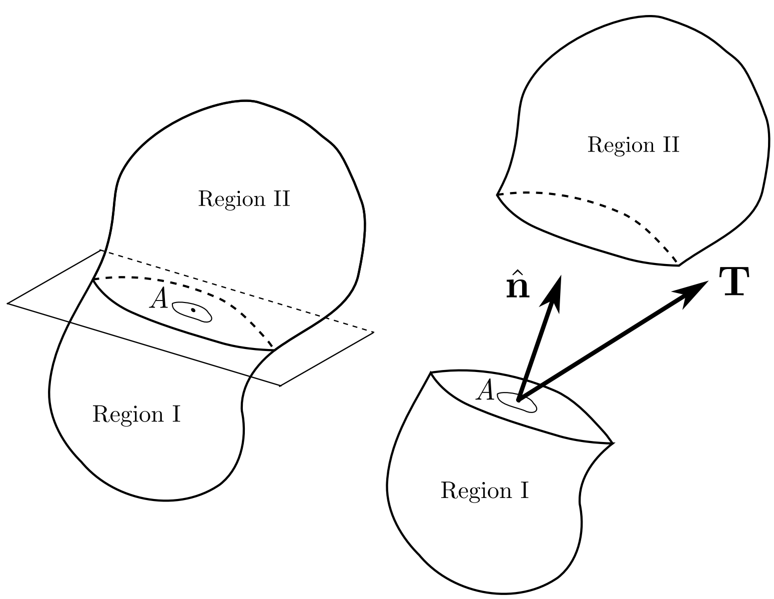

Figure 1: Concept of traction

Note that the traction vector depends on the location of P as well as the choice of the plane on which A is located. Since the orientation of this plane is given by $\hat{\mathbf{n}}$, so $\mathbf{T}$ depends the position vector of P *and* $\hat{\mathbf{n}}$. If $\hat{\mathbf{n}}$ is different, $\mathbf{T}$ will be different.

Note also that just as Region II exerts forces on Region I, so also Region I exerts forces on Region II over the area; these forces must necessarily be equal in magnitude and opposite in direction from Newton's third law (important: it is not necessary for the whole body to be in equilibrium for these forces to be equal and opposite). Thus, in a fashion identical to what was discussed previously, a stress vector or traction vector can be defined on the plane and considering the unit vector, $-\hat{\mathbf{n}}$. We then have

$$\mathbf{T}\left(\mathbf{x},\hat{\mathbf{n}}\right) = -\mathbf{T}\left(\mathbf{x},-\hat{\mathbf{n}}\right), \tag{7}$$

where it is important to note that the position vector $\mathbf{x}$ is the same for both traction vectors because we are considering the same point P just from two different sides.

2.2 Surface tractions

The nature of the action between two bodies in contact is assumed to be of the same nature as the action between two portions of the same body separated by an imaginary surface. If the point P in the previous discussion is moved to a point P' on the bounding surface of the body with the position vector $\mathbf{x}$ changing to $\mathbf{x}'$ and $\hat{\mathbf{n}}$ changing to $\hat{\mathbf{n}}'$ that coincides with the unit outward normal at P', the resulting traction vector $\mathbf{T}'\left(\mathbf{x}',\hat{\mathbf{n}}'\right)$ is referred to as the *surface traction*.

VERY IMPORTANT: Whether it is the traction across an imaginary plane inside a body or the surface traction which acts at the actual bounding surface of a body, the direction of the traction vector does not, in general, coincide with that of $\hat{\mathbf{n}}$.

2.3 Principle of equilibrium of tractions

Consider Eq. (5) but written as:

$$\int_{V_\mathrm{m}} \left(\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho\mathbf{b}\right)\,\mathrm{d}V = \int_{S_\mathrm{m}} \mathbf{T}\,\mathrm{d}S \tag{8}$$

Let us denote a characteristic linear dimension of V_m as l, with l^3 understood to be equal to V_m. Now, denoting the mean value of the integrand over V_m by $\langle\rangle$, we can express the above equation as

$$\langle\rangle l^3 = \int_{S_\mathrm{m}} \mathbf{T}\,\mathrm{d}S. \tag{9}$$

Now, let the volume be shrunk to a very small size by letting l tend to 0. Then dividing throughout by l^2 and taking the limit $l \to 0$, we have

$$\lim_{l\to 0}\int_{S_\mathrm{m}} \mathbf{T}\,\mathrm{d}S = 0. \tag{10}$$

The above equation says that the forces due to the tractions on the area elements of the surface of an infinitesimal body are, approximately, a system in equilibrium.

The most important thing to note here is that this equilibrium of forces due to tractions holds even if the infinitesimal body is itself not in static equilibrium, i.e. even if it is accelerating.

This result is referred to as the principle of equilibrium of tractions on small volumes.

2.4 Different planes, different tractions - at the same point

Consider a plane passing through a point with position vector $\mathbf{x}$. Let the unit normal vector to the plane be $\hat{\mathbf{n}}$. Referred to a particular choice of coordinate axes $Ox_1x_2x_3$, let the components of $\hat{\mathbf{n}}$ be (n_1, n_2, n_3). The traction vector at $\mathbf{x}$ across this plane will be denoted by $\mathbf{T}(\mathbf{x},\hat{\mathbf{n}})$. Henceforth, we will denote it by $\mathbf{T}_{(n)}$. If we consider a different plane, the unit normal vector, $\hat{\mathbf{n}}$ will change and so will the traction $\mathbf{T}_{(n)}$.

It is extremely important to note that this change occurs even though we are still considering the same point having position vector, $\mathbf{x}$. Since through any given point, an infinite number of different planes may be considered, so we can have infinite number of different traction vectors at the same point – each such traction vector being distinguished by the corresponding $\hat{\mathbf{n}}$. As a special case, if we choose a plane such that the unit normal vector to it is parallel to the x_1–direction, then $\hat{\mathbf{n}} \equiv \hat{\mathbf{e}}_1$ with components $(1,0,0)$. We denote the traction vector corresponding to this plane as $\mathbf{T}_{(1)}$. Similarly, if planes are chosen such that the unit normal vectors to them are parallel to the x_2– and x_3–directions, then the corresponding traction vectors will be denoted by $\mathbf{T}_{(2)}$ and $\mathbf{T}_{(3)}$, respectively. Note again that all these various traction vectors $\mathbf{T}_{(n)}$, $\mathbf{T}_{(1)}$, $\mathbf{T}_{(2)}$, and $\mathbf{T}_{(3)}$ are at the same point with position vector, $\mathbf{x}$.

2.5 Traction in terms of stress

We use the law of equilibrium of surface tractions to express the traction at any point across any plane in terms of the components of the tractions across planes that are parallel to the coordinate planes.

Consider the equilibrium of a tetrahedral portion of the body having one vertex at O and the three edges that meet at this vertex to be parallel to the coordinate axes.

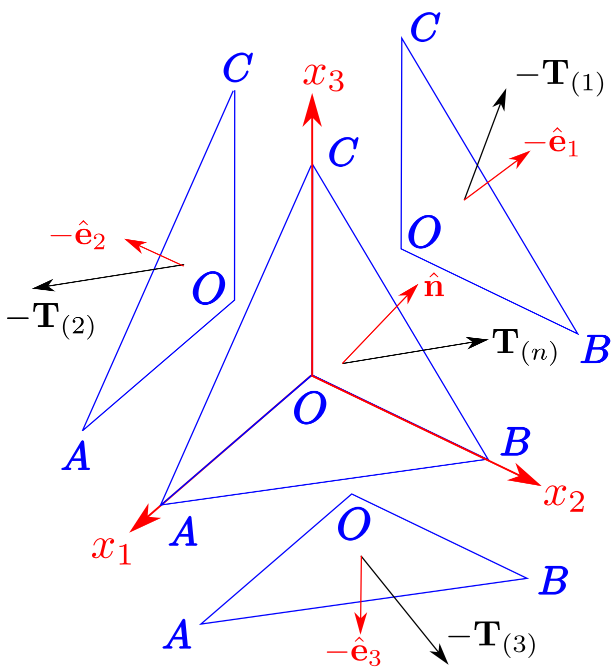

Figure 2: Stress equilibrium

Referring to Fig. 2, for the force equilibrium along direction-1, we have:

$$T_{(n)1}\Delta A - T_{(1)1}\Delta A_1 - T_{(2)1}\Delta A_2 - T_{(3)1}\Delta A_3 = 0, \tag{11}$$

where ΔA, ΔA_1, ΔA_2, and ΔA_3 are, respectively, the areas of triangles ABC, OCB, OAC, and OAB. Further, $T_{(n)1}$, $T_{(1)1}$, $T_{(2)1}$, and $T_{(3)1}$ are, respectively, the components of $\mathbf{T}_{(n)}$, $\mathbf{T}_{(1)}$, $\mathbf{T}_{(2)}$, and $\mathbf{T}_{(3)}$ along the x_1–direction. Now,

$$\Delta A_1 = n_1\Delta A, \quad \Delta A_2 = n_2\Delta A, \quad \Delta A_3 = n_3\Delta A. \tag{12}$$

Therefore,

$$T_{(n)1} - T_{(1)1}n_1 - T_{(2)1}n_2 - T_{(3)1}n_3 = 0 \tag{13}$$

So, writing generally for any direction-i, we have

$$\begin{aligned} T_{(n)i} &= T_{(1)i}n_1 + T_{(2)i}n_2 + T_{(3)i}n_3, \\ \text{or,} \quad T_{(n)i} &= T_{(j)i}n_j \qquad \text{(using indical notation)} \end{aligned} \tag{14}$$

Here, $T_{(j)i}$ represents the component of the traction vector T_j along the i-th direction, and is denoted, alternatively, as σ_{ji}. Thus,

$$T_{(n)i} = \sigma_{ji}n_j, \tag{15}$$

which we identify as a dot product (because one of the indices is repeated) and so rewrite in vector (or, compact notation) as

$$\mathbf{T}_{(n)} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \tag{16}$$

or, equivalently in matrix representation as

$$[\mathbf{T}_{(n)}] = [\boldsymbol{\sigma}]^{\mathsf{T}}[\hat{\mathbf{n}}], \tag{17}$$

where the ‘T’ in the superscript refers to transpose. To understand why the transpose comes, refer to the “Mathematical Preliminaries” document.

VERY IMPORTANT: We state (without proving) that conservation of angular momentum in the absence of body couples leads to the conclusion that the stress tensor is symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$.

In expanded form, we have from Eq. (17)

$$T_{(n)1} = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3, \tag{18a}$$
$$T_{(n)2} = \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3, \tag{18b}$$
$$T_{(n)3} = \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3. \tag{18c}$$

Eqs (15), (16), (17), and (18) are different forms of what are referred to as the Cauchy’s formula (sometimes Cauchy’s stress theorem or Cauchy’s law).

3 The differential form of equation of motion

We now go back to the integral form of the momentum conservation equation, i.e. Eq. (5) and substitute the expression for the traction vector in terms of the stress; thus

$$\begin{aligned} \int_{V_\mathrm{m}} \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}V &= \int_{V_\mathrm{m}} \rho\mathbf{b}\,\mathrm{d}V + \int_{S_\mathrm{m}} \mathbf{T}\,\mathrm{d}S \\ &= \int_{V_\mathrm{m}} \rho\mathbf{b}\,\mathrm{d}V + \int_{S_\mathrm{m}} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\,\mathrm{d}S \\ &= \int_{V_\mathrm{m}} \rho\mathbf{b}\,\mathrm{d}V + \int_{V_\mathrm{m}} \nabla \cdot \boldsymbol{\sigma}\,\mathrm{dV} \quad \text{(Using divergence theorem)} \end{aligned} \tag{19}$$

Thus we can write

$$\int_{V_\mathrm{m}} \left(\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho\mathbf{b} - \nabla \cdot \boldsymbol{\sigma}\right)\,\mathrm{d}V = 0. \tag{20}$$

Now using the same arguments about the arbitrariness of the volume selected as those we had used to obtain the conservation of mass equation, we can set the integrand in the above equation to zero. Thereby we obtain the differential form of the conservation of momentum equation:

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = \rho \mathbf{b} + \nabla \cdot \boldsymbol{\sigma}, \tag{21}$$

which is referred to as *Cauchy's equation of motion*. It is applicable to any deformable continuous medium whether it is a fluid or a solid or anything else that cannot even be strictly categorized as being either fluid or solid.

4 Static fluid behaviour

For static fluid:

$$\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = 0. \tag{22}$$

$$\text{Therefore,} \quad \rho \mathbf{b} + \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}. \tag{23}$$

Now in the static fluid we know that

$$\mathbf{T}_{(n)} = -p\hat{\mathbf{n}} \quad \text{or,} \quad T_{(n)i} = -pn_i. \tag{24}$$

But, from Eq. (15), we have that $T_{(n)i} = \sigma_{ji} n_j$. Therefore

$$\sigma_{ji} n_j = -pn_i. \tag{25}$$

Now, we use the power of Kronecker delta as a substitution operator to write

$$\sigma_{ji} n_j = -pn_j \delta_{ji}, \tag{26}$$

so that we obtain finally

$$\sigma_{ji} = -p\delta_{ji}, \quad \text{or,} \quad \boldsymbol{\sigma} = -p\mathbf{I}, \tag{27}$$

where **I** is the identity matrix.

Now we substitute Eq. (27) in (23) to obtain:

$$\nabla \cdot (-p\mathbf{I}) + \rho \mathbf{b} = 0. \tag{28}$$

We can write the previous using indical notation and again exploit the power of the Kronecker delta as a substitution operator as follows

$$\begin{aligned} & \nabla \cdot (-p\mathbf{I}) + \rho \mathbf{b} = 0 \\ \text{or,} \quad & \frac{\partial}{\partial x_j}(-p\delta_{ji}) + \rho b_i = 0 \\ \text{or,} \quad & -\frac{\partial p}{\partial x_i} + \rho b_i = 0 \\ \text{or,} \quad & -\nabla p + \rho \mathbf{b} = 0. \end{aligned} \tag{29}$$