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Mathematical Preliminaries

Vectors

Compact notation : $\vec{v}$, $\underset{\sim}{v}$

Component notation : $\underset{\sim}{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 \rightarrow \sum_{i=1}^{3} v_i\hat{e}_i \rightarrow v_i\hat{e}_i$

$$\underset{\sim}{v} = v_1'\hat{e}_1' + v_2'\hat{e}_2' + v_3'\hat{e}_3' \rightarrow \sum_{i=1}^{3} v_i'\hat{e}_i' \rightarrow v_i'\hat{e}_i'$$

Matrix representation: $[\underset{\sim}{v}] = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ OR $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

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Index notation

$$\underset{\sim}{v} = v_i \quad , \; i = 1, 2, 3$$

Rules of index notation

1. No summation sign or unit vectors are written
2. A repeated index indicates summation

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

3. A repeated index is like a dummy index and it can be replaced by any other letter

$$a_i b_i = a_j b_j = a_p b_p = \dots$$

4. An index repeated more than once is <u>meaningless</u>

$a_i b_i c_i$ X

5. The number of <u>free indices</u> denotes the order of the tensor of that term

$a_i \rightarrow$ tensor of order 1 (vector)

$b_i c_i \rightarrow$ tensor of order 0 (scalar)

$A_{kkj} \rightarrow$ tensor of order 1

$T_{lm} \rightarrow$ tensor of order 2

$C_{ijkl} \rightarrow$ tensor of order 4

Mathematical Preliminaries ... contd.

$$\hat{e}_1 \cdot \hat{e}_1 = 1 \,,\quad \hat{e}_1 \cdot \hat{e}_2 = 0 \,,\ \ldots\ \hat{e}_2 \cdot \hat{e}_3 = 0,\ \hat{e}_3 \cdot \hat{e}_3 = 1$$

$$\hat{e}_i \cdot \hat{e}_j = \begin{cases} 1 \,, & i = j \\ 0 \,, & i \neq j \end{cases}$$

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$$

↳ Kronecker Delta

Dot Product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_i \hat{e}_i) \cdot (b_j \hat{e}_j) = a_i b_j \, \hat{e}_i \cdot \hat{e}_j = a_i b_j \delta_{ij} = a_i b_i = a_j b_j$$

$$(b_j \delta_{ij} = b_i) \qquad (a_i \delta_{ij} = a_j)$$

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$$\begin{aligned}
\left[\underset{\sim}{a} \cdot \underset{\sim}{b}\right] &= \left[\underset{\sim}{a}\right]^T \left[\underset{\sim}{b}\right] \\
&= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^T \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3 \\
&= a_i b_i
\end{aligned}$$

Mathematical Preliminaries ... contd.

$$[\underset{\sim}{a}][\underset{\sim}{b}]^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix}$$

$= a_i b_j \rightarrow$ Tensor of order 2

$$\equiv \underset{\sim}{a} \otimes \underset{\sim}{b}$$

Outer product OR Dyadic product

$\underset{\sim}{b} \otimes \underset{\sim}{a} \neq \underset{\sim}{a} \otimes \underset{\sim}{b}$ In fact, $[\underset{\sim}{b} \otimes \underset{\sim}{a}] = [\underset{\sim}{a} \otimes \underset{\sim}{b}]^T$

$$\underset{\approx}{T} \otimes \underset{\sim}{v} = \underset{\approx\!\!\approx}{A} \rightarrow T_{ij} \; v_k$$

2nd order

1st order

3rd order

Contraction

Dot product is a special case of contraction

$a_i \; b_j \rightarrow$ Contraction $\rightarrow a_i b_i$ OR $a_j b_j$

$T_{ij} \; v_k \rightarrow$ Contraction $\rightarrow T_{ij} v_j$ OR $T_{ik} v_k$ (set $j = k$)

$T_{ij} v_i$ OR $T_{kj} v_k$ (set $i = k$)

$\rightarrow$ Not equal

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$$T_{ik}v_k = T_{i1}v_1 + T_{i2}v_2 + T_{i3}v_3$$

$$\equiv \begin{bmatrix} T_{11}v_1 + T_{12}v_2 + T_{13}v_3 \\ T_{21}v_1 + T_{22}v_2 + T_{23}v_3 \\ T_{31}v_1 + T_{32}v_2 + T_{33}v_3 \end{bmatrix} = [\underset{\approx}{T}][\underset{\sim}{v}] \equiv \underset{\approx}{T} \cdot \underset{\sim}{v}$$

$$T_{kj}v_k = T_{1j}v_1 + T_{2j}v_2 + T_{3j}v_3$$

$$\equiv \begin{bmatrix} T_{11}v_1 + T_{21}v_2 + T_{31}v_3 \\ T_{12}v_1 + T_{22}v_2 + T_{32}v_3 \\ T_{13}v_1 + T_{23}v_2 + T_{33}v_3 \end{bmatrix} = [\underset{\approx}{T}]^T[\underset{\sim}{v}] \equiv \underset{\approx}{T}^T \cdot \underset{\sim}{v}$$

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Mathematical Preliminaries ... contd.

$$\hat{e}_1 \otimes \hat{e}_1 \equiv [\hat{e}_1][\hat{e}_1]^T$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \hat{i} & 1 & 0 & 0 \\ \hat{j} & 0 & 1 & 0 \\ \hat{k} & 0 & 0 & 1 \end{array}$$

$$\hat{e}_1 \otimes \hat{e}_2 \equiv [\hat{e}_1][\hat{e}_2]^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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$$\underset{\approx}{T} = T_{ij}\,\hat{e}_i \otimes \hat{e}_j$$

$$= T_{11}\,\hat{e}_1 \otimes \hat{e}_1 + T_{12}\,\hat{e}_1 \otimes \hat{e}_2 + T_{13}\,\hat{e}_1 \otimes \hat{e}_3 + \dots\; T_{32}\,\hat{e}_3 \times \hat{e}_2 + T_{33}\,\hat{e}_3 \times \hat{e}_3$$

$$\equiv \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{32} \end{bmatrix}$$

$$\underset{\approx}{T} \cdot \underset{\sim}{v} = \left(T_{ij}\,\hat{e}_i \otimes \hat{e}_j\right) \cdot \left(v_k\,\hat{e}_k\right)$$

$$= T_{ij}\,v_k\,\hat{e}_i \otimes \hat{e}_j \cdot \hat{e}_k$$

$$= T_{ij}\,v_k\,\hat{e}_i\,\delta_{jk} = T_{ik}\,v_k\,\hat{e}_i \quad \text{OR} \quad T_{ij}\,v_j\,\hat{e}_i$$

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$$\underset{\approx}{T}^T \cdot \underset{\sim}{v} = \left(T_{ij}\, \hat{e}_j \otimes \hat{e}_i\right) \cdot \left(v_k \hat{e}_k\right)$$

$$= T_{ij}\, v_k\, \hat{e}_j \otimes \hat{e}_i \cdot \hat{e}_k$$

$$= T_{ij}\, v_k\, \hat{e}_j\, \delta_{ik}$$

$$= T_{kj}\, v_k\, \hat{e}_j \quad \text{OR} \quad T_{ij}\, v_i\, \hat{e}_j$$

$$\underset{\approx}{A} \cdot \underset{\approx}{B} = \left(A_{ij}\, \hat{e}_i \otimes \hat{e}_j\right) \cdot \left(B_{kl}\, \hat{e}_k \otimes \hat{e}_l\right)$$

$$= A_{ij}\, B_{kl}\, \hat{e}_i \otimes \hat{e}_j \cdot \hat{e}_k \otimes \hat{e}_l$$

$$= A_{ij}\, B_{kl}\, \delta_{jk}\, \hat{e}_i \otimes \hat{e}_l \quad \text{OR} \quad A_{ij}\, B_{jl}\, \hat{e}_i \otimes \hat{e}_l$$

$$= A_{ik}\, B_{kl}\, \hat{e}_i \otimes \hat{e}_l$$

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$$\underset{\approx}{A}^T \cdot \underset{\approx}{B} = \left(A_{ij}\,\hat{e}_j \otimes \hat{e}_i\right) \cdot \left(B_{kl}\,\hat{e}_k \otimes \hat{e}_l\right)$$

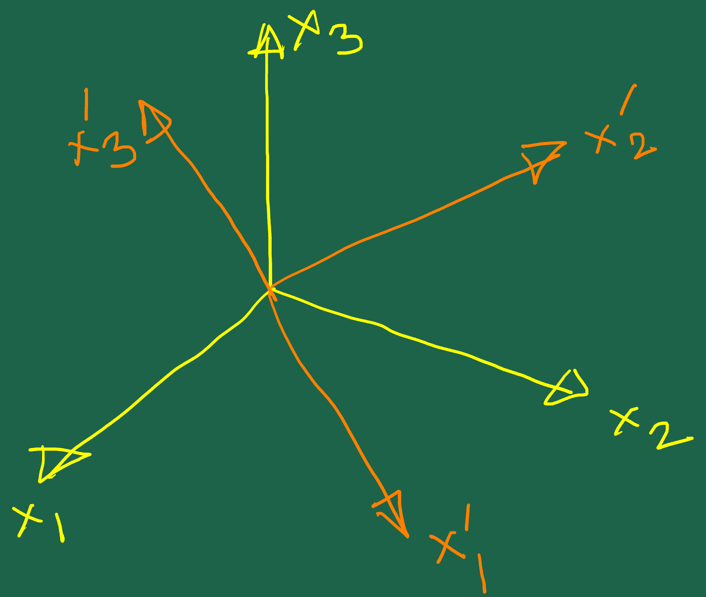
$$= A_{ij} B_{kl}\,\hat{e}_j \otimes \hat{e}_i \cdot \hat{e}_k \otimes \hat{e}_l$$

$$= A_{ij} B_{kl}\,\delta_{ik}\,\hat{e}_j \otimes \hat{e}_l$$

$$= A_{kj} B_{kl}\,\hat{e}_j \otimes \hat{e}_l \quad \text{OR} \quad A_{ij} B_{il}\,\hat{e}_j \otimes \hat{e}_l$$

Mathematical Preliminaries ... contd.

Coordinate Transformations



$$\hat{e}'_1 = Q_{11}\hat{e}_1 + Q_{12}\hat{e}_2 + Q_{13}\hat{e}_3$$

$$\hat{e}'_2 = Q_{21}\hat{e}_1 + Q_{22}\hat{e}_2 + Q_{23}\hat{e}_3$$

$$\hat{e}'_3 = Q_{31}\hat{e}_1 + Q_{32}\hat{e}_2 + Q_{33}\hat{e}_3$$

$$\begin{bmatrix} \hat{e}'_1 \\ \hat{e}'_2 \\ \hat{e}'_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

$$\cos(x'_1, x_1) = Q_{11}$$

$$\cos(x'_1, x_2) = Q_{12}$$

$$\cos(x'_2, x_3) = Q_{23}$$

$$\Rightarrow \hat{e}'_i = Q_{ij}\hat{e}_j$$

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$$\hat{e}_1 = Q_{11}\hat{e}_1' + Q_{21}\hat{e}_2' + Q_{31}\hat{e}_3'$$

$$\hat{e}_2 = Q_{12}\hat{e}_1' + Q_{22}\hat{e}_2' + Q_{32}\hat{e}_3'$$

$$\hat{e}_3 = Q_{13}\hat{e}_1' + Q_{23}\hat{e}_2' + Q_{33}\hat{e}_3'$$

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3' \end{bmatrix}$$

$$\Rightarrow \quad \hat{e}_i = Q_{ji}\,\hat{e}_j'$$

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$$\underset{\sim}{v} = v_i \hat{e}_i = v_i' \hat{e}_i' = v_j' \hat{e}_j'$$

$$\hat{e}_i' = Q_{ij} \hat{e}_j$$

$$\hat{e}_j' = Q_{jp} \hat{e}_p$$

$$= Q_{ji} \hat{e}_i$$

$$v_i \hat{e}_i = v_j' \hat{e}_j'$$

$$\Rightarrow v_i \hat{e}_i = v_j' Q_{ji} \hat{e}_i$$

$$\Rightarrow v_i = v_j' Q_{ji}$$

$$\Rightarrow v_i = Q_{ji} v_j' \longrightarrow v_i = Q_{ji} v_j'$$

$$= Q_{ji} Q_{jk} v_k$$

$$v_j' \hat{e}_j' = v_i \hat{e}_i$$

$$\Rightarrow v_j' \hat{e}_j' = v_i Q_{ji} \hat{e}_j'$$

$$\Rightarrow v_j' = Q_{ji} v_i = Q_{jk} v_k$$

$$\therefore v_k \delta_{ik} = Q_{ji} Q_{jk} v_k$$

$$\Rightarrow Q_{ji} Q_{jk} = \delta_{ik}$$

$$\underset{\approx}{Q}^T \cdot \underset{\approx}{Q} = \underset{\approx}{I}$$

$$[\underset{\approx}{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mathematical Preliminaries … contd.

$$v'_i = Q_{ip} v_p$$

$$T'_{ij} = a'_i b'_j$$

$$= Q_{ip} a_p Q_{jq} b_q$$

$$= Q_{ip} Q_{jq} a_p b_q$$

$$= Q_{ip} Q_{jq} T_{pq}$$

$$A'_{ijk} = Q_{ip} Q_{jq} Q_{kr} A_{pqr}$$