Problem Sheet 4: Material Behaviour

1. Verify that the form $C_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$ is indeed isotropic. *Hint:* Proceed by the same method as in **Q6** in the Problem Sheet on Mathematical Preliminaries.)

2. The constitutive law for a linear, elastic, isotropic solid is given by $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$. From this relation find the inverse relation, i.e. ε_{ij} in terms of σ_{ij}. To do this, first set $i = j = p$, utilize the fact that $\varepsilon_{pp} \equiv \varepsilon_{kk}$, and then obtain the expression for ε_{kk}; next substitute this expression back in the above equation to find ε_{ij}.

3. By choosing specific values of the indices in the expression of C_{ijkl} for a linear, elastic, isotropic material, express λ and G in terms of the various components of C_{ijkl} and, equivalently, $\overline{C}_{pq}$. Are these expressions unique?

4. Compare the relation found in the previous question with the following

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

to show that $E = \dfrac{G\left(3\lambda + 2G\right)}{\lambda + G}$ and $\nu = \dfrac{\lambda}{2\left(\lambda + G\right)}$. Subsequently show that $\lambda = \dfrac{E\nu}{\left(1+\nu\right)\left(1-2\nu\right)}$ and $G = \dfrac{E}{2\left(1+\nu\right)}$.

5. For a hydrostatic state of compression $\sigma_{ij} = -p\delta_{ij}$ (p being a positive constant), show that the dilatation (refer the problem sheet on kinematics) is given by $-p/K$, where K is referred to as the bulk modulus. Also show

$$K = \frac{E}{3\left(1-2\nu\right)} \equiv \frac{3\lambda + 2G}{3}.$$

If $\nu \to 1/2$, what happens to K? Also, what happens to the volumetric deformation? What can you say about the compressibility of the material? How does the first invariant of strain feature in all this?

6. If the Young's modulus E, the bulk modulus K, and the shear modulus μ are required to be positive, show that the Poisson's ratio ν must satisfy the inequality

$$-1 < \nu < \frac{1}{2}.$$

For most real materials, however, $0 < \nu < 1/2$. Show that this more restrictive inequality implies $\lambda > 0$. Materials that have negative Poisson's ratio are referred to as *auxetic* materials.

7. For a linear, elastic, isotropic solid material, show that the principal axes of strain coincide with the principal axes of stress. Thus show that the principal stresses ($\sigma^{(p)}$) can be expressed in terms of the principal strains ($\varepsilon^{(p)}$) as $\sigma^{(p)} = \lambda J_1 + 2G\varepsilon^{(p)}$, where J_1 is the first strain invariant. (*Hint:* Consider the two eigenvalue problems: $(\sigma_{ij} - \sigma\delta_{ij})\, n_i = 0$ and $(\varepsilon_{ij} - \varepsilon\delta_{ij})\, n'_i = 0$, where σ and ε are the general principal stress and principal strain, respectively; n_i and n'_i are the principal directions corresponding to the stress and the strain, respectively. Use the linear, elastic, isotropic constitutive law to draw the link between the two eigenvalue problems.)

8. Consider an incompressible elastic material. It was to be shown in **Q3a** of the Problem Sheet on Kinematics that $\varepsilon_{kk} = 0$ when the change in volume is zero.

   (a) Show that the constraint of $\varepsilon_{kk} = 0$ implies that the Poisson's ratio, $\nu = \dfrac{1}{2}$.

   (b) Referring to the expressions of the Lamé parameter, λ and the bulk modulus, K found earlier, show that both λ and K tend to infinity when $\nu \to \infty$.

   (c) Show that the constitutive relation, $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$ for such an incompressible material should contain an indeterminate term.

   (d) The constitutive relation with the indeterminate term is usually written as $\sigma_{ij} = -p\delta_{ij} + 2G\varepsilon_{ij}$; show that $p = -\dfrac{1}{3}\sigma_{kk}$.

9. Strain gauges are little devices used to measure the strain on the free surface of a body, along a particular direction. Usually three strain gauges are combined together to form what is called a strain rosette. Following the measurement of strains along three directions, various pieces of information regarding the state of strain and stress may be determined.

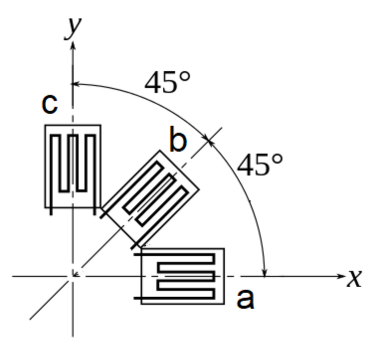

Figure 1: Rectangular strain rosette

Consider a rectangular rosette, as shown in Figure 1 that is cemented to the *free surface* of an airplane wing. The material of the airplane wing is an aluminium alloy with Young's modulus, $E = 72$ GPa and Poisson's ratio, $\nu = 0.33$. Under load, the strain readings are $\varepsilon_a = 0.00250$, $\varepsilon_b = 0.00140$, $\varepsilon_c = -0.00125$.

(a) Determine the state of strain, i.e. find the various strain components.

(b) Determine the principal stresses.

(c) Determine the orientation of the principal planes.

(d) Determine the maximum shear stress.

(e) Determine the orientation of the plane on which the maximum shear stress acts.

[(a) $\varepsilon_{xx} = 0.00250$, $\varepsilon_{yy} = -0.00125$, $\varepsilon_{xy} = 0.000775$; (b) $p_1 = 176.99$ MPa, $p_2 = -42.67$ MPa; (c) At 11.23° to the x-axis or y-axis; (d) $\tau_{\max} = 109.83$ MPa; (e) At 56.23° to the x-axis.]

10. Using the constitutive relation for a linear, elastic, isotropic solid in the mechanical equilibrium equations ($\nabla \cdot \boldsymbol{\sigma} = 0$ OR $\dfrac{\partial \sigma_{ij}}{\partial X_j} = 0$) where body forces are absent, show that the following equation is obtained:

$$(\lambda + G)\nabla(\nabla \cdot \boldsymbol{u}) + G\nabla^2 \boldsymbol{u} = 0 \quad \text{OR} \quad (\lambda + G)\frac{\partial}{\partial X_i}\left(\frac{\partial u_k}{\partial X_k}\right) + G\frac{\partial^2 u_i}{\partial X_j^2} = 0.$$