Problem Sheet 3: Stress

1. Consider a state of stress in which the non-zero stress components are σ_{xx}, σ_{yy}, and σ_{xy} only; this state of stress being referred to the xyz coordinate system. Consider another set of coordinate axes $x'y'z'$ with the z' axis coinciding with the z axis and the x' axis located counterclockwise through angle θ from the x axis. Using the formulae for T^N and T^S, show that the following equations that you learnt in first year can be recovered:

$$
\begin{aligned}
\sigma_{x'x'} &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos(2\theta) + \sigma_{xy}\sin(2\theta),\\
\sigma_{y'y'} &= \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2} - \sigma_{xy}\sin(2\theta),\\
\sigma_{x'y'} &= -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin(2\theta) + \sigma_{xy}\cos(2\theta).
\end{aligned}
$$

2. Consider again the situation described in Problem 1 with the only change that σ_{zz} is also non-zero (in addition to σ_{xx}, σ_{yy}, and σ_{xy} being non-zero). Show that the expressions for $\sigma_{x'x'}$, $\sigma_{y'y'}$ and $\sigma_{x'y'}$ are again the same as in Problem 1.

3. Consider yet again the situation described Problem 1, but this time obtain the expressions for $\sigma_{x'x'}$, $\sigma_{y'y'}$ and $\sigma_{x'y'}$ using the transformation rules for tensors of order 2 discussed in Mathematical Preliminaries.

4. For the 2D state of stress $[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$, the principal stresses can be found again from the requirement that on the principal planes, the shear stresses must be zero. Using the expressions from Problem 1, show that the principal stresses are given by

$$p_1, p_2 = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}.$$

Further, show that the maximum shear stress is given by

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}.$$

Finally, express p_1, p_2, and $\tau_{\max}$ in terms of stress invariants (appropriately written for the 2D case).

5. Let $\mathbf{T}_x$, $\mathbf{T}_y$, and $\mathbf{T}_z$ be stress (or, traction) vectors perpendicular, respectively, to the x, y, and z axes. Show that the sum of the squares of the magnitudes of these stress vectors is an invariant that is expressible in terms of the stress invariants I_1 and I_2.

   $[I_1^2 - 2I_2]$

6. From the strength of materials approach, for a beam of circular cross-section, we have the following

$$\sigma_{xx} = -\frac{My}{I}, \quad \sigma_{xy} = \frac{V(R^2 - y^2)}{3I}, \quad \sigma_{yy} = \sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0,$$

   where R is the radius of the cross-section, $I = \pi R^4/4$, M is the bending moment, V is the shear force, and $\mathrm{d}M/\mathrm{d}x = V$. Assuming zero body forces, show that the stress field does not satisfy the mechanical equilibrium equations.

7. A one-dimensional problem of a prismatic bar loaded under its own weight can be modelled by the stress field $\sigma_{xx} \equiv \sigma_{xx}(x)$, $\sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$ with body forces $F_x = \rho g$, $F_y = F_z = 0$, ρ is the density. Using mechanical equilibrium relations, show that the non-zero stress will be given by $\sigma_{xx} = \rho g(l - x)$ where l is the length of the prismatic bar.

8. The state of stress at a point is given by the components of stress tensor σ_{ij}. A plane is defined by the direction cosines of the normal $\left(1/2, 1/2, 1/\sqrt{2}\right)$. State the general conditions for which the traction on the plane has the same direction as the x_2–axis and a magnitude of 1.

   $[\frac{1}{\sqrt{2}}\sigma_{11} + \frac{1}{\sqrt{2}}\sigma_{12} + \sigma_{13} = 0,\ \frac{1}{\sqrt{2}}\sigma_{12} + \frac{1}{\sqrt{2}}\sigma_{22} + \sigma_{23} = \sqrt{2},\ \frac{1}{\sqrt{2}}\sigma_{13} + \frac{1}{\sqrt{2}}\sigma_{23} + \sigma_{33} = 0.]$

9. Referring to coordinate axes aligned along the principal directions and with the principal stresses denoted by $\sigma^{(1)}$, $\sigma^{(2)}$, $\sigma^{(3)}$, we have for a plane characterized by the unit normal $[\hat{\mathbf{n}}] = [n_1\ n_2\ n_3]^{\mathsf{T}}$:

$$\begin{aligned}
(T^N)^2 + (T^S)^2 &= (\sigma^{(1)}n_1)^2 + (\sigma^{(2)}n_2)^2 + (\sigma^{(3)}n_3)^2, \\
T^N &= \sigma^{(1)}n_1^2 + \sigma^{(2)}n_2^2 + \sigma^{(3)}n_3^2, \\
1 &= n_1^2 + n_2^2 + n_3^2.
\end{aligned}$$

   Solve the above system of equations for n_1^2, n_2^2, and n_3^2 to show that:

$$\begin{aligned}
n_1^2 &= \frac{(T^N - \sigma^{(2)})(T^N - \sigma^{(3)}) + (T^S)^2}{(\sigma^{(1)} - \sigma^{(2)})(\sigma^{(1)} - \sigma^{(3)})}, \\
n_2^2 &= \frac{(T^N - \sigma^{(3)})(T^N - \sigma^{(1)}) + (T^S)^2}{(\sigma^{(2)} - \sigma^{(3)})(\sigma^{(2)} - \sigma^{(1)})}, \\
n_3^2 &= \frac{(T^N - \sigma^{(1)})(T^N - \sigma^{(2)}) + (T^S)^2}{(\sigma^{(3)} - \sigma^{(1)})(\sigma^{(3)} - \sigma^{(2)})}.
\end{aligned}$$

Take $\sigma^{(1)} > \sigma^{(2)} > \sigma^{(3)}$ and noting that $n_1^2, n_2^2, n_3^2 \geq 0$, show that

$$(T^N - \sigma^{(2)})(T^N - \sigma^{(3)}) + (T^S)^2 \geq 0,$$
$$(T^N - \sigma^{(3)})(T^N - \sigma^{(1)}) + (T^S)^2 \leq 0,$$
$$(T^N - \sigma^{(1)})(T^N - \sigma^{(2)}) + (T^S)^2 \geq 0.$$

Recast the above inequalities in the following forms:

$$\left(T^N - \frac{\sigma^{(2)} + \sigma^{(3)}}{2}\right)^2 + (T^S)^2 \geq \left(\frac{\sigma^{(2)} - \sigma^{(3)}}{2}\right)^2,$$
$$\left(T^N - \frac{\sigma^{(3)} + \sigma^{(1)}}{2}\right)^2 + (T^S)^2 \leq \left(\frac{\sigma^{(3)} - \sigma^{(1)}}{2}\right)^2,$$
$$\left(T^N - \frac{\sigma^{(1)} + \sigma^{(2)}}{2}\right)^2 + (T^S)^2 \geq \left(\frac{\sigma^{(1)} - \sigma^{(2)}}{2}\right)^2.$$

Considering the above inequalities to depict regions in a $T^N - T^S$, plot out the boundaries of those regions and the intersection of the valid regions. Note that these boundaries are in the form of circles; they are referred to as the Mohr's circles for a 3D state of stress.

10. Considering the Mohr's circles for a 3D state of stress, as discussed in the previous problem, answer the following:
    (a) What is the maximum shear stress?
    (b) What is the normal stress associated with the maximum shear stress? Think about the opposite situation: what is the shear stress associated with the principal stresses?
    (c) Determine the plane, in terms of n_1, n_2, and n_3, on which the maximum shear stress occurs.

11. A widely used method to represent the stress field in a body is in terms of the effective or von Mises stress which is defined as $\sigma_{\text{eff}} \equiv \sigma_{\text{vonMises}} = \sqrt{\frac{3}{2}\sigma_{ij}^{\text{D}}\sigma_{ij}^{\text{D}}}$, where σ_{ij}^{D} represents the deviatoric part of the stress tensor. Show that

$$\sigma_{\text{eff}} \equiv \sigma_{\text{vonMises}} = \frac{3}{\sqrt{2}}\tau_{\text{oct}},$$

where τ_{oct} is the octahedral shear stress. *Hint:* First express $\sigma_{\text{eff}} \equiv \sigma_{\text{vonMises}}$ in terms of stress invariants.