## Material Behaviour Stress Strain Relations*

## 1 The Need for Material Behaviour Relations

All of us are familiar with material behaviour relations which are often referred to as constitutive relations also. For instance, the famous Hooke's law. However, let us try to motivate the discussion of material behaviour relations by discussing why we need them in the first place. Towards that end, let us take stock of what equations and unknown variables we have till now, and we want we need in order to have a closed system of equations.

Just like in fluid mechanics, in solid mechanics too, we need to ensure that mass conservation is satisfied. We can proceed exactly as was done in fluid mechanics to obtain the following

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\nabla \cdot(\rho \boldsymbol{v})=0, \tag{1}
\end{equation*}
$$

which is, of course, the continuity equation. Here, $\rho$ is the mass density of the material that can, in general, vary with time and position, and $v$ is the velocity. It is extremely important to note that this equation which was introduced in the context of studying fluids holds true for solids too. Mass conservation must be satisfied, whether it is fluid or solid.

Next, again like in fluid mechanics, we need to ensure the balance of linear momentum. This balance has already been discussed in the previous chapter on "Stress", and it led to the Cauchy's equations of motion

$$
\begin{equation*}
\rho \frac{\mathrm{D} \boldsymbol{v}}{\mathrm{D} t}=\nabla \cdot \boldsymbol{\sigma}+\rho \boldsymbol{b} \tag{2}
\end{equation*}
$$

[^0]where $\sigma$ is the stress tensor and $\boldsymbol{b}$ is the body force per unit volume. We had also discussed that for a body in static equilibrium, this equation reduces to the following form, referred to as the stress equilibrium equations:
\[

$$
\begin{equation*}
\nabla \cdot \boldsymbol{\sigma}+\rho \boldsymbol{b}=0 \tag{3}
\end{equation*}
$$

\]

First, in the continuity equation (1), we have the unknown density $\rho$ and the three unknown components of the velocity vector $v$. Alternatively, instead of the unknown velocity components, we can say that the unknowns are the three components of the displacement vector $\boldsymbol{u}$. We note that the velocity can be immediately found once the displacement is found using the kinematic definition: $\boldsymbol{v}:=\frac{\mathrm{D} \boldsymbol{u}}{\mathrm{D} t}$. In any case, what we have are $(1+3=) 4$ unknowns and 1 equation.

Next, in the stress equilibrium equations (3), we have the six unknown components of the stress tensor $\sigma$ and three equations. So, in total, we have $(1+3+6=) 10$ unknowns and $(1+3=) 4$ equations. There is, therefore, a discrepancy of 6 equations. These are what we need to find.

It is to fill this discrepancy of 6 equations that we need to use the material behaviour or constitutive relations.

We can use six constitutive relations in the form $\sigma=f(\varepsilon)$. But note that these six relations are actually expressions of the six components of $\sigma$ in terms of the six components of $\varepsilon$. Thus, in introducing the six equations, we have apparently ended up introducing another six unknowns! But remember that we already have our six strain-displacement relations.

So, overall, we have 16 unknowns: $\rho$, the three components of the displacement vector $(\boldsymbol{u})$, the six components of the strain tensor $(\boldsymbol{\varepsilon})$, and the six components of the stress tensor $(\sigma)$. We also have 16 equations: the continuity equation, the three mechanical equilibrium equations, and the six strain-displacement relations.

In order to arrive at a specific form for the constitutive relations, we will use the following assumptions:

- Linearity
- No rate or history effects
- Uniformity or Homogeneity

These assumptions lead to the following relations:

$$
\begin{equation*}
\sigma_{i j}=C_{i j k l} \varepsilon_{k l} \tag{4}
\end{equation*}
$$

where $C_{i j k l} \equiv \mathbb{C}$ is a 4 th order tensor called the elasticity tensor. These relations can be simplified somewhat by further considerations as follows.

## 2 Minor Symmetry

We first interchange $i$ and $j$ in Eq. (4) to obtain:

$$
\begin{equation*}
\sigma_{j i}=C_{j i k l} \varepsilon_{k l} . \tag{5}
\end{equation*}
$$

We subtract Eq. (5) from Eq. (4) to obtain

$$
\begin{align*}
0 & =\left(C_{i j k l}-C_{j i k l}\right) \varepsilon_{k l},  \tag{6}\\
& \Longrightarrow C_{i j k l}=C_{j i k l} . \tag{7}
\end{align*}
$$

The symmetricity about the first two indices mean that out of the $(3 \times 3 \times 3 \times$ $3)=81$ components of the fourth order elasticity tensor only $(3 \times 2 \times 3 \times 3)=$ 54 are independent. Proceeding further, if we next interchange $k$ and $l$ in Eq. (4), we obtain

$$
\begin{equation*}
\sigma_{i j}=C_{i j l k} \varepsilon_{l k} . \tag{8}
\end{equation*}
$$

But since we know that $\varepsilon_{k l}=\varepsilon_{l k}$, so we must have

$$
\begin{equation*}
\sigma_{i j}=C_{i j l k} \varepsilon_{k l} . \tag{9}
\end{equation*}
$$

Now, subtracting Eq. (9) from Eq. (4), we obtain

$$
\begin{align*}
0 & =\left(C_{i j k l}-C_{i j l k}\right) \varepsilon_{k l},  \tag{10}\\
& \Longrightarrow C_{i j k l}=C_{i j l k} \tag{11}
\end{align*}
$$

Again, just as before, the symmetricity about the third and fourth indices mean that of the previously obtained 54 independent components of the elasticity tensor, we actually have $(3 \times 2 \times 3 \times 2)=36$ components that are really independent.

We note that $C_{i j k l}=C_{j i k l}$ and $C_{i j k l}=C_{i j l k}$ are referred to as the minor symmetries.

The minor symmetries are utilized to simplify the matrix representation of the stress-strain relations through what is referred to as the Voigt notation.

## 3 Voigt Notation

In the Voigt notation, the following mappings are used to denote the index pairs:

$$
\begin{gathered}
11 \mapsto 1, \quad 22 \mapsto 2, \quad 33 \mapsto 3, \\
23 \equiv 32 \mapsto 4, \quad 13 \equiv 31 \mapsto 5, \quad 12 \equiv 21 \mapsto 6
\end{gathered}
$$

Using these index pair mappings, the six independent components of the stress matrix and the strain matrix are represented in Voigt notation as the components of column matrices with six rows:

$$
\left[\begin{array}{ccc}
\sigma_{11} & \sigma_{12} & \sigma_{13}  \tag{12}\\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{array}\right] \mapsto\left[\begin{array}{c}
\sigma_{1} \\
\sigma_{2} \\
\sigma_{3} \\
\sigma_{4} \\
\sigma_{5} \\
\sigma_{6}
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{lll}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{array}\right] \mapsto\left[\begin{array}{l}
\varepsilon_{1} \\
\varepsilon_{2} \\
\varepsilon_{3} \\
\varepsilon_{4} \\
\varepsilon_{5} \\
\varepsilon_{6}
\end{array}\right]
$$

The components of the elasticity tensor are rewritten as:

$$
\begin{equation*}
C_{122} \mapsto \bar{C}_{12}, C_{1233} \mapsto \bar{C}_{63} \text {, etc. } \tag{13}
\end{equation*}
$$

Thus, the $6 \times 1$ stress and the $6 \times 1$ strain column matrices are connected through a $6 \times 6$ matrix consisting of the 36 independent components of the elaticity matrix rewritten in Voigt notation:

$$
\left[\begin{array}{l}
\sigma_{1}  \tag{14}\\
\sigma_{2} \\
\sigma_{3} \\
\sigma_{4} \\
\sigma_{5} \\
\sigma_{6}
\end{array}\right]=\left[\begin{array}{llllll}
\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} & \bar{C}_{15} & \bar{C}_{16} \\
\bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} & \bar{C}_{25} & \bar{C}_{26} \\
\bar{C}_{31} & \bar{C}_{32} & \bar{C}_{33} & \bar{C}_{34} & \bar{C}_{35} & \bar{C}_{36} \\
\bar{C}_{11} & \bar{C}_{42} & \bar{C}_{43} & \bar{C}_{44} & \bar{C}_{45} & \bar{C}_{46} \\
\bar{C}_{51} & \bar{C}_{52} & \bar{C}_{53} & \bar{C}_{54} & \bar{C}_{55} & \bar{C}_{56} \\
\bar{C}_{62} & \bar{C}_{63} & \bar{C}_{64} & \bar{C}_{65} & \bar{C}_{66}
\end{array}\right]\left[\begin{array}{c}
\varepsilon_{1} \\
\varepsilon_{2} \\
\varepsilon_{3} \\
2 \varepsilon_{4} \\
2 \varepsilon_{5} \\
2 \varepsilon_{6},
\end{array}\right]
$$

where we must be careful to note the factor " 2 " in front of the shear strain components.

It is also extremely important to note that based on just the minor symmetries, the $6 \times 6$ matrix is not symmetric, i.e $\bar{C}_{12} \neq \bar{C}_{21}, \bar{C}_{13} \neq \bar{C}_{31}$, etc. It would be symmetric only under what is referred to as major symmetry, i.e. when $C_{i j k l}=C_{k l i j}$.

## 4 Major Symmetry

The major symmetry arises if the stress tensor can be written as

$$
\begin{equation*}
\sigma_{i j}=\frac{\partial U_{0}}{\partial \varepsilon_{i j}} \tag{15}
\end{equation*}
$$

where $U_{0}$ is the strain energy density. ${ }^{\dagger}$
Now, from the previous relation, we have

$$
\begin{equation*}
\frac{\partial \sigma_{i j}}{\partial \varepsilon_{k l}}=\frac{\partial^{2} U_{0}}{\partial \varepsilon_{k l} \partial \varepsilon_{i j}} \tag{16}
\end{equation*}
$$

[^1]Again, we can write

$$
\begin{equation*}
\sigma_{k l}=\frac{\partial U_{0}}{\partial \varepsilon_{k l}} \tag{17}
\end{equation*}
$$

and again we have as before

$$
\begin{equation*}
\frac{\partial \sigma_{k l}}{\partial \varepsilon_{i j}}=\frac{\partial^{2} U_{0}}{\partial \varepsilon_{i j} \partial \varepsilon_{k l}} . \tag{18}
\end{equation*}
$$

But, we must have $\frac{\partial^{2} U_{0}}{\partial \varepsilon_{k l} \partial \varepsilon_{i j}}=\frac{\partial^{2} U_{0}}{\partial \varepsilon_{i j} \partial \varepsilon_{k l}}$ based on the continuity of the partial derivatives of the strain energy density. Consequently, we have

$$
\begin{equation*}
\frac{\partial \sigma_{i j}}{\partial \varepsilon_{k l}}=\frac{\partial \sigma_{k l}}{\partial \varepsilon_{i j}} \tag{19}
\end{equation*}
$$

and so if we have $\sigma_{i j}=C_{i j k l} \varepsilon_{k l}$ and $\sigma_{k l}=C_{k l i j} \varepsilon_{i j}$, then we end up with

$$
\begin{align*}
& C_{i j k l}=C_{k l i j}  \tag{20}\\
& \text { or, } \bar{C}_{p q} \tag{21}
\end{align*}=\bar{C}_{q p} .
$$

Based on this major symmetry, if we go back to the Voigt notation, we can immediately conclude that the $6 \times 6$ matrix is indeed symmetric, so that the 36 independent components actually reduce to 21 independent components. Further reductions in the number of independent components are possible based on symmetry arguments. (We are not going to discuss them here.) We note that the "highest" amount of symmetry is possible for an isotropic material where the material properties are are completely direction-independent. For such an isotropic material, the 21 independent components reduce to just 2 independent components.

## 5 Linear, Elastic, Isotropic Behaviour

We note a result from a tensor algebra which states that a general fourthorder isotropic tensor can be expressed in terms of the Kronecker delta as follows ${ }^{\ddagger}$

$$
\begin{equation*}
C_{i j k l}=\alpha \delta_{i j} \delta_{k l}+\beta \delta_{i k} \delta_{j l}+\gamma \delta_{i l} \delta_{j k} \tag{22}
\end{equation*}
$$

[^2]It appears from the above that there are 3 constants involved instead of the 2 that was previously mentioned. Let's see how we can reduce the 3 to 2 .

First, we interchange $i$ and $j$, so that

$$
\begin{equation*}
C_{j i k l}=\alpha \delta_{j i} \delta_{k l}+\beta \delta_{j k} \delta_{i l}+\gamma \delta_{j l} \delta_{i k}, \tag{23}
\end{equation*}
$$

We subtract Eq. (23) from Eq. (22) to obtain:

$$
\begin{equation*}
0=(\beta-\gamma) \delta_{i k} \delta_{j l}+(\gamma-\beta) \delta_{i l} \delta_{j k} \tag{24}
\end{equation*}
$$

The above relation can be true in general only if $\beta=\gamma$.
Similarly, if we interchange $j$ and $l$ in Eq. (22), we will again end up with $\beta=\gamma$.

So, in reality there are only 2 independent coefficients in Eq. (22)

$$
C_{i j k l}=\alpha \delta_{i j} \delta_{k l}+\beta\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)
$$

It is conventional to use the symbols $\lambda$ and $G$ instead of $\alpha$ and $\beta$; thus

$$
\begin{equation*}
C_{i j k l}=\lambda \delta_{i j} \delta_{k l}+G\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right) \tag{25}
\end{equation*}
$$

Now, using this equation in the stress-strain relation, we have

$$
\begin{array}{lrl} 
& \sigma_{i j} & =C_{i j k l} \varepsilon_{k l} \\
\text { or, } & \sigma_{i j} & =\left\{\lambda \delta_{i j} \delta_{k l}+G\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)\right\} \varepsilon_{k l}, \\
\text { or, } & \sigma_{i j} & =\lambda \varepsilon_{k k} \delta_{i j}+2 G \varepsilon_{i j} \tag{26}
\end{array}
$$

This is the constitutive relation for a linear, elastic, isotropic solid.
This equation represents the general constitutive behaviour of a linear, elastic, isotropic solid. The constants $\lambda$ and $G$ are referred to as the Lamé parameters.


[^0]:    *Notes prepared by Jeevanjyoti Chakraborty. Contact: jeevan@mech.iitkgp.ac.in

[^1]:    ${ }^{\dagger}$ Discussions involving the strain energy density will be done in the course Advanced Mechanics of Solids (ME60402). Students may refer to notes here: http://www.facweb.iitkgp.ac.in/~jeevanjyoti/teaching/advmechsolids/ 2020/notes/energy.pdf

[^2]:    $\ddagger$ In the problem sheet, you will be asked to verify this result.

