Stress[*][†]

1 Traction across a plane at a point

Consider an area A in a given plane and containing a point P within a body as shown in Fig. 1. Suppose the plane divides the body into two regions, Region I and Region II. Consider Region I. Draw a normal $\hat{\mathbf{n}}$ to the plane at P and pointing from Region I towards Region II. Over the area A, Region II exerts forces on Region I. Suppose this system of forces is statically equivalent to a force $\mathbf{F}$ acting at P in a definite direction and a couple $\mathbf{C}$ about a definite axis. Let us make the area A small, ensuring that the point P is always inside it. Then the force $\mathbf{F}$ and the couple $\mathbf{C}$ tend to zero limits and the direction of $\mathbf{F}$ tends to a limiting direction. We assume that as A tends to zero, the number $|\mathbf{F}|/A$ tends to a non-zero limit while $|\mathbf{C}|/A$ tends to 0 (which is sensible because smaller the area, the smaller will be the distance from the definite axis referred to earlier leading to a couple that vanishes).

We define a vector

$$\mathbf{T} = \lim_{A \to 0} \frac{\mathbf{F}}{A} \tag{1}$$

called the *stress vector* or the *traction vector*.[‡]

Note that the traction vector depends on the location of P as well as the choice of the plane on which A is located. Since the orientation of this plane is given by $\hat{\mathbf{n}}$, so $\mathbf{T}$ depends the position vector of P *and* $\hat{\mathbf{n}}$. If $\hat{\mathbf{n}}$ is different, $\mathbf{T}$ will be different.

*Notes prepared by Jeevanjyoti Chakraborty. Contact: jeevan@mech.iitkgp.ac.in

†The Dual Degree students who were in Section 1 of Mechanics of Solids in Autumn, 2019 will find a lot common in these notes. However, unlike the previous chapter, there a number of new things!

‡The way in which most modern mechanicians present the concepts of traction and stress can be traced back to the way it was presented in the classic, "A Treatise on the Mathematical Theory of Elasticity" by A. E. H. Love. My way is no exception.

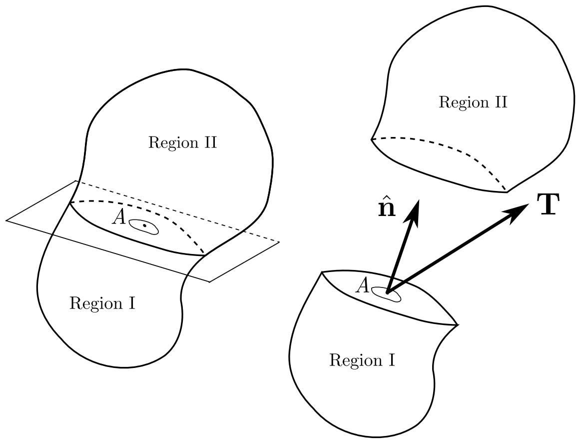

Figure 1: Traction

Note also that just as Region II exerts forces on Region I, so also Region I exerts forces on Region II over the area; these forces must necessarily be equal in magnitude and opposite in direction from Newton's third law (important: it is not necessary for the whole body to be in equilibrium for these forces to be equal and opposite). Thus, in a fashion identical to what was discussed previously, a stress vector or traction vector can be defined on the plane and considering the unit vector, $-\hat{\mathbf{n}}$. We then have

$$\mathbf{T}\left(\mathbf{x}, \hat{\mathbf{n}}\right) = -\mathbf{T}\left(\mathbf{x}, -\hat{\mathbf{n}}\right), \tag{2}$$

where it is important to note that the position vector $\mathbf{x}$ is the same for both traction vectors because we are considering the same point P just from two different sides.

2 Surface tractions

The nature of the action between two bodies in contact is assumed to be of the same nature as the action between two portions of the same body separated by an imaginary surface. If the point P in the previous discussion is moved to a point P' on the bounding surface of the body with the position vector $\mathbf{x}$ changing to $\mathbf{x}'$ and $\hat{\mathbf{n}}$ changing to $\hat{\mathbf{n}}'$ that coincides with the unit outward normal at P', the resulting traction vector $\mathbf{T}'(\mathbf{x}', \hat{\mathbf{n}}')$ is referred to as the *surface traction.*

Very important: Whether it is the traction across an imaginary plane inside a body or the surface traction which acts at the actual bounding surface of a body, the direction of the traction vector does not, in general, coincide with that of $\hat{\mathbf{n}}$.

The traction vector can be decomposed into a component normal to the plane (defined by $\hat{\mathbf{n}}$) and a component parallel to the plane.

3 Connection between traction vector and stress

We are going to establish the connection between traction vector and stress. In order to do that we proceed via the balance of linear momentum wherein the balance of mass will be embedded also.

3.1 Balance of Linear Momentum

Consider the balance of linear momentum applied to a general material volume element in integral form

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_{\mathrm{m}}(t)} \rho \mathbf{v}\ \mathrm{d}V = \int_{V_{\mathrm{m}}(t)} \rho \mathbf{b}\ \mathrm{d}V + \int_{S_{\mathrm{m}}(t)} \mathbf{T}\ \mathrm{d}S, \tag{3}$$

where $V_{\mathrm{m}}(t)$ is the domain contained in the material volume element, S_{m} is the bounding surface, ρ is the density of the material, $\mathbf{v}$ is the velocity, $\mathbf{b}$ is the body force per unit mass acting at a generic point within the volume element, and $\mathbf{T}$ is the traction acting at a generic point on the bounding surface.

Using the Reynolds' transport theorem on the l.h.s. we have:

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\mathrm{m}(t)} \rho\mathbf{v}\,\mathrm{d}V &= \int_{V_\mathrm{m}(t)} \frac{\partial(\rho\mathbf{v})}{\partial t}\,\mathrm{d}V + \int_{S_\mathrm{m}(t)} (\rho\mathbf{v})\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S, \\
&= \int_{V_\mathrm{m}(t)} \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\{(\rho\mathbf{v})\otimes\mathbf{v}\}\right]\mathrm{d}V \qquad (4)
\end{aligned}
$$

(Using Gauss' divergence theorem)

$$
= \int_{V_\mathrm{m}(t)} \left[\mathbf{v}\left\{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right\} + \rho\left\{\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right\}\right]\mathrm{d}V \qquad (5)
$$

Now, note that from the balance (or, conservation) of mass, we have the following continuity equation:

$$
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0. \qquad (6)
$$

It is very important to note that while we first came across the continuity equation in fluid mechanics, but this equation is not restricted to the scope of fluid mechanics only. It is equally applicable to solid mechanics; in general it is applicable to all of continuum mechanics. Using Eq. (6) in Eq. (5), we have

$$
\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\mathrm{m}(t)} \rho\mathbf{v}\,\mathrm{d}V = \int_{V_\mathrm{m}(t)} \rho\left\{\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right\}\mathrm{d}V
$$

We further note that the combination of terms, $\dfrac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}$ is nothing but the material time derivative of the velocity, $\dfrac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}$, which is nothing but the acceleration itself. So, we have:

$$
\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_\mathrm{m}(t)} \rho\mathbf{v}\,\mathrm{d}V = \int_{V_\mathrm{m}(t)} \rho\left\{\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right\}\mathrm{d}V \qquad (7)
$$

Substituting the above in Eq. (3), we have

$$
\int_{V_\mathrm{m}(t)} \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}V = \int_{V_\mathrm{m}(t)} \rho\mathbf{b}\,\mathrm{d}V + \int_{S_\mathrm{m}(t)} \mathbf{T}\,\mathrm{d}S. \qquad (8)
$$

In order to establish the connection between the traction vector and stress, we are going to apply this form of the balance of linear momentum equation to a special kind of material volume shaped in the form of a tetrahedron. Considering this special geometrical shape helps us to take advantage of certain geometrical features as well to establish the desired connection; this advantage will come to light in the subsequent discussions.

3.2 Balance of linear momentum applied to a tetrahedron

Consider the equilibrium of a tetrahedron having one vertex at the origin O with the three edges meeting at this vertex to be oriented along the coordinate axes. Now, the application of the balance of linear momentum in integral form, Eq. (8) to this general tetrahedron may seem rather daunting. However, we are going to simplify this application by using the *mean value theorem of integration.* Let us state this theorem without using too much mathematical jargon as follows:

> ***Mean value theorem of integration:*** *The value of the integral of a continuous function over a certain domain is equal to the value of the integrand somewhere in the domain multiplied by the size of the domain.*

So, practically speaking, what is this theorem telling us?

First consider a 1D domain L: Let X be a generic point in the domain; suppose L is bounded by $X = a$ and $X = b$. Then, for a function, $f(X)$, the mean value theorem of integration tells us that

$$\int_a^b f(X)\, \mathrm{d}x = f(X^*)\Delta L,$$

where $a \leq X^* \leq b$ and $\Delta L = (b-a)$ is, of course, the size of the 1D domain, i.e. its length.

Similarly, consider a 2D domain S: Let $\mathbf{X}$ having components (X_1, X_2) be a generic point in the domain. Then, for a function, $f(\mathbf{X})$, the mean value theorem of integration tells us that

$$\int_S f(\mathbf{X})\ \mathrm{d}S = f(\mathbf{X}^*)\Delta S,$$

where $\mathbf{X}^*$ is some point in S, and ΔS is the size of the 2D domain, i.e. its area.

Finally, consider a 3D domain V: Let $\mathbf{X}$ having components (X_1, X_2, X_3) be a generic point in the domain. Then, for a function, $f(\mathbf{X})$, the mean value theorem of integration tells us that

$$\int_V f(\mathbf{X})\ \mathrm{d}V = f(\mathbf{X}^*)\Delta V,$$

where $\mathbf{X}^*$ is some point in V, and ΔV is the size of the 3D domain, i.e. its volume.

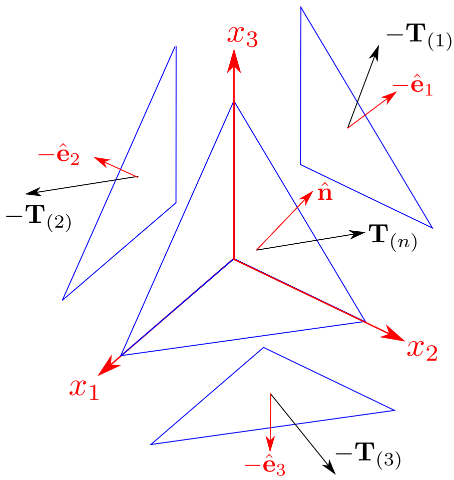

Figure 2: Stress equilibrium

In order to apply the balance of linear momentum, Eq. (8), to the tetrahedron, we are going to need the mean value theorem of integration for 2D and 3D domains. Let us first rewrite Eq. (8) by collecting the volume integrals together. Thus, we have:

$$\int_{V_{\mathrm{m}}(t)} \rho \left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \mathbf{b} \right)\ \mathrm{d}V = \int_{S_{\mathrm{m}}(t)} \mathbf{T}\ \mathrm{d}S. \tag{9}$$

The mean value theorem of integration for 3D domain can be directly applied over the entire volume of the tetrahedron OABC on the l.h.s. For the r.h.s, however, the mean value theorem of integration for 2D domain needs to be applied *piecewise* over the four triangles, ABC, OCB, OAC, and OAB.

We thus obtain:

$$\begin{aligned}\left[\rho\left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}-\mathbf{b}\right)\right]^{P^*_{OABC}}\Delta V_{OABC} = \mathbf{T}(P^*_{ABC},\hat{\mathbf{n}})\Delta ABC \\ + \mathbf{T}(P^*_{OCB},-\hat{\mathbf{e}}_1)\Delta OCB \\ + \mathbf{T}(P^*_{OAC},-\hat{\mathbf{e}}_2)\Delta OAC \\ + \mathbf{T}(P^*_{OAB},-\hat{\mathbf{e}}_3)\Delta OAB,\end{aligned} \tag{10}$$

where P^*_{OABC}, P^*_{ABC}, P^*_{OCB}, P^*_{OAC}, and P^*_{OAB} are some *specific points*, in the tetrehedron OABC, in the triangle OCB, in the triangle OAC, and in the triangle OAB, respectively. Further, the unit outward normal to triangle ABC is $\hat{\mathbf{n}}$, while the unit outward normals to the triangles OCB, OAB, and OAB are $(-\hat{\mathbf{e}}_1)$, $(-\hat{\mathbf{e}}_2)$, and $(-\hat{\mathbf{e}}_3)$, respectively. Note that $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are the unit vectors along the *positive* X_1, X_2, and X_3 axes, respectively.

Referring to Eq. (2), we have the following:

$$\mathbf{T}(P^*_{OCB},-\hat{\mathbf{e}}_1) = -\mathbf{T}(P^*_{OCB},\hat{\mathbf{e}}_1), \tag{11a}$$
$$\mathbf{T}(P^*_{OAC},-\hat{\mathbf{e}}_2) = -\mathbf{T}(P^*_{OAC},\hat{\mathbf{e}}_2), \tag{11b}$$
$$\mathbf{T}(P^*_{OAB},-\hat{\mathbf{e}}_3) = -\mathbf{T}(P^*_{OAB},\hat{\mathbf{e}}_3), \tag{11c}$$

We note that the volume of the tetrahedron OABC is:

$$\Delta V_{OABC} = \frac{1}{3}h\Delta ABC, \tag{12}$$

where h is the "height" of the tetrahedron, i.e. the length of the perpendicular from point O to the triangle ABC.

Furthermore, the triangles OCB, OAC, and OAB can be viewed as the projection of the triangle ABC on to the plane X_2X_3, the plane X_3X_1, and the plane X_1X_2, respectively. Then we note that

$$\Delta OCB = (\Delta ABC)n_1, \tag{13a}$$
$$\Delta OAC = (\Delta ABC)n_2, \tag{13b}$$
$$\Delta OAB = (\Delta ABC)n_3, \tag{13c}$$

where n_1, n_2, and n_3 are the components of the unit outward $\hat{\mathbf{n}}$ to the triangle ABC, i.e. $\hat{\mathbf{n}} = n_1\hat{\mathbf{e}}_1 + n_2\hat{\mathbf{e}}_2 + n_3\hat{\mathbf{e}}_3$. The above three relations can be readily obtained by first considering the area, ΔABC as a vector directed along $\hat{\mathbf{n}}$ and then taking the dot product with $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, respectively.

Substituting equations (11), (12) and (13) in Eq. (10), we obtain

$$\begin{aligned}\left[\rho\left(\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \mathbf{b}\right)\right](P^*_{OABC}, \hat{\mathbf{n}})\frac{1}{3}h\Delta ABC = \mathbf{T}(P^*_{ABC}, \hat{\mathbf{n}})\Delta ABC \\ - \mathbf{T}(P^*_{OCB}, \hat{\mathbf{e}}_1)(\Delta ABC)n_1 \\ - \mathbf{T}(P^*_{OAC}, \hat{\mathbf{e}}_2)(\Delta ABC)n_2 \\ - \mathbf{T}(P^*_{OAB}, \hat{\mathbf{e}}_3)(\Delta ABC)n_3.\end{aligned} \tag{14}$$

We cancel out the ΔABC from the l.h.s and the r.h.s, and then take the limit $h \longrightarrow 0$. In the process of taking this limit, the triangle ABC moves increasingly closer to the origin O. And, in the limiting condition, the different points P^*_{OABC}, P^*_{ABC}, P^*_{OCB}, P^*_{OAC}, and P^*_{OAB} "converge" to the same point O. Very importantly, in this limiting condition, the l.h.s vanishes (because of the presence of h) whereas the r.h.s does not. Therefore, we end up with the following equation:

$$0 = \mathbf{T}(O, \hat{\mathbf{n}}) - \mathbf{T}(O, \hat{\mathbf{e}}_1)n_1 - \mathbf{T}(O, \hat{\mathbf{e}}_2)n_2 - \mathbf{T}(O, \hat{\mathbf{e}}_3)n_3 \tag{15}$$

We drop the repeated dependence on O and use a shortened form as follows:

$$\begin{aligned}\mathbf{T}(O, \hat{\mathbf{n}}) &\equiv \mathbf{T}_{(n)}, \\ \mathbf{T}(O, \hat{\mathbf{e}}_1) &\equiv \mathbf{T}_{(1)}, \\ \mathbf{T}(O, \hat{\mathbf{e}}_2) &\equiv \mathbf{T}_{(2)}, \\ \mathbf{T}(O, \hat{\mathbf{e}}_3) &\equiv \mathbf{T}_{(3)}.\end{aligned}$$

Then, from Eq. (15), we obtain:

$$\mathbf{T}_{(n)} = \mathbf{T}_{(1)} n_1 + \mathbf{T}_{(2)} n_2 + \mathbf{T}_{(3)} n_3. \tag{16}$$

This equation is a vector equation, and in terms of the traction vector components along the three coordinate axis directions, we have the following:

$$T_{(n)1} = T_{(1)1} n_1 + T_{(2)1} n_2 + T_{(3)1} n_3, \tag{17a}$$
$$T_{(n)2} = T_{(1)2} n_1 + T_{(2)2} n_2 + T_{(3)2} n_3, \tag{17b}$$
$$T_{(n)3} = T_{(1)3} n_1 + T_{(2)3} n_2 + T_{(3)3} n_3 \tag{17c}$$

Each of the nine traction vector components in the r.h.s of the three preceding equations are what are referred to as the stress components. Our original objective of establishing the connection between the traction vector and stress is thus achieved.

In index notation, the preceding three equations can be written as

$$T_{(n)i} = T_{(j)i} n_j, \tag{18}$$

where $T_{(j)i}$ represents the component of the traction vector $T_{(j)}$ along the i-th direction, and is denoted, alternatively, as σ_{ji}. Thus, we have

$$T_i = \sigma_{ji} n_j, \tag{19}$$

where we have dropped the explicit denotion of n. We identify this relation as a dot product (because one of the indices is repeated) and so rewrite the equivalent vector (or, compact) notation as

$$\mathbf{T} = \boldsymbol{\sigma}^{\mathsf{T}} \cdot \hat{\mathbf{n}}, \tag{20}$$

or, the equivalent matrix representation as

$$[\mathbf{T}] = [\boldsymbol{\sigma}]^{\mathsf{T}} [\hat{\mathbf{n}}], \tag{21}$$

where the 'T' in the superscript refers to the transpose.

VERY IMPORTANT: We state (without proving) that conservation of angular momentum in the absence of body couples leads to the conclusion that the stress tensor is symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$.

In expanded form, we have from Eq. (21)

$$T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3, \tag{22a}$$

$$T_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3, \tag{22b}$$

$$T_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3. \tag{22c}$$

Eqs (19), (20), (21), and (22) are different forms of what are referred to as the Cauchy's formula (sometimes Cauchy's stress theorem or Cauchy's law).

4 Cauchy's equation of motion and mechanical equilibrium equations

Going back to Eq. (8), we have

$$\begin{aligned} \int_{V_\mathrm{m}} \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\, \mathrm{d}V &= \int_{V_\mathrm{m}} \rho \mathbf{b}\, \mathrm{d}V + \int_{S_\mathrm{m}} \mathbf{T}\, \mathrm{d}S \\ &= \int_{V_\mathrm{m}} \rho \mathbf{b}\, \mathrm{d}V + \int_{S_\mathrm{m}} \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \\ &= \int_{V_\mathrm{m}} \rho \mathbf{b}\, \mathrm{d}V + \int_{V_\mathrm{m}} \nabla \cdot \boldsymbol{\sigma}\, \mathrm{dV} \quad \text{(Using divergence theorem)} \end{aligned} \tag{23}$$

Thus we can write

$$\int_{V_\mathrm{m}} \left(\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho \mathbf{b} - \nabla \cdot \boldsymbol{\sigma} \right) \mathrm{d}V = 0, \tag{24}$$

or, using the arbitrariness of the material volume we have

$$\boxed{\rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho \mathbf{b} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}.} \tag{25}$$

If the body is in equilibrium, we have

$$\boxed{\rho \mathbf{b} + \nabla \cdot \boldsymbol{\sigma} = 0.} \tag{26}$$

Referring to a rectangular Cartesian coordinate system, Eq. (26) can be expressed in component form as

$$\boxed{\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 &= 0, && \text{(27a)}\\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 &= 0, && \text{(27b)}\\
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 &= 0. && \text{(27c)}
\end{aligned}}$$

VERY IMPORTANT: Eq. (26) or, equivalently, the three equations collectively in in Eqs (27) is known as the mechanical equilibrium equations.

5 Normal and shear components of traction

It has already been pointed out that, in general, the traction vector $\mathbf{T}$ acting at a point in a plane with unit normal $\hat{\mathbf{n}}$ is not parallel to $\hat{\mathbf{n}}$. So, it is possible to resolve $\mathbf{T}$ into components parallel and perpendicular to $\hat{\mathbf{n}}$.

We denote the component parallel to $\hat{\mathbf{n}}$ as T^N and call it the normal component. We have

$$T^N = \mathbf{T} \cdot \hat{\mathbf{n}} = (\boldsymbol{\sigma}^\mathsf{T} \cdot \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}, \tag{28}$$

or, in indical notation

$$T^N = T_i n_i = \sigma_{ji} n_j n_i, \tag{29}$$

or, in matrix representation

$$T^N = \left([\sigma]^\mathsf{T}[\hat{\mathbf{n}}]\right)^\mathsf{T}[\hat{\mathbf{n}}] \equiv [\hat{\mathbf{n}}]^\mathsf{T}[\boldsymbol{\sigma}][\hat{\mathbf{n}}] \tag{30}$$

or, in expanded form

$$T^N = \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 + 2\sigma_{12} n_1 n_2 + 2\sigma_{23} n_2 n_3 + 2\sigma_{13} n_1 n_3. \tag{31}$$

Note that since T^N is the component of the traction $\mathbf{T}$ along $\hat{\mathbf{n}}$, T^N may be equivalently denoted by σ_{nn}, i.e. $T^N \equiv \sigma_{nn}$.

Note also that the expressions for T^N (or, σ_{nn}) are exactly like the ones we had found, in the previous chapter, for the engineering strain along a particular direction.

Likewise, we denote the component of $\mathbf{T}$ perpendicular to $\hat{\mathbf{n}}$ and lying in the same plane as $\mathbf{T}$ and $\hat{\mathbf{n}}$ as T^S and call it the shear component. We have

$$\left(T^S\right)^2 = |\mathbf{T}|^2 - \left(T^N\right)^2 \tag{32}$$

$$\text{or,} \quad \left(T^S\right)^2 = \left(T_1^2 + T_2^2 + T_3^2\right) - \left(T^N\right)^2. \tag{33}$$

Substituting the expressions for T_1, T_2, and T_3 from Eqs (22) and the expression for T^N from Eq. (31), we can obtain T^S.

When we use Eq. (32), we are implicitly saying that T^N and T^S are in the same plane contained by $\mathbf{T}$ and $\hat{\mathbf{n}}$. So another way of finding T^S would be by taking the dot product $\mathbf{T} \cdot \hat{\mathbf{e}}_s$, where $\hat{\mathbf{e}}_s$ is perpendicular to $\hat{\mathbf{n}}$ and is contained in the plane formed by $\mathbf{T}$ and $\hat{\mathbf{n}}$.

Let us now try to find an expression for $\hat{\mathbf{e}}_s$. The unit vector that is perpendicular to both $\mathbf{T}$ and $\hat{\mathbf{n}}$ is $\dfrac{\mathbf{T} \times \hat{\mathbf{n}}}{|\mathbf{T} \times \hat{\mathbf{n}}|}$. Now, $\hat{\mathbf{e}}_s$ is the unit vector that should be perpendicular to both this newly found unit vector and $\hat{\mathbf{n}}$, so that $\hat{\mathbf{e}}_s = \hat{\mathbf{n}} \times \left(\dfrac{\mathbf{T} \times \hat{\mathbf{n}}}{|\mathbf{T} \times \hat{\mathbf{n}}|}\right)$.

In terms of unit vectors, therefore, we have the following:

$$\mathbf{T} = T^N \hat{\mathbf{n}} + T^S \hat{\mathbf{e}}_s. \tag{34}$$

Just as we had used σ_{nn} to denote the component of $\mathbf{T}$ along $\hat{\mathbf{n}}$, we can use σ_{ns} to denote the component of $\mathbf{T}$ along $\hat{\mathbf{e}}_s$.

Now, consider another unit vector, say, $\hat{\mathbf{e}}_t$ that is lying in the plane perpendicular to $\hat{\mathbf{n}}$, has a common origin as $\hat{\mathbf{n}}$ and $\mathbf{T}$ but, unlike $\hat{\mathbf{e}}_s$, is not coplanar with $\mathbf{T}$ and $\hat{\mathbf{n}}$. If we take the dot product $\mathbf{T} \cdot \hat{\mathbf{e}}_t$, then the resulting component would also lie on the plane perpendicular to $\hat{\mathbf{n}}$. It is important to note that while $\hat{\mathbf{e}}_s$ is unique, we can have infinite such $\hat{\mathbf{e}}_t$. In fact, $\hat{\mathbf{e}}_s$ is a special case of $\hat{\mathbf{e}}_t$ distinguished by its requirement to be coplanar with $\mathbf{T}$ and $\hat{\mathbf{n}}$.

The component of $\mathbf{T}$ given by the dot product $\mathbf{T} \cdot \hat{\mathbf{e}}_s$ is also a shearing component of $\mathbf{T}$. However, $T^S = \mathbf{T} \cdot \hat{\mathbf{e}}_s$ is a special shearing component distinguished by its requirement to be coplanar with $\mathbf{T}$ and $\hat{\mathbf{n}}$.

Just as we had used σ_{nn} and σ_{ns} to denote the components of $\mathbf{T}$ along $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_s$, respectively, we use σ_{nt} to denote the component of $\mathbf{T}$ along $\hat{\mathbf{e}}_t$.

VERY IMPORTANT: In the previous chapter on Kinematics, the shear component of strain tensor was physically interpreted by referring to elemental line segments along two perpendicular directions. Similarly, here, the shearing components of $\mathbf{T}$ can be related to two perpendicular directions. We can say that $\sigma_{nt} = \mathbf{T} \cdot \hat{\mathbf{e}}_t$ is related $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_t$ while $\sigma_{ns} \equiv T^S = \mathbf{T} \cdot \hat{\mathbf{e}}_s$ is related to $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_s$. We have the following:

$$
\begin{aligned}
\sigma_{nt} &= [\mathbf{T} \cdot \hat{\mathbf{e}}_t], \\
&= [\mathbf{T}]^{\mathsf{T}} [\hat{\mathbf{e}}_t], \\
&= [\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}]^{\mathsf{T}} [\hat{\mathbf{e}}_t], \\
&= \left([\boldsymbol{\sigma}]^{\mathsf{T}} [\hat{\mathbf{n}}]\right)^{\mathsf{T}} [\hat{\mathbf{e}}_t], \\
&= [\hat{\mathbf{n}}]^{\mathsf{T}} [\boldsymbol{\sigma}] [\hat{\mathbf{e}}_t].
\end{aligned}
$$

Similarly, we have $\sigma_{ns} \equiv T^S = [\hat{\mathbf{n}}]^{\mathsf{T}} [\boldsymbol{\sigma}] [\hat{\mathbf{e}}_s]$.

6 Principal stress

In the previous chapter on Kinematics, after we had found an expression for the normal or engineering strain along a given direction (or, unit vector) in terms of a given strain tensor, we had set about the problem of finding the directions along which the normal strain was maximum – these strains being referred to as the principal strains. We are at a corresponding point in this chapter. We have in our hands the expression for the normal stress $\sigma_{nn} \equiv T^N$, and we set about the following problem:

Given a state of stress $\boldsymbol{\sigma}$ *referred to coordinate axes along the directions* $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, *and* $\hat{\mathbf{e}}_3$, which $\hat{\mathbf{n}}$ maximizes $\sigma_{nn} \equiv T^N$?

We could proceed exactly as in the previous chapter by using the method of Lagrange multiplier but we take up another method as follows:

We first note that for T^N to be maximum, $\mathbf{T}$ must be parallel to $\hat{\mathbf{n}}$.

Now, referred to $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, we have:

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_1 n_1 + \hat{\mathbf{e}}_2 n_2 + \hat{\mathbf{e}}_3 n_3.$$

Therefore, $\mathbf{T} \parallel \hat{\mathbf{n}}$ with its magnitude entirely contributed by T^N must be

$$\begin{aligned} &\mathbf{T} = T^N \hat{\mathbf{n}}, \\ \text{or,} \qquad &\mathbf{T} = \hat{\mathbf{e}}_1(T^N n_1) + \hat{\mathbf{e}}_2(T^N n_2) + \hat{\mathbf{e}}_3(T^N n_3). \end{aligned} \tag{35}$$

The plane defined by $\hat{\mathbf{n}}$ is the principal plane and T^N is the principal stress. Henceforth, we will denote T^N by σ. Using this new notation, Eq. (35) can be written in component form as

$$T_1 = \sigma n_1, \tag{36a}$$
$$T_2 = \sigma n_2, \tag{36b}$$
$$T_2 = \sigma n_3. \tag{36c}$$

Now, subtracting the three Cauchy's formula equations, Eq. (22)(a), (b), and (c) respectively, from Eqs. (36)(a), (b), and (c) gives us

$$(\sigma_{11} - \sigma)\, n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = 0, \tag{37a}$$
$$\sigma_{12} n_1 + (\sigma_{22} - \sigma)\, n_2 + \sigma_{32} n_3 = 0, \tag{37b}$$
$$\sigma_{13} n_1 + \sigma_{23} n_2 + (\sigma_{33} - \sigma)\, n_3 = 0. \tag{37c}$$

For non-trivial solutions of n_1, n_2, and n_3, we must have

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{vmatrix} = 0. \tag{38}$$

On expanding,

$$\begin{aligned} \sigma^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\, \sigma^2 + \left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2\right) \sigma \\ - \left(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2\right) = 0, \\ \text{or,} \qquad \sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0, \end{aligned} \tag{39}$$

where

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33}, && \text{(40a)}\\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2\\
&= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{33} & \sigma_{31} \\ \sigma_{31} & \sigma_{11} \end{vmatrix}, && \text{(40b)}\\
I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2,\\
&= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}. && \text{(40c)}
\end{aligned}
$$

Here, I_1, I_2, and I_3 are stress invariants. These are not the only stress invariants. Other invariants can be formed from them. For instance, $2I_1^2 - 6I_2$ is another stress invariant.

There are a couple of important facts associated with principal stresses and principal directions that follow from general theory of eigenvalues (covered in Mathematics II in First Year):

(i) Eigenvalues of a real, symmetric matrix are real. The stress matrix is real and symmetric. So the principal stresses are always real (as one would, of course, expect!)

(ii) The eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. So if the principal stress values are all different, then the principal directions are mutually perpendicular to each other.

7 State of stress referred to principal directions

We can choose to orient the coordinate axes along three mutually perpendicular principal directions. In that case, the state of stress shapes up (in matrix representation) as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix}, \tag{41}$$

where $\sigma^{(1)}$, $\sigma^{(2)}$, $\sigma^{(3)}$ are the three principal stresses. The fact that the stress matrix referred to axes that are directed along the principal directions must be diagonal is embedded in the definition of the principal stress itself. The principal stresses are defined to be the normal components of the traction vectors on those planes where the traction vector is parallel to the unit normal to the plane itself. In other words, a component along a direction perpendicular to this unit normal (i.e. along the plane itself) must be necessarily zero. So, if the principal directions themselves are chosen as the coordinate axes, then the traction vector corresponding to each principal plane will be entirely along the axis perpendicular to the plane and along the plane there will be no component - meaning that along the other two coordinate axes which necessarily must lie on the plane, there can be no component of the traction. Thus, stress components along these two directions (the shear directions) must be zero.

8 Octahedral stress

Consider the coordinate axes aligned along the principal directions. A plane that is equally aligned to these axes is called an octahedral plane. For such a plane, $|n_1| = |n_2| = |n_3|$. Now, since we must have $n_1^2+n_2^2+n_3^2 = 1$, therefore

$$|n_1| = |n_2| = |n_3| = \frac{1}{\sqrt{3}}. \tag{42}$$

Note that there can be eight such planes and together they form an octahedron.

Normal and shear stress on each of these planes are referred to as octahedral normal stress and octahedral shear stress

$$
\begin{aligned}
\sigma_{\text{oct}} &= \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 \\
&= \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) = \frac{1}{3} I_1. \qquad (43) \\
\tau_{\text{oct}} &= \left(\sigma_{11} - \sigma_{22}\right)^2 n_1^2 n_2^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 n_2^2 n_3^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 n_3^2 n_1^2 \\
&= \frac{1}{9}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2\right] \\
&= \frac{1}{9}\left[2\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)^2 - 6\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)\right] \\
&= \frac{1}{9}\left(2I_1^2 - 6I_2\right) \qquad (44)
\end{aligned}
$$

From the last equation we have

$$
\left|\tau_{\text{oct}}\right| = \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2} \qquad (45)
$$

9 Decomposition into mean and deviatoric parts

A general state of stress matrix can be decomposed as follows:

$$
[\boldsymbol{\sigma}] \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{\text{m}} & 0 & 0 \\ 0 & \sigma_{\text{m}} & 0 \\ 0 & 0 & \sigma_{\text{m}} \end{bmatrix}}_{[\boldsymbol{\sigma}^{\text{M}}]} + \underbrace{\begin{bmatrix} \sigma_{11} - \sigma_{\text{m}} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{\text{m}} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{\text{m}} \end{bmatrix}}_{[\boldsymbol{\sigma}^{\text{D}}]}, \qquad (46)
$$

where $\boldsymbol{\sigma}_{\text{m}}$ is taken as $\sigma_{\text{m}} = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) \equiv \frac{1}{3} I_1$ with I_1 being the first stress invariant of $\boldsymbol{\sigma}$. Equivalently, $\sigma_{\text{m}} = \frac{1}{3}\text{tr}(\boldsymbol{\sigma})$ where tr is the trace. In the above, $\boldsymbol{\sigma}^{\text{M}}$ is the mean stress tensor and $\boldsymbol{\sigma}^{\text{D}}$ is the deviatoric stress tensor. Note also that the first stress invariant of the deviatoric stress tensor, $\boldsymbol{\sigma}^{\text{D}}$, is 0 while the first invariant of the mean stress tensor, $\boldsymbol{\sigma}^{\text{M}}$ is I_1 itself.