Mathematical Preliminaries *

1 Vector representations

A vector quantity exists independent of its mathematical representation. But if we have to work with it, we do need to represent it mathematically. There can be various ways of representing a vector:

Direct or compact notation: $\mathbf{v}$

Component form: $v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3$ OR, $\sum_{i=1}^{3} v_i\hat{\mathbf{e}}_i$

Index or indical notation: v_i

Matrix representation: $[\mathbf{v}] \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

2 Dot product

Let us now look at the dot product:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= a_i b_i \\ &= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\ &= [\mathbf{a}]^{\mathsf{T}}[\mathbf{b}] \end{aligned}$$

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In component form:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= (a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3)\cdot(b_1\hat{\mathbf{e}}_1 + b_2\hat{\mathbf{e}}_2 + b_3\hat{\mathbf{e}}_3) \\
&= (a_i\hat{\mathbf{e}}_i)\cdot\left(b_j\hat{\mathbf{e}}_j\right) \\
&= a_i b_j \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j \\
&= a_i b_j \delta_{ij} \\
&= a_i b_i
\end{aligned}$$

A special property of the Kronecker delta is its “ability" to substitute indices, and it is this property that has been used in the last step above.

3 Dyadic product

We have seen what happens in the case of the dot product: $[\mathbf{a}]^\mathsf{T}[\mathbf{b}]$. Next, what happens if we do $[\mathbf{a}][\mathbf{b}]^\mathsf{T}$?

We obtain: $\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$.

This entity is represented by a_ib_j. In terms of a single variable two indices are used; for instance T_{ij}.

In direct or compact notation, it is written as $\mathbf{a}\otimes\mathbf{b}$. Note that $\mathbf{a}\otimes\mathbf{b} \neq \mathbf{b}\otimes\mathbf{a}$. In fact, $\mathbf{a}\otimes\mathbf{b} = (\mathbf{b}\otimes\mathbf{a})^\mathsf{T}$.

In component form, we have: $(a_i\mathbf{e}_i)\otimes\left(b_j\mathbf{e}_j\right) = a_ib_j\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)$

Note:

$$\begin{aligned}
\mathbf{e}_1\otimes\mathbf{e}_1 &= \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \\
\mathbf{e}_1\otimes\mathbf{e}_2 &= \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},
\end{aligned}$$

and so on.

The dyadic product $\otimes$ sums up the order of the tensors. Thus, if we have a vector $\mathbf{v}$ and two second-order tensors $\mathbf{T}$ and $\mathbf{S}$, then $\mathbf{T} \otimes \mathbf{v} = \mathcal{A}$ is a tensor of order 3, and $\mathbf{S} \otimes \mathbf{T} = \mathbb{C}$ is a tensor of order 4.

4 Contraction

We can also have operations between tensors that reduce the order.

Consider $\mathbf{T} \otimes \mathbf{v} \equiv T_{ij} v_k$. Now set $k = j$. This operation is referred to as contraction. The resulting entity $T_{ij} v_j$ has only one free index and so it is a tensor of order 1, i.e. a vector. Contraction reduced the order from 2 to 1.

The dot product was a special case of contraction.

Contraction can be between any two indices. Consider again $\mathbf{T} \otimes \mathbf{v} \equiv T_{ij} v_k$. Instead of setting $k = j$, we set $k = i$ this time. Then the resulting entity $T_{ij} v_i$ is again a tensor or order 1. However, importantly, $T_{ij} v_j \neq T_{ij} v_i$ which can easily be verified by expanding. In the direct or compact notation, we have

$$
\begin{aligned}
T_{ij} v_j &\equiv \mathbf{T}\mathbf{v} \\
T_{ij} v_i &\equiv \mathbf{T}^{\mathsf{T}}\mathbf{v}
\end{aligned}
$$

Contraction between tensors of order 2 can have various possibilities:

$$
\begin{aligned}
\mathbf{S}\mathbf{T} &\equiv S_{ij} T_{jl}, \\
\mathbf{S}^{\mathsf{T}}\mathbf{T} &\equiv S_{ji} T_{jl}, \\
\mathbf{S}\mathbf{T}^{\mathsf{T}} &\equiv S_{ij} T_{lj}, \\
\mathbf{S}^{\mathsf{T}}\mathbf{T}^{\mathsf{T}} &\equiv S_{ji} T_{lj}
\end{aligned}
$$

We can work out the preceding expressions in a more explicit way using the component form. For instance:

$$
\begin{aligned}
\mathbf{Tv} &\equiv T_{ij}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot v_k\hat{\mathbf{e}}_k \\
&\equiv T_{ij}v_k\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \\
&\equiv T_{ij}v_k\delta_{jk}\hat{\mathbf{e}}_i \\
&\equiv T_{ij}v_j\hat{\mathbf{e}}_i
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{ST} &\equiv S_{ij}\left(\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j\right) \cdot T_{kl}\left(\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l\right) \\
&\equiv S_{ij}T_{kl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_l \\
&\equiv S_{ij}T_{kl}\delta_{jk}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l \\
&\equiv S_{ij}T_{kl}\hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_l
\end{aligned}
$$