Problem Sheet 2: Kinematics

1. For arbitrary constants A, B, C, determine the strain tensor (ε_{ij}) and rotation tensor (Ω_{ij}) for the following displacement fields:

   (a) $u = Axy$, $v = Bxz^2$, $w = C(x^2 + y^2)$

   (b) $u = Ax^2$, $v = Bxy$, $w = Cxyz$

   (c) $u = Ayz^3$, $v = Bxy^2$, $w = C(x^2 + y^2)$

   If you are familiar with Python, find these using the symbolic facilities in `SymPy`.

$$(a) \quad [\boldsymbol{\varepsilon}] = \begin{bmatrix} Ay & \frac{Ax}{2} + \frac{Bz^2}{2} & Cx \\ \frac{Ax}{2} + \frac{Bz^2}{2} & 0 & Bxz + Cy \\ Cx & Bxz + Cy & 0 \end{bmatrix}$$

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \frac{Ax}{2} - \frac{Bz^2}{2} & -Cx \\ -\frac{Ax}{2} + \frac{Bz^2}{2} & 0 & Bxz - Cy \\ Cx & -Bxz + Cy & 0 \end{bmatrix}$$

$$(b) \quad [\boldsymbol{\varepsilon}] = \begin{bmatrix} 2Ax & \frac{1}{2}By & \frac{1}{2}Cyz \\ \frac{1}{2}By & Bx & \frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & Cxy \end{bmatrix}$$

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & -\frac{1}{2}By & -\frac{1}{2}Cyz \\ \frac{1}{2}By & 0 & -\frac{1}{2}Cxz \\ \frac{1}{2}Cyz & \frac{1}{2}Cxz & 0 \end{bmatrix}$$

$$(c) \quad [\boldsymbol{\varepsilon}] = \begin{bmatrix} 0 & \frac{Az^3}{2} + \frac{By^2}{2} & \frac{3A}{2}yz^2 + Cx \\ \frac{Az^3}{2} + \frac{By^2}{2} & 2Bxy & Cy \\ \frac{3A}{2}yz^2 + Cx & Cy & 0 \end{bmatrix}$$

$$[\boldsymbol{\Omega}] = \begin{bmatrix} 0 & \frac{Az^3}{2} - \frac{By^2}{2} & \frac{3A}{2}yz^2 - Cx \\ -\frac{Az^3}{2} + \frac{By^2}{2} & 0 & -Cy \\ -\frac{3A}{2}yz^2 + Cx & Cy & 0 \end{bmatrix}$$

2. A two-dimensional problem of a rectangular bar stretched by uniform end loadings results in the following strain field:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & -C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where C_1 and C_2 are constants. Assuming the field depends only on x and y, integrate the strain-displacement relations to determine the displacement components and identify any rigid-body motion terms.

[$u_x = C_1 x + Ky + D$, $u_y = -C_2 y - Kx + E$; rigid body rotation about z-axis: $\omega_z = -K$; translation along x: D; translation along y: E]

3. Consider a cubical volume element with three of its edges at any one chosen vertex oriented along the principal directions. The original length of each side is $dX^{(k)}$ and the final length is $dx^{(k)}$, such that

$$dx^{(k)} = dX^{(k)}(1 + \varepsilon^{(k)}), \quad k = 1, 2, 3 \quad \text{(no summation implied)}$$

where $\varepsilon^{(k)}$ is the principal strain in the k-direction.

(a) *Dilatation* is defined as the relative change in volume:

$$\text{Dilatation} = \frac{\text{(final vol.)} - \text{(initial vol.)}}{\text{(initial vol.)}}.$$

Find an expression for *dilatation* assuming that $|\varepsilon^{(k)}| \ll 1$. Rewrite this expression in terms of the components of displacement gradient $u_{i,j}$.

(b) If the initial and final volumes are V_0 and V_f respectively with corresponding densities ρ_0 and ρ_f (assumed uniform throughout the volumes), then mass conservation gives the relation $\rho_\mathrm{f} V_\mathrm{f} = \rho_0 V_0$. Using this relation and the result of part (a), relate ρ_0, ρ_f, and the divergence of the displacement, $u_{i,i}$. What happens when $\rho_0 = \rho_\mathrm{f}$, i.e. when the density is constant?

4. Since dilatation can be expressed solely in terms of the normal strain components (refer previous problem), these normal strain components are said to be responsible for *changes in volume* while the shearing strains are responsible for *changes in shape.* Often, the (infinitesimal) strain tensor is decomposed into two parts: the mean normal strain $\boldsymbol{\varepsilon}_\mathrm{M}$, which accounts for volumetric change, and the deviatoric strain ε_D, which accounts for shape change.

(a) Define $\varepsilon_\mathrm{M} = \frac{1}{3}(\nabla \cdot \mathbf{u})\,\mathbb{I}$, where $\mathbb{I}$ is the identity tensor; or, equivalently, $(\varepsilon_M)_{ij} = \frac{1}{3} u_{i,i}\delta_{ij}$. Find the matrix representation of ε_D.

(b) The definition of ε_{M} ensures that the mean normal strain represents a state of equal elongation in all directions. Under this state of strain the elemental volume deforms in such a way that the shape remains *similar* to the original shape. Since ε_{M} accounts for the volumetric strain, the volumetric change associated with ε_{D} should be zero. Check if this is so by finding the dilatation of ε_{D}.

5. The problem of finding the principal strains at a point reduces to the eigenvalue problem:

$$(\varepsilon_{ij} - \lambda_{ij})\, n_j = 0, \quad \text{or, equivalently, in matrix form} \quad ([\varepsilon] - \lambda[\mathbb{I}])\,[\hat{\mathbf{n}}] = 0.$$

Non-trivial solutions of this problem may be found by using the condition that

$$\det\left([\varepsilon] - \lambda[\mathbb{I}]\right) = 0$$

where $\det([A])$ means determinant of $[A]$. The above gives us:

$$\begin{vmatrix} \varepsilon_{11} - \lambda & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \lambda & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \lambda \end{vmatrix} = 0,$$

that reduces to

$$\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3 = 0.$$

(a) Find expressions for J_1, J_2, and J_3 in terms of the components of ε_{ij}.

(b) J_1, J_2, and J_3 are referred to as the *strain invariants*. What do you think is the motivation behind calling them *invariants*? *Hint:* Principal strains pertain to the actual physical situation while the components of ε_{ij} are a consequence of the choice of our coordinate axes.

(c) The principal strain tensor is such that in its matrix representation the diagonal elements are the λ's while the off-diagonal elements are zero. Find expressions for J_1, J_2, and J_3 in terms of the principal strains λ_1, λ_2, and λ_3.

6. The strain field at a point $P(x, y, z)$ in an elastic body is given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 20 & 3 & 2 \\ 3 & -10 & 5 \\ 2 & 5 & -8 \end{bmatrix} \times 10^{-6}.$$

Determine the following values:

(a) The strain invariants

(b) The principal strains

(c) The mean normal strain and the deviatoric strain

[(a) $J_1 = 2 \times 10^{-6}$, $J_2 = -318 \times 10^{-12}$, $J_3 = 1272 \times 10^{-18}$; (b) $\lambda_1 = 20.5 \times 10^{-6}$, $\lambda_2 = -14.1 \times 10^{-6}$, $\lambda_3 = -4.39 \times 10^{-6}$]

7. Consider a strain field such that

$$\varepsilon_{11} = Ax_2^2, \quad \varepsilon_{22} = Ax_1^2, \quad \varepsilon_{12} = Bx_1x_2, \quad \varepsilon_{33} = \varepsilon_{32} = \varepsilon_{31} = 0.$$

Find the relationship between A and B such that it is possible to obtain a single-valued continuous displacement field which corresponds to the given strain field. [$B = 2A$]

8. Consider the strain-displacement relations in a rectangular Cartesian coordinate system and verify that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{1}$$

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{2}$$

Extend the ideas of these two equations to obtain

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \tag{3}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} \tag{4}$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{5}$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right) \tag{6}$$

These six equations are referred to as the compatibility equations.

9. The six compatibility equations in the previous question are not actually independent. To see this, first obtain from Eqs. (2), (5), and (6) the following:

$$\frac{\partial^4 \varepsilon_{xx}}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z}\right) \tag{7}$$

$$\frac{\partial^4 \varepsilon_{yy}}{\partial z^2 \partial x^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x}\right) \tag{8}$$

$$\frac{\partial^4 \varepsilon_{zz}}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y}\right). \tag{9}$$

Next, add Eqs. (7) and (8) and compare with what you obtain after differentiating Eq. (1) w.r.t z twice. *This comparison shows that Eqs. (7), (8), and (9) are really the three independent equations.*

10. Show that if the rotation is zero throughout a body (irrotational deformation), the displacement vector is the gradient of a scalar potential function. *Hint*: Use the idea from irrotational fluid flow.

11. This problem will involve two important results related to principal strains and principal directions

    (a) Using the property that the strain tensor is symmetric, show that eigenvectors corresponding to unequal eigenvalues are orthogonal to each other. *Hint:* Consider two different eigenvectors $n_j^{(1)}$ and $n_j^{(2)}$ corresponding to the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. Then write the eigenvalue problem equations for each case. Your aim is to show that $n_j^{(1)} n_j^{(2)} = 0$ when $\lambda^{(1)} \neq \lambda^{(2)}$; so multiply the equations appropriately to obtain this product, use the symmetry property, rearrange the indices, and proceed.

    (b) Show that the state of strain referred to a set of coordinate axes that are aligned along the principal directions is purely diagonal. In other words show that referred to these coordinate axes, the shear strains are zero.

12. In polar coordinates, the 2D-strains are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta}\right) + \frac{u_r}{r}, \quad \varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right).$$

    Using these relations, determine the two-dimensional strains for the following displacement fields:

    (a) $u_r = \frac{A}{r}$, $u_\theta = B\cos\theta$

    (b) $u_r = Ar^2$, $u_\theta = Br\sin\theta$

    (c) $u_r = A\sin\theta + B\cos\theta$, $u_\theta = A\cos\theta - B\sin\theta + Cr$

    where A, B, C are arbitrary constants.

[(a) $\boldsymbol{\varepsilon} = \begin{bmatrix} -\frac{A}{r^2} & -\frac{B}{2r}\cos(\theta) \\ -\frac{B}{2r}\cos(\theta) & \frac{1}{r^2}\left(A - Br\sin(\theta)\right) \end{bmatrix}$ (b) $\boldsymbol{\varepsilon} = \begin{bmatrix} 2Ar & 0 \\ 0 & Ar + B\cos(\theta) \end{bmatrix}$ (c) $\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$]

13. For the deformation field given by $x_1 = X_1$, $x_2 = X_2 + AX_3$, and $x_3 = X_3 + AX_2$, determine the displaced location of the material particles which originally lie along the plane circular surface given by $X_1 = 0$ and $X_2^2 + X_3^2 = \dfrac{1}{1 - A^2}$. Compare the equation for the displaced location with the general conic section equation and determine its nature. Plot the original circle and the displaced shape.

[$(1 + A^2)x_2^2 - 4Ax_2x_3 + (1 + A^2)x_3^2 = (1 - A^2)$. It is an ellipse.]

14. Consider an infinitesimal deformation, described by the displacement vector of the form, $u_i = A_{ij}X_j$. Here, the coefficients A_{ij} are so small that their products may be neglected in comparison to the coefficients themselves. Show that the total deformation obtained through two such successive infinitesimal deformations is equivalent to the sum of the two individual deformations. In the process you should also be able to show that the order of applying such deformations does not alter the final configuration.

15. In continuum mechanics theory, the deformation gradient is defined as $F_{ij} := \dfrac{\partial x_i}{\partial X_j}$. The Green-St. Venant strain tensor is defined as $\mathbf{E}_{\mathrm{G}} := \dfrac{1}{2}\left(\mathbf{F}^{\mathsf{T}} \cdot \mathbf{F} - \mathbf{I}\right)$. Show by expanding using indicial notation that the Green-St. Venant strain tensor, $\mathbf{E}_{\mathrm{G}}$ is the same as the finite strain tensor, $\mathbf{E}$ defined in the notes.

16. In a state of plane strain, we have $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$. If the displacement field is given by $u_1 = Ax_1 + 3x_2$, $u_2 = 3x_1 - Bx_2$, and $u_3 = 5$, show that the deformation is isochoric (no volume change) if $A = B$.

17. For the deformation given by $x_1 = X_1$, $x_2 = X_2$, and $x_3 = X_3 + 2X_2/\sqrt{3}$, determine the direction of a line element lying on the X_2X_3 plane along which the engineering strain is zero.

[Either $\hat{\mathbf{N}} = \pm\hat{\mathbf{e}}_2$ OR $\hat{\mathbf{N}} = \pm\hat{\mathbf{e}}_3$]