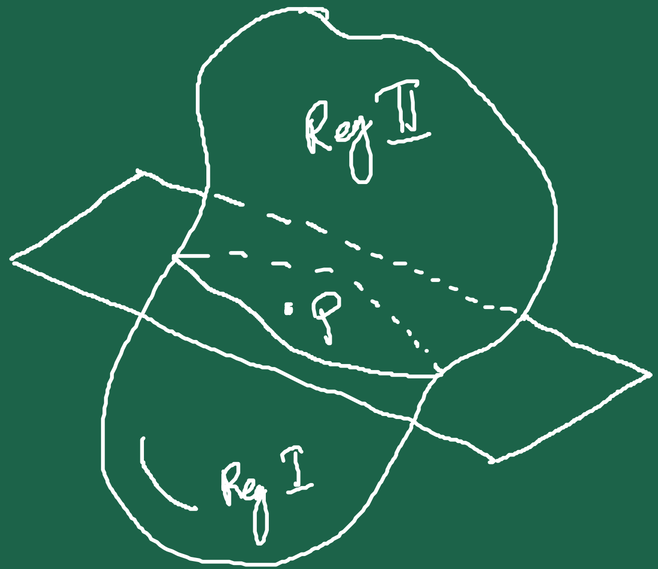
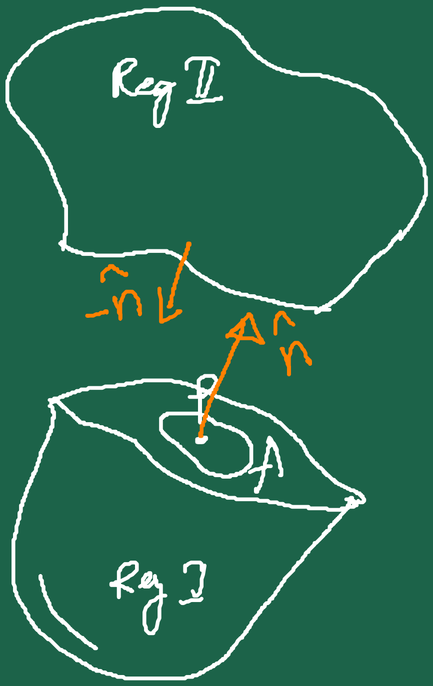
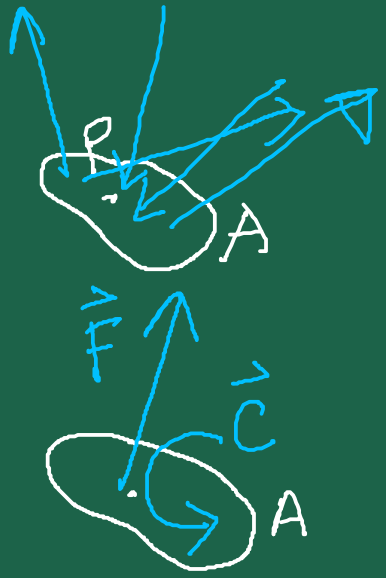
STRESS

TRACTION ACROSS A PLANE AT A POINT







$A \to 0$

$$\lim_{A \to 0} \frac{\vec{F}}{A} \to \text{finite value}$$

Considering Reg II

$$\vec{T}(\vec{x}, -\hat{n}) = -\vec{T}(\vec{x}, \hat{n})$$

$$\vec{T}(\vec{x}, \hat{n}) := \lim_{A \to 0} \frac{\vec{F}}{A}$$

↳ Traction vector OR Stress Vector

STRESS . . . CONTD.

STATE OF STRESS AT A POINT

Balance of linear momentum

$$\underbrace{\frac{D}{Dt}\int_V \rho\vec{v}\, dV} = \int_V \rho\vec{b}\, dV + \int_S \vec{T}\, dS$$

$$\frac{D}{Dt}\int_V \rho\vec{v}\, dV \xrightarrow{RTT} = \int_V \frac{\partial(\rho\vec{v})}{\partial t}\, dV + \int_S (\rho\vec{v})\, \vec{v}\cdot\hat{n}\, dS$$

$$= \int_V \left[\frac{\partial(\rho\vec{v})}{\partial t} + \nabla\cdot\left\{(\rho\vec{v})\otimes\vec{v}\right\}\right] dV$$

$$\frac{D}{Dt}\int_V (\rho\vec{v})\, dV = \int_V \left[\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla\cdot\left\{(\rho\vec{v})\otimes\vec{v}\right\}\right] dV$$

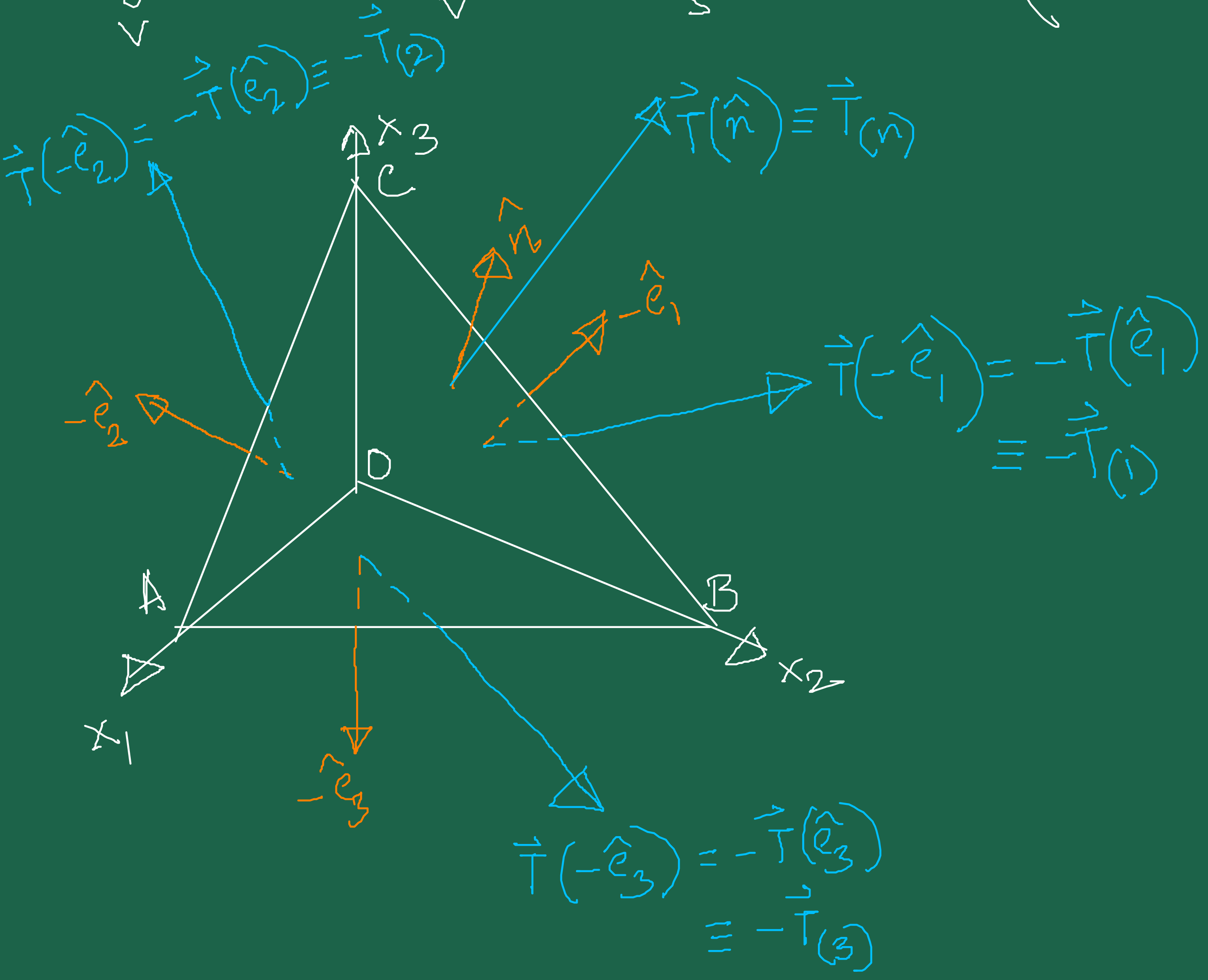
$$= \int_V \left[\rho\frac{\partial\vec{v}}{\partial t} + \frac{\partial\rho}{\partial t}\vec{v} + \left\{\nabla\cdot(\rho\vec{v})\right\}\vec{v} + \rho\vec{v}\cdot\nabla\vec{v}\right] dV$$

$$= \int_V \left[\vec{v}\left\{\underline{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})}\right\} + \rho\left\{\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right\}\right] dV$$

By conservation of mass: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$

$$\therefore \frac{D}{Dt}\int_V (\rho\vec{v})\, dV = \int_V \rho\left\{\frac{\partial\vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v}\right\} dV = \int_V \rho\frac{D\vec{v}}{Dt}\, dV$$

$$\int_V \rho \frac{D\vec{v}}{Dt}\, dV = \int_V \rho \vec{b}\, dV + \int_S \vec{T}\, dS \Rightarrow \int \rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) dV = \int_S \vec{T}\, dS$$

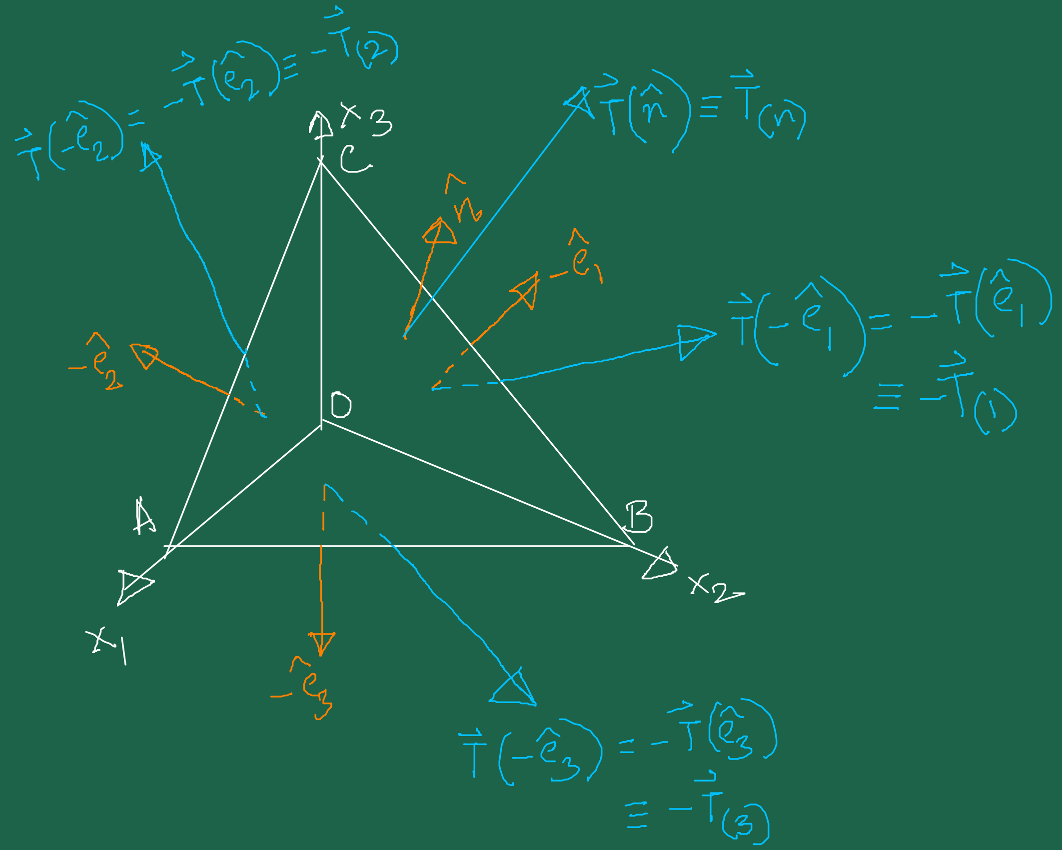


For applying the balance of lin. mom. eqn. to the tetrahedron, we will use the mean value theorem for integration

$$\int_{x_1}^{x_2} f(x)\, dx = f(x^*)(x_2 - x_1), \quad x^* \in [x_1, x_2]$$

$$\int_S f(\vec{r})\, dS = f(\vec{r}^*)\, \Delta S, \quad \vec{r}^* \in S$$

$$\int_V f(\vec{r})\, dV = f(\vec{r}^*)\, \Delta V, \quad \vec{r}^* \in V$$



$$\hat{n} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$$

$$\Delta OCB = (\Delta ABC)n_1$$

$$\Delta OAC = (\Delta ABC)n_2$$

$$\Delta OAB = (\Delta ABC)n_3$$

$$\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right)^{*}\frac{1}{3}h\,\Delta ABC = \vec{T}^{*}_{(n)}\Delta ABC + \left(-\vec{T}^{*}_{(1)}\right)\underbrace{\cancel{\Delta OCB}}_{(\Delta ABC)n_1} + \left(-\vec{T}^{*}_{(2)}\right)\underbrace{\cancel{\Delta OAC}}_{(\Delta ABC)n_2} + \left(-\vec{T}^{*}_{(3)}\right)\underbrace{\cancel{\Delta OAB}}_{(\Delta ABC)n_3}$$

Take $h \to 0$

$$0 = \vec{T}_{(n)} - \vec{T}_{(1)}n_1 - \vec{T}_{(2)}n_2 - \vec{T}_{(3)}n_3$$

(In the limit $h \to 0$, all the traction vectors are evaluated effectively at pt. O)

$$\therefore \vec{T}_{(n)} = \vec{T}_{(1)}n_1 + \vec{T}_{(2)}n_2 + \vec{T}_{(3)}n_3$$

$$\vec{T}_{(n)} = \vec{T}_{(1)} n_1 + \vec{T}_{(2)} n_2 + \vec{T}_{(3)} n_3$$

$$\left\{\begin{aligned} T_{(n)1} &= T_{(1)1} n_1 + T_{(2)1} n_2 + T_{(3)1} n_3 \\ T_{(n)2} &= T_{(1)2} n_1 + T_{(2)2} n_2 + T_{(3)2} n_3 \\ T_{(n)3} &= T_{(1)3} n_1 + T_{(2)3} n_2 + T_{(3)3} n_3 \end{aligned}\right.$$

$$\left.\begin{aligned} T_{(n)1} &= \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\ T_{(n)2} &= \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\ T_{(n)3} &= \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \end{aligned}\right\} \rightarrow T_{(n)i} = \sigma_{ji} n_j$$

$$\boxed{T_i = \sigma_{ji} n_j}$$

stress tensor $\rightarrow \underset{\approx}{\sigma} = \sigma_{ij} \rightarrow$ 2nd order tensor

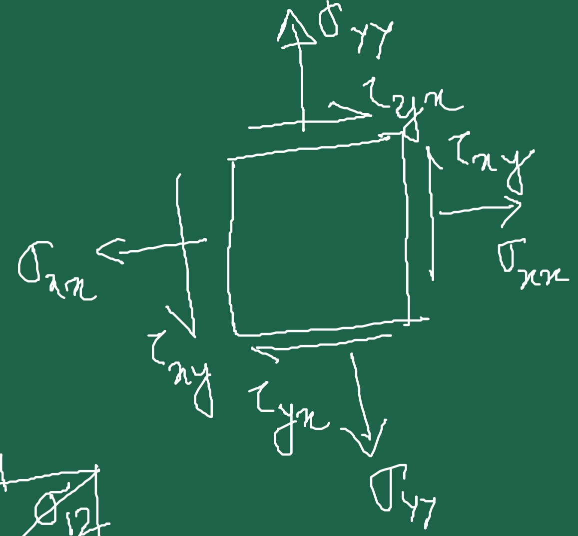
$$T_i = \sigma_{ji}\, n_j$$

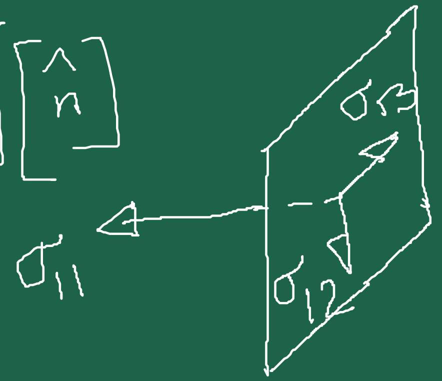
CAUCHY'S FORMULA OR LAW OR RELATION OR STRESS THEOREM

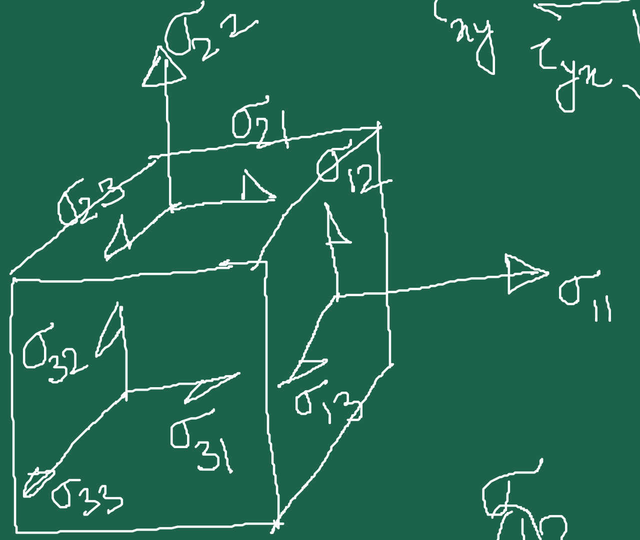
$$\vec{T} = \underset{\sim}{\sigma}^{T} \cdot \hat{n}$$

$$\underset{\sim}{\sigma} \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

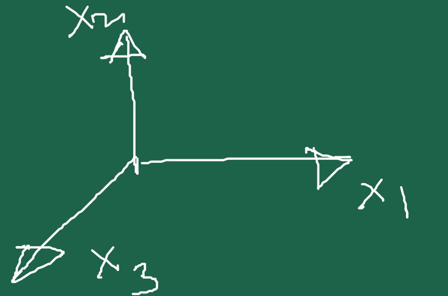
$$[\vec{T}] = [\underset{\sim}{\sigma}]^{T} [\hat{n}]$$







σ_{12}



STRESS ... CONTD.

Balance of angular momentum → when no body couples

↳ $\boxed{\underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T}$ → $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$

$$\vec{T} = \underset{\approx}{\sigma}^T \cdot \hat{n} = \underset{\approx}{\sigma} \cdot \hat{n}$$

Getting back to the balance of linear momentum eqn:

$$\int_V \rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) dV = \int_S \vec{T}\, dS = \int_S \underset{\approx}{\sigma} \cdot \hat{n}\, dS$$

$$\Rightarrow \int_V \rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) dV = \int_V \nabla \cdot \underset{\approx}{\sigma}\, dV \Rightarrow \int_V \left[\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) - \nabla \cdot \underset{\approx}{\sigma}\right] dV = 0$$

(Using Gauss divergence theorem)

$$\int_V \left[\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) - \nabla \cdot \underset{\sim}{\sigma} \right] dV = 0$$

Because this must be true for any arbitrary volume,

we must have:

$$\rho\left(\frac{D\vec{v}}{Dt} - \vec{b}\right) - \nabla \cdot \underset{\sim}{\sigma} = 0$$

$$\Rightarrow \boxed{\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \underset{\sim}{\sigma} + \rho \vec{b}} \rightarrow$$ Cauchy's eqn of motion

For static equilibrium: $\rho \dfrac{D\vec{v}}{Dt} = 0$

$$\therefore \quad \nabla \cdot \underset{\sim}{\sigma} + \rho \vec{b} = 0$$

$$\nabla \cdot \underset{\sim}{\sigma} + \rho \vec{b} = 0$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = 0$$

STRESS EQUILIBRIUM EQUATIONS

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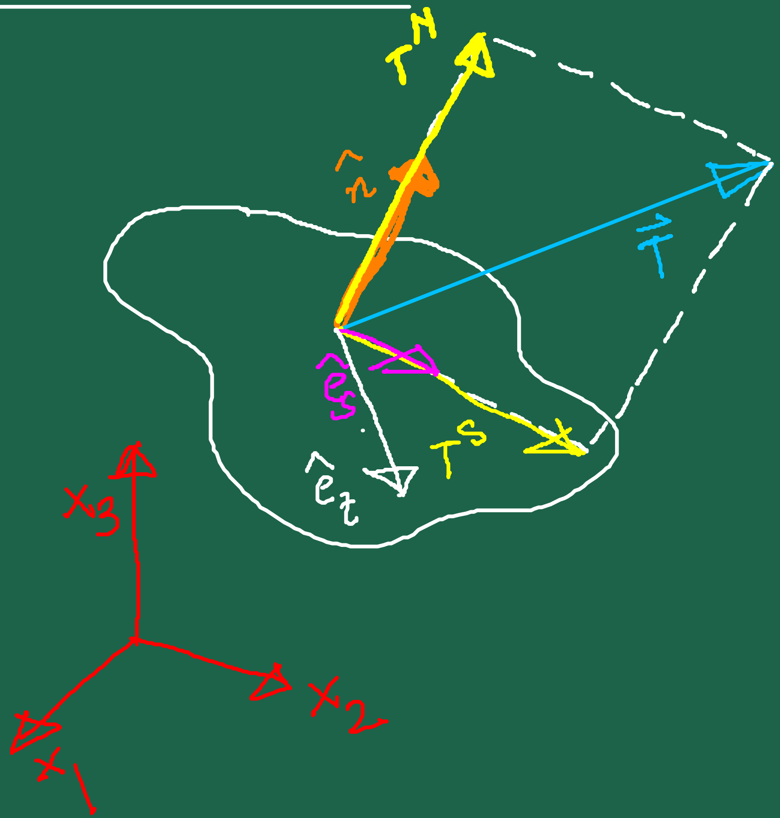
Normal and Shear Components of Traction

$T^N = \vec{T} \cdot \hat{n}$

↳ Normal component of $\vec{T}$



$T^S = \vec{T} \cdot \hat{e}_s$

$\hat{e}_s \overset{?}{\longleftrightarrow} \{\vec{T}, \hat{n}\}$

$$T^N = \vec{T} \cdot \hat{n}$$
$$= (\underset{\approx}{\sigma}^T \cdot \hat{n}) \cdot \hat{n}$$

$$T^N = [\vec{T} \cdot \hat{n}]$$
$$= [\vec{T}]^T [\hat{n}]$$
$$= [\underset{\approx}{\sigma}^T \cdot \hat{n}]^T [\hat{n}]$$
$$= ([\underset{\approx}{\sigma}^T][\hat{n}])^T [\hat{n}]$$
$$= [\hat{n}]^T [\underset{\approx}{\sigma}] [\hat{n}]$$

$$T^N = (\underset{\sim}{\sigma}^T \cdot \hat{n}) \cdot \hat{n} \equiv [\hat{n}]^T [\underset{\sim}{\sigma}] [\hat{n}]$$

$$(T^S)^2 = |\vec{T}|^2 - (T^N)^2$$

$$\vec{T} = T^N \hat{n} + T^S \hat{e}_s$$

$$T^S = [\vec{T} \cdot \hat{e}_s] = [\vec{T}]^T [\hat{e}_s]$$

$$= [\underset{\sim}{\sigma}^T \cdot \hat{n}]^T [\hat{e}_s]$$

$$= [\hat{n}]^T [\underset{\sim}{\sigma}] [\hat{e}_s]$$

$$T^t = [\vec{T} \cdot \hat{e}_t]$$

$$= [\hat{n}] [\underset{\sim}{\sigma}] [\hat{e}_t]$$

Suppose $\hat{n} \equiv \hat{e}_1 \equiv [\hat{e}_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow T^N = [\hat{e}_1]^T [\underset{\sim}{\sigma}] [\hat{e}_1] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$= \sigma_{11}$$

To interpret σ_{12} :

$$\hat{n} \equiv \hat{e}_1 \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{e}_t \equiv \hat{e}_2 \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T^t = [\hat{n}]^T [\underset{\sim}{\sigma}] [\hat{e}_t]$$

$$= [\hat{e}_1]^T [\underset{\sim}{\sigma}] [\hat{e}_2]$$

$$= \sigma_{12}$$

STRESS ... CONTD.

Principal Stresses

$$\hat{n} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$$

For T^N to be maximum, we must have $\vec{T} \parallel \hat{n}$

$$\vec{T} = T^N \hat{n} = \sigma \hat{n}$$

$$\Rightarrow \vec{T} = (T^N n_1)\hat{e}_1 + (T^N n_2)\hat{e}_2 + (T^N n_3)\hat{e}_3$$

$$T_1 = \sigma n_1 \quad - (\#_1)$$

$$T_2 = \sigma n_2 \quad - (\#_2)$$

$$T_3 = \sigma n_3 \quad - (\#_3)$$

However, $\vec{T} = \underset{\sim}{\sigma} \cdot \hat{n}$

$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \quad - (*_1)$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \quad - (*_2)$$

$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3 \quad - (*_3)$$

$(*) - (\#)$

$$\Rightarrow \begin{cases} 0 = (\sigma_{11} - \sigma) n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ 0 = \sigma_{12} n_1 + (\sigma_{22} - \sigma) n_2 + \sigma_{23} n_3 \\ 0 = \sigma_{13} n_1 + \sigma_{23} n_2 + (\sigma_{33} - \sigma) n_3 \end{cases}$$

$$\begin{bmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For non-trivial solutions

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma \end{vmatrix} = 0$$

$$\Rightarrow \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{33} & \sigma_{13} \\ \sigma_{13} & \sigma_{11} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

I_1, I_2, I_3 are <u>stress invariants</u>.

Stress ... contd.

Principal Stresses ... contd.

Fact 1: Eigenvalues of a real, symmetric matrix must be real. Since the stress components are real, and the stress matrix is symmetric, therefore the principal stresses must be real.

Fact 2: Eigenvectors corresponding to distinct eigenvalues must be perpendicular to each other. So, if the values of the principal stresses are distinct, then the principal directions are perpendicular to each other.

State of stress referred to principal directions

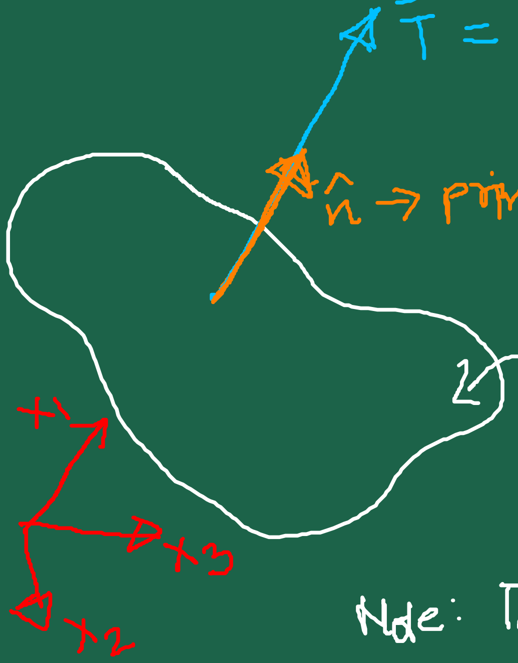
Here, we orient our coordinate axes along the principal directions.

$$\begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix}$$

$$\vec{T} = T^N \hat{n} \equiv \sigma \hat{n}$$

$$T^T = T^S = 0$$

$\hat{n} \rightarrow$ principal dir^n



Principal Plane ($\parallel x_2 x_3$)

Note: The x_2x_3 plane here will be $\parallel$ to the principal plane

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \rightarrow \text{Roots}: \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}$$

Stress ... contd.

Octahedral Stress

We first consider a situation where the coordinate axes are oriented along the principal directions.

Octahedral plane

Consider a plane which is equally inclined to the coordinate axes (or, equivalently to the principal directions).

$$\hat{n} = n_1\hat{e}_1 + n_2\hat{e}_2 + n_3\hat{e}_3$$

$$|n_1| = |n_2| = |n_3| \qquad \text{But } n_1^2 + n_2^2 + n_3^2 = 1$$

$$\therefore \ |n_1| = |n_2| = |n_3| = \frac{1}{\sqrt{3}}$$

Octahedral Normal Stress

$$T^N = [\hat{n}]^T [\underset{\sim}{\sigma}] [\hat{n}]$$

$$= \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= \sigma^{(1)} n_1^2 + \sigma^{(2)} n_2^2 + \sigma^{(3)} n_3^2$$

$$= \frac{1}{3}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right) \quad \left(\because |n_1| = |n_2| = |n_3| = \frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{3} I_1 \rightarrow \text{Extremely important}$$

Octahedral Shear Stress

$$\left(T^S\right)^2 = \left|\vec{T}\right|^2 - \left(T^N\right)^2$$

$$\left[\vec{T}\right] = \left[\underset{=}{\sigma}^T \cdot \hat{n}\right] = \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^{(1)} n_1 \\ \sigma^{(2)} n_2 \\ \sigma^{(3)} n_3 \end{bmatrix}$$

$$T_1 = \sigma^{(1)} n_1 , \quad T_2 = \sigma^{(2)} n_2 , \quad T_3 = \sigma^{(3)} n_3$$

$$(T^S)^2 = |\vec{T}|^2 - (T^N)^2$$

$$= T_1^2 + T_2^2 + T_3^2 - \left\{\frac{1}{3}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right)\right\}^2$$

$$= \left(\sigma^{(1)} n_1\right)^2 + \left(\sigma^{(2)} n_2\right)^2 + \left(\sigma^{(3)} n_3\right)^2 - \frac{1}{9}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right)^2$$

$$= \frac{1}{3}\left\{\left(\sigma^{(1)}\right)^2 + \left(\sigma^{(2)}\right)^2 + \left(\sigma^{(3)}\right)^2\right\} - \frac{1}{9}\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right)^2$$

$$= \frac{1}{9}\Big[3\left(\sigma^{(1)}\right)^2 + 3\left(\sigma^{(2)}\right)^2 + 3\left(\sigma^{(3)}\right)^2 - \left(\sigma^{(1)}\right)^2 - \left(\sigma^{(2)}\right)^2 - \left(\sigma^{(3)}\right)^2$$
$$- 2\sigma^{(1)}\sigma^{(2)} - 2\sigma^{(2)}\sigma^{(3)} - 2\sigma^{(3)}\sigma^{(1)}\Big]$$

$$= \frac{1}{9}\Big[2\left(\sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)}\right)^2 - 6\sigma^{(1)}\sigma^{(2)} - 6\sigma^{(2)}\sigma^{(3)} - 6\sigma^{(3)}\sigma^{(1)}\Big]$$

$$(T^s)^2 = \frac{1}{9}\left[2 I_1^2 - 6 I_2\right]$$

$$|T^s| = \frac{\sqrt{2}}{3}\left(I_1^2 - 3 I_2\right)^{1/2}$$

Both octahedral normal and shear stresses are expressed in terms of stress invariants.

Stress ... contd.

Mean and Deviatoric Stress

$$\underset{\sim}{\sigma} \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix}}_{\text{Mean Stress}} + \underbrace{\begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\sigma_m \end{bmatrix}}_{\text{Deviatoric Stress}}$$

where $\sigma_m = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) = \frac{1}{3} I_1$

1st Stress Invariant