3D Elasticity ... contd.

Mathematical Precursor to Kelvin's Problem

Poisson's equation

$$\nabla^2 \vec{F} = \vec{f}$$

Particular soln :

$$\vec{F}^{(p)}(\vec{x}) = -\frac{1}{4\pi}\int_V \frac{\vec{f}(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d\vec{x}'^3$$

Complete soln :

$$\vec{F} = \vec{F}^{(p)}(\vec{x}) + \vec{F}^{(h)}(\vec{x})$$

$$\nabla^2 F^{(h)} = 0$$

3D Dirac Delta

$$\delta(\vec{x}) = 0 \quad \forall \ \vec{x} \neq \vec{0}$$

$$\delta(x) = 0 \quad \forall \ x \neq 0$$

$$\delta(\vec{x}-\vec{\xi}) = 0 \quad \forall \ \vec{x} \neq \vec{\xi}$$

$$\int_V \vec{T}(\vec{x})\,\delta(\vec{x}'-\vec{\xi})\, d\vec{x}'^3 = \vec{T}(\vec{\xi}) \quad , \quad \vec{\xi} \in V$$

Consider the Poisson's eqn:

$$\vec{\nabla}^2 \vec{F} = \vec{A}\, \delta(\vec{x} - \vec{\xi}),$$

where $\vec{A}$ is some constant, $\vec{\xi}$ is the position vector of a fixed point.

The particular solⁿ:

$$\vec{F}^{(P)}(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\vec{A}\, \delta(\vec{x}' - \vec{\xi})}{|\vec{x} - \vec{x}'|}\, d\vec{x}'^3$$

$$= -\frac{\vec{A}}{4\pi} \int_V \frac{\delta(\vec{x}' - \vec{\xi})}{|\vec{x} - \vec{x}'|}\, d\vec{x}'^3$$

$$= -\frac{\vec{A}}{4\pi} \frac{1}{|\vec{x} - \vec{\xi}|}$$

3D Elasticity ... contd.

Kelvin's Problem

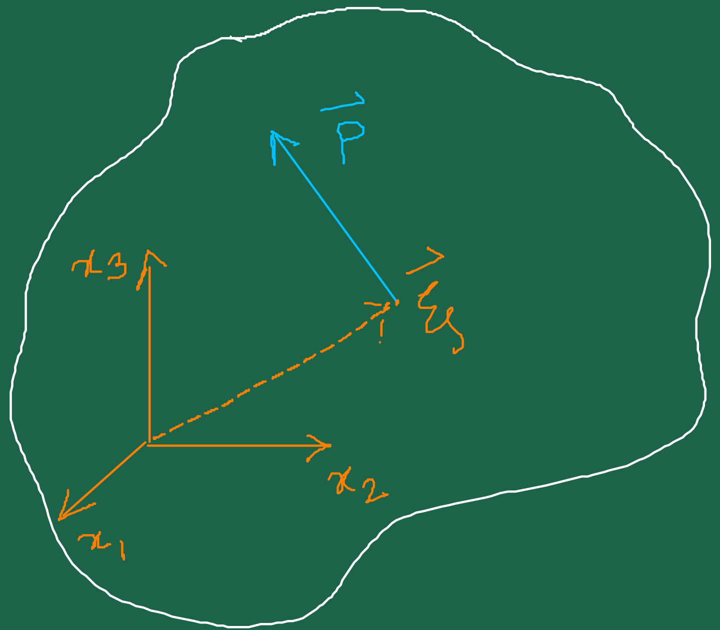

Fig. 1

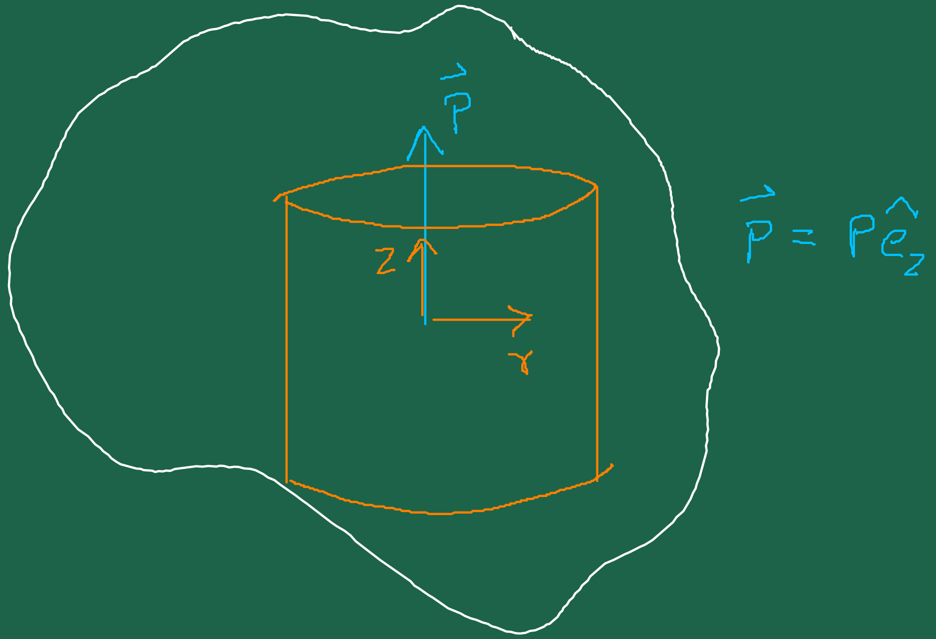

Fig. 2

At $|\vec{x}| \to \infty$, $\vec{u} = 0$

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Solution of the Kelvin's Problem using the Papkovich-Neuber representation

$$\nabla^2 \vec{B} = \frac{\rho \vec{b}}{2(1-\nu)} = \frac{\vec{P}\,\delta(\vec{x}-\vec{\xi}_s)}{2(1-\nu)}$$

$$\nabla^2 \beta = -\frac{\vec{x}\cdot\rho\vec{b}}{2(1-\nu)} = -\frac{\vec{x}\cdot\vec{P}\,\delta(\vec{x}-\vec{\xi}_s)}{2(1-\nu)}$$

The particular solutions to these Poisson's eqns are:

$$\vec{B}^{(P)}(\vec{x}) = -\frac{1}{4\pi}\int \frac{\vec{P}\,\delta(\vec{x}'-\vec{\xi}_s)}{2(1-\nu)\,|\vec{x}-\vec{x}'|}\,d\vec{x}'^3 = -\frac{\vec{P}}{8\pi(1-\nu)}\int \frac{\delta(\vec{x}'-\vec{\xi}_s)}{|\vec{x}-\vec{x}'|}\,d\vec{x}'^3$$

$$= -\frac{\vec{P}}{8\pi(1-\nu)\,|\vec{x}-\vec{\xi}_s|}$$

$$\beta^{(P)}(\vec{x}) = -\frac{1}{4\pi}\int \frac{-\vec{x}'\cdot\vec{P}\,\delta(\vec{x}'-\vec{\xi}_s)}{2(1-\nu)\,|\vec{x}-\vec{x}'|}\,d\vec{x}'^3 = +\frac{\vec{\xi}_s\cdot\vec{P}}{8\pi(1-\nu)\,|\vec{x}-\vec{\xi}_s|}$$

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Missing Link: How the particular solution becomes the total sol^n?

$$\vec{u} = \frac{1}{2G}\left[\nabla\left(\beta + \vec{x}\cdot\vec{B}\right) - 4(1-\nu)\vec{B}\right]$$

$$\Rightarrow 2G\vec{u} = \nabla\left(\beta + \vec{x}\cdot\vec{B}\right) - 4(1-\nu)\vec{B}$$

$$= \nabla\left(\frac{\vec{\xi}\cdot\vec{P}}{8\pi(1-\nu)|\vec{x}-\vec{\xi}|} - \frac{\vec{x}\cdot\vec{P}}{8\pi(1-\nu)|\vec{x}-\vec{\xi}|}\right) + \cancel{4(1-\nu)}\frac{\vec{P}}{8\pi\cancel{(1-\nu)}|\vec{x}-\vec{\xi}|}$$

$$= -\nabla\left\{\frac{(\vec{x}-\vec{\xi})\cdot\vec{P}}{8\pi(1-\nu)|\vec{x}-\vec{\xi}|}\right\} + \frac{\vec{P}}{2\pi|\vec{x}-\vec{\xi}|}$$

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Now, we change our viewpoint to Fig. 2, where

$$\vec{\xi} = \vec{0}$$

$$\vec{x} = r\hat{e}_r + z\hat{e}_z$$

$$|\vec{x}-\vec{\xi}| = |r\hat{e}_r + z\hat{e}_z| = \sqrt{r^2+z^2}$$

$$2G\vec{u} = -\nabla\left\{\frac{(\vec{x}-\vec{\xi})\cdot\vec{P}}{8\pi(1-\nu)|\vec{x}-\vec{\xi}|}\right\} + \frac{\vec{P}}{2\pi|\vec{x}-\vec{\xi}|}$$

$$\Rightarrow 2G(u_r\hat{e}_r + u_z\hat{e}_z) = -\nabla\left\{\frac{(r\hat{e}_r + z\hat{e}_z)\cdot P\hat{e}_z}{8\pi(1-\nu)\sqrt{r^2+z^2}}\right\} + \frac{P\hat{e}_z}{2\pi\sqrt{r^2+z^2}}$$

$$= -\nabla\left\{\frac{Pz}{8\pi(1-\nu)\sqrt{r^2+z^2}}\right\} + \frac{P\hat{e}_z}{2\pi\sqrt{r^2+z^2}}$$

$$= -\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_z \frac{\partial}{\partial z}\right)\left\{\frac{Pz}{8\pi(1-\nu)\sqrt{r^2+z^2}}\right\} + \frac{P\hat{e}_z}{2\pi\sqrt{r^2+z^2}}$$

$$2G\,u_r = -\frac{\partial}{\partial r}\left(\frac{Pz}{8\pi(1-\nu)\sqrt{r^2+z^2}}\right) \quad ; \quad u_z = -\frac{\partial}{\partial z}\left(\frac{Pz}{8\pi(1-\nu)\sqrt{r^2+z^2}}\right) + \frac{P}{2\pi\sqrt{r^2+z^2}}$$

$$= -\frac{Pz}{8\pi(1-\nu)}\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+z^2}}\right)$$

$$= -\frac{Pz}{8\pi(1-\nu)}\left(-\frac{1}{2}\right)\left(r^2+z^2\right)^{-1/2-1} 2r$$

$$= \frac{Pzr}{8\pi(1-\nu)\left(r^2+z^2\right)^{3/2}}$$

$$2G\, u_r = \frac{P z r}{8\pi(1-\nu)\, r^3 \dfrac{(r^2+z^2)^{3/2}}{r^3}}$$

$$= \frac{P z \, {}^1\!/_{r^2}}{8\pi(1-\nu)\left(\dfrac{r^2+z^2}{r^2}\right)^{3/2}}$$

$$= \frac{P z \, {}^1\!/_{r^2}}{8\pi(1-\nu)\left(1+\dfrac{z^2}{r^2}\right)^{3/2}}$$

$$\lim_{r\to\infty} 2G\, u_r = 0$$

This sol^n for u_r comes from $\vec{u}$ considering Fig. 2, and this in turn comes from the $\vec{u}$ of Fig. 1. And that $\vec{u}$ was obtained just from the particular solutions $\vec{B}^{(P)}$ and $\beta^{(P)}$. So, really, we have been considering is $\vec{u}^{(P)}$. This $\vec{u}^{(P)}$ is certainly Navier-Lamé eqns including the body force $\left(\vec{P}\,\delta(\vec{x}-\vec{\xi}_1)\right)$ and it turns out that this solution satisfies the far-field conditions for the original $\vec{u}$.

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