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Material Behaviour

Balance of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

1 eqn, 4 unknowns

ρ ; v_1, v_2, v_3 OR u_1, u_2, u_3

$\vec{v} := \frac{D\vec{u}}{Dt}$

Balance of lin. momentum

$$\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \underset{\sim}{\sigma} + \rho \vec{b}$$

$$0 = \nabla \cdot \underset{\sim}{\sigma} + \rho \vec{b}$$

3 eqns, 6 unknowns

$\hookrightarrow \underset{\sim}{\sigma}$

$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$

4 eqns, 10 unknowns

6 eqns will be provided through a description of the material behaviour

Eg. $\underset{\approx}{\sigma} = f(\underset{\approx}{\varepsilon})$

But, we have ended up introducing 6 more unknowns $(\underset{\approx}{\varepsilon})$

However, we already know the 6 strain-displacement relations

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{or} \quad \underset{\approx}{\varepsilon} = \frac{1}{2}\left[\nabla\vec{u} + (\nabla\vec{u})^T\right]$$

$(4+6+6) = 16$ eqns , $(10+6) = 16$ unknowns

Some assumptions to describe material behaviour

Linearity

No rate or history effects

Uniformity or Homogeneity

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

$$C_{ijkl} \rightarrow 3 \times 3 \times 3 \times 3 = 81$$

Minor Symmetries

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad \text{— } (\#_1)$$

Interchange i & j

$$\sigma_{ji} = C_{jikl}\,\varepsilon_{kl} \quad \text{— } (\#_2)$$

$(\#_1) - (\#_2)$

$$\Rightarrow \quad \sigma_{ij} - \sigma_{ji} = \left(C_{ijkl} - C_{jikl}\right)\varepsilon_{kl}$$

But $\sigma_{ij} = \sigma_{ji}$

$$\therefore \quad 0 = \left(C_{ijkl} - C_{jikl}\right)\varepsilon_{kl} \Rightarrow C_{ijkl} = C_{jikl}$$

$\hookrightarrow 3 \times 2 \times 3 \times 3 = 54$

Next interchange k & l

$$\sigma_{ij} = C_{ijlk}\,\varepsilon_{lk} \quad \text{---} \ (\#_3)$$

$(\#_1) - (\#_3)$

$$\Rightarrow \ 0 = C_{ijkl}\,\varepsilon_{kl} - C_{ijlk}\,\varepsilon_{lk}$$

But $\varepsilon_{kl} = \varepsilon_{lk}$

$$\therefore \ 0 = \left(C_{ijkl} - C_{ijlk}\right)\varepsilon_{kl}$$

$$\Rightarrow C_{ijkl} = C_{ijlk}$$

$\hookrightarrow 3\times2\times3\times2 = 36$

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$$\left.\begin{aligned} &* \; C_{ijkl} = C_{jikl} \\ &* \; C_{ijkl} = C_{ijlk} \end{aligned}\right\} \rightarrow \text{Minor Symmetries}$$

Voigt Notation

$11 \mapsto 1 \;,\quad 22 \mapsto 2 \;,\quad 33 \mapsto 3$

$23 \equiv 32 \mapsto 4 \;,\quad 13 \equiv 31 \mapsto 5 \;,\quad 12 \equiv 21 \mapsto 6$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \longmapsto \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \;,\quad \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \longmapsto \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

$$C_{1122} \mapsto \bar{C}_{12} \quad , \quad C_{1233} \mapsto \bar{C}_{63}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} & \bar{C}_{16} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} & \bar{C}_{26} \\ \bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} & \bar{C}_{36} \\ \bar{C}_{41} & \bar{C}_{42} & \bar{C}_{43} & \bar{C}_{44} & \bar{C}_{45} & \bar{C}_{46} \\ \bar{C}_{51} & \bar{C}_{52} & \bar{C}_{53} & \bar{C}_{54} & \bar{C}_{55} & \bar{C}_{56} \\ \bar{C}_{61} & \bar{C}_{62} & \bar{C}_{63} & \bar{C}_{64} & \bar{C}_{65} & \bar{C}_{66} \end{bmatrix}}_{\text{As of now, it is \underline{not} symmetric}} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ 2\varepsilon_4 \\ 2\varepsilon_5 \\ 2\varepsilon_6 \end{bmatrix}$$

$$\bar{C}_{12} \overset{?}{=} \bar{C}_{21}$$

$$C_{1122} \overset{?}{=} C_{2211}$$

Major Symmetry

$\rightarrow \quad \sigma_{ij} = \dfrac{\partial U_0}{\partial \varepsilon_{ij}}$, U_0 : strain energy density

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial^2 U_0}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}}$$

$\rightarrow \quad \sigma_{kl} = \dfrac{\partial U_0}{\partial \varepsilon_{kl}}$

$$\frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}} = \frac{\partial^2 U_0}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$$

But we must have

$$\frac{\partial^2 U_0}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = \frac{\partial^2 U_0}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}$$

$$\Rightarrow \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}}$$

$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$, $\sigma_{kl} = C_{klij} \varepsilon_{ij}$

$C_{ijkl} = C_{klij} \rightarrow$ Major Symmetry

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Based on the major symmetry the 36 independent components of the elasticity tensor further reduce to 21 independent components

Isotropy — Material behaviour is the same in all directions

$\hookrightarrow$ 21 independent comp. $\rightarrow$ 2 independent comp.

Isotropy

$$C_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \quad — (*_1)$$

Interchange i & j

$$C_{jikl} = \alpha \delta_{ji} \delta_{kl} + \beta \delta_{jk} \delta_{il} + \gamma \delta_{jl} \delta_{ik} \quad — (*_2)$$

$$(*_1) - (*_2)$$

$$\Rightarrow 0 = (\beta - \gamma) \underset{11}{\delta_{ik}} \underset{22}{\delta_{jl}} + (\gamma - \beta) \underset{12}{\delta_{il}} \underset{21}{\delta_{jk}}$$

$$\therefore \beta = \gamma$$

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$$C_{ijkl} = \alpha\,\delta_{ij}\delta_{kl} + \beta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$C_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\,\varepsilon_{kl} \\ &= \left\{\lambda\,\delta_{ij}\delta_{kl} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right\}\varepsilon_{kl} \\ &= \lambda\,\varepsilon_{kk}\delta_{ij} + G\left(\varepsilon_{ij} + \varepsilon_{ji}\right)\end{aligned}$$

$$\boxed{\sigma_{ij} = \lambda\,\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}} \quad \longleftrightarrow \quad \text{Generalized Hooke's law}$$