3D Elasticity

Previous chapter → 2D Elasticity

Plane Stress

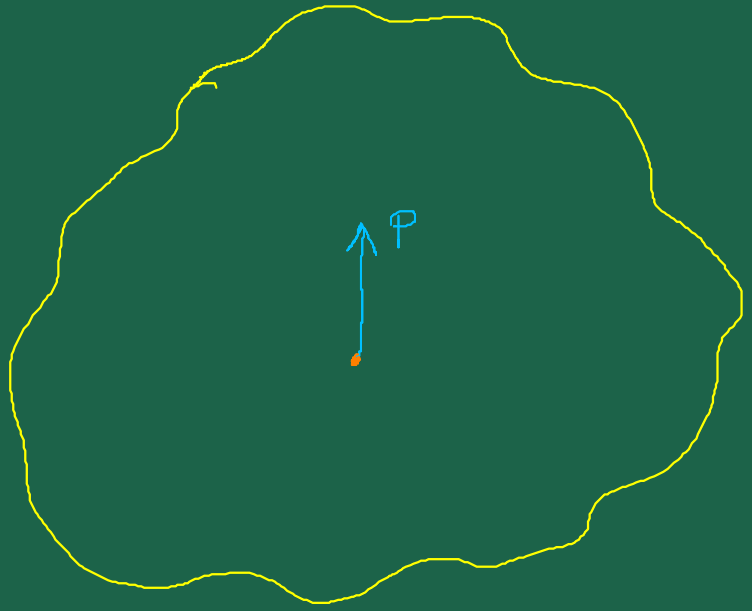
Plane Strain

} Airy Stress Function

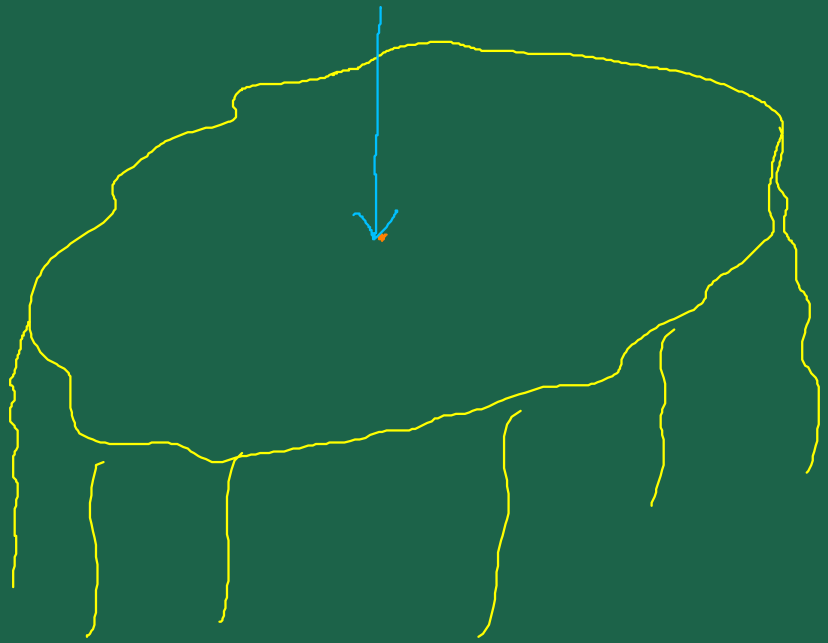
Method of Potentials

↳ Displacement Potentials

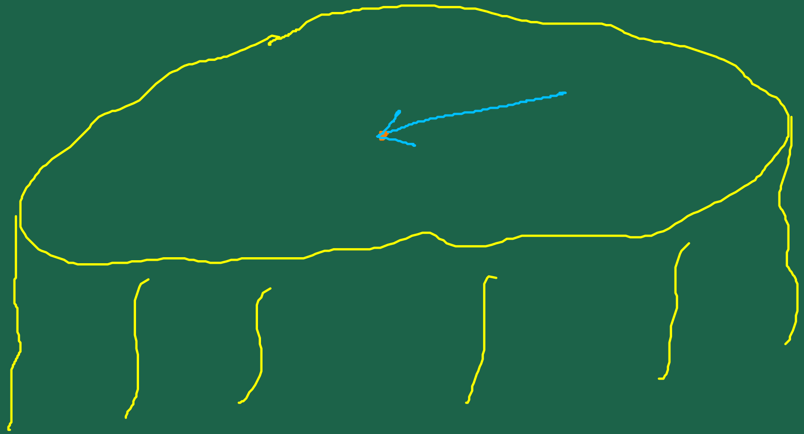
↳ Problems ?



Kelvin's Problem



Boussinesq



Cerruti

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In the displacement potential method:

Express the displacement field $\vec{u}$ in terms of some scalar and vector potentials.

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \vec{b} = 0$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

2nd order diff. eqn in terms of $\vec{u}$

3D Elasticity ... contd.

Helmholtz Representation Theorem: Any sufficiently smooth vector field can be represented as the sum of the gradient of a scalar potential and the curl of a vector potential.

$$\vec{u} = \underbrace{\nabla\phi}_{\text{Irrotational part}} + \underbrace{\nabla\times\vec{\Omega}}_{\text{Solenoidal part}} \; ; \quad \begin{array}{l}\phi : \text{scalar potential} \\ \vec{\Omega} : \text{vector potential}\end{array}$$

Irrotational part $\rightarrow$ $\underbrace{\nabla\times(\nabla\phi)=0}_{\text{Identically}}$

Solenoidal part $\rightarrow$ $\underbrace{\nabla\cdot(\nabla\times\vec{\Omega})=0}_{\text{Identically}}$

\# $\nabla\cdot\underset{\sim}{\sigma} + \rho\vec{b} = 0 \longrightarrow (\lambda+G)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} + \rho\vec{b} = 0$

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Omega}$$

$$\nabla\cdot\vec{u} = \nabla\cdot\nabla\phi + \nabla\cdot(\nabla\times\vec{\Omega})$$

$$\Rightarrow \boxed{\nabla\cdot\vec{u} = \nabla^2\phi}$$

$$\nabla\times\vec{u} = \nabla\times\nabla\phi + \nabla\times(\nabla\times\vec{\Omega})$$

$$= \nabla\times(\nabla\times\vec{\Omega})$$

vector identity:

$$\nabla\times(\nabla\times\vec{\Omega}) = \nabla(\nabla\cdot\vec{\Omega}) - \nabla^2\vec{\Omega} \qquad (\nabla^2 \equiv \nabla\cdot\nabla)$$

curl curl = grad div − div grad

Imp: Take $\boxed{\nabla\cdot\vec{\Omega} = 0}$

$$\therefore \nabla\times(\nabla\times\vec{\Omega}) = -\nabla^2\vec{\Omega}$$

$$\therefore \boxed{\nabla\times\vec{u} = -\nabla^2\vec{\Omega}}$$

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$$(\lambda+G)\nabla(\nabla\cdot\vec{u}) + G\nabla^2\vec{u} + \rho\vec{b} = 0$$

$$\nabla\times\nabla\times\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u}$$

$$\Rightarrow \nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times\nabla\times\vec{u}$$

$$(\lambda+G)\nabla(\nabla\cdot\vec{u}) + G\nabla\nabla\cdot\vec{u} - G\nabla\times\nabla\times\vec{u} + \rho\vec{b} = 0$$

$$\Rightarrow (\lambda+2G)\nabla(\nabla\cdot\vec{u}) - G\nabla\times\nabla\times\vec{u} + \rho\vec{b} = 0$$

$$\Rightarrow (\lambda+2G)\nabla\nabla^2\phi + G\nabla\times\nabla^2\vec{\Omega} + \rho\vec{b} = 0$$

Consider $\rho\vec{b} = 0$ (and don't forget $\nabla\cdot\vec{\Omega} = 0$)

$$(\lambda+2G)\,\nabla(\nabla^2\phi) + G\,\nabla\times\nabla^2\vec{\Omega} = 0 \quad \text{— } (\#)$$

Take the divergence of (#)

$$(\lambda+2G)\,\nabla\cdot\nabla(\nabla^2\phi) + G\,\underbrace{\nabla\cdot\left(\nabla\times\nabla^2\vec{\Omega}\right)}_{=0\ (\because\ \nabla\cdot\vec{\Omega}=0)} = 0$$

$$\Rightarrow \quad \nabla^2(\nabla^2\phi) = 0 \Rightarrow \boxed{\nabla^4\phi = 0} \quad \text{— } (\#\#_1)$$

Take the curl of (#)

$$(\lambda+2G)\,\underbrace{\nabla\times\nabla(\nabla^2\phi)}_{=0\ \text{identically}} + G\,\nabla\times\left(\nabla\times\nabla^2\vec{\Omega}\right) = 0$$

$$\Rightarrow \quad \nabla\times\nabla\times\nabla^2\vec{\Omega} = 0$$

$$\Rightarrow \quad \nabla\,\underbrace{\nabla\cdot(\nabla^2\vec{\Omega})}_{=0} - \nabla^2(\nabla^2\vec{\Omega}) = 0 \Rightarrow \boxed{\nabla^4\vec{\Omega} = 0} \quad (\#\#_2)$$

3D Elasticity ... contd.

Lamé's Potential

$$\vec{u} = \frac{1}{2G}\nabla\phi \quad \longrightarrow \quad \nabla\cdot\vec{u} = \frac{1}{2G}\nabla^2\phi$$

$$\nabla\times\vec{u} = \frac{1}{2G}\nabla\times\nabla\phi = 0 \text{ (identically)}$$

$\therefore$ Lamé's potential is strictly limited to irrotational displacement fields.

$$\nabla\cdot\underset{\sim}{\sigma} + \rho\vec{b} = 0$$

$$\Rightarrow (\lambda+G)\nabla\nabla\cdot\vec{u} + G\nabla^2\vec{u} + \rho\vec{b} = 0$$

$$\Rightarrow (\lambda+2G)\nabla\nabla\cdot\vec{u} - G\nabla\times\underbrace{\nabla\times\vec{u}}_{=0} + \rho\vec{b} = 0$$

$$\Rightarrow (\lambda+2G)\nabla(\nabla^2\phi) + \rho\vec{b} = 0$$

Consider $\rho_b = 0$

$\therefore \nabla(\nabla^2\phi) = 0$

In rectangular Cartesian coordinate system

$$\frac{\partial}{\partial x}(\nabla^2\phi) = 0 \Rightarrow \nabla^2\phi = f(y,z)$$

$$\frac{\partial}{\partial y}(\nabla^2\phi) = 0 \Rightarrow \frac{\partial}{\partial y}\left(f(y,z)\right) = 0 \Rightarrow f(y,z) \equiv g(z)$$

$$\frac{\partial}{\partial z}(\nabla^2\phi) = 0 \Rightarrow \frac{\partial}{\partial z}\left(g(z)\right) = 0 \Rightarrow g(z) \equiv c \text{ (a const.)}$$

$\therefore \nabla^2\phi = c$ (In the form of Poisson's eqn.)

$\nabla^2\phi = 0$

3D Elasticity ... contd.

Helmholtz displacement potential
Lamé's strain potential

} $\vec{u}$ in terms of 1st order derivatives of potentials

Galerkin potential

$$\vec{u} = \frac{1}{2G}\left[2(1-\nu)\nabla^2\vec{V} - \nabla(\nabla\cdot\vec{V})\right] \quad ; \quad \vec{V} : \text{Galerkin vector potential}$$

$$(\lambda+G)\nabla\nabla\cdot\vec{u} + G\nabla^2\vec{u} + \rho\vec{b} = 0$$

$$\Rightarrow \frac{\lambda+G}{2G}\nabla\left[2(1-\nu)\nabla^2(\nabla\cdot\vec{V}) - \nabla^2(\nabla\cdot\vec{V})\right] + \frac{1}{2}\left[2(1-\nu)\nabla^4\vec{V} - \nabla\nabla^2(\nabla\cdot\vec{V})\right] + \rho\vec{b} = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{\lambda}{G}+1\right)(1-2\nu)\nabla\nabla^2(\nabla\cdot\vec{V}) + (1-\nu)\nabla^4\vec{V} - \frac{1}{2}\nabla\nabla^2(\nabla\cdot\vec{V}) + \rho\vec{b} = 0$$

$$\frac{\lambda}{G}+1=\frac{\cancel{E}\nu}{\cancel{(1+\nu)}(1-2\nu)}\,\frac{2\cancel{(1+\nu)}}{\cancel{E}}+1=\frac{2\nu}{1-2\nu}+1=\frac{1}{1-2\nu}$$

$$\therefore\ \cancel{\frac{1}{2}\nabla\nabla(\nabla\cdot\vec{v})}+(1-\nu)\nabla^4\vec{v}-\cancel{\frac{1}{2}\nabla\nabla(\nabla\cdot\vec{v})}+\rho\vec{b}=0$$

$$\Rightarrow\quad \nabla^4\vec{v}=-\frac{\rho\vec{b}}{1-\nu}$$

Consider $\rho\vec{b}=0$

$$\therefore\ \boxed{\nabla^4\vec{v}=0}$$

3D Elasticity ... contd.

Papkovich – Neuber Representation

Motivation: # To include body forces
To set up the framework in the form of Poisson's eqns.

When the Helmholtz displacement potential was substituted in the Navier-Lamé equations, we had obtained the following:

$$(\lambda+2G)\nabla\nabla^2\varphi + G\nabla\times\nabla^2\vec{\Omega} + \rho\vec{b} = 0$$

$$\lambda+2G = \frac{E\nu}{(1+\nu)(1-2\nu)} + 2G = \frac{2G\nu}{1-2\nu} + 2G = \frac{2G(\nu+1-2\nu)}{1-2\nu} = \frac{2G(1-\nu)}{1-2\nu}$$

$$\frac{2G(1-\nu)}{1-2\nu}\nabla\nabla^2\varphi + G\nabla\times\nabla^2\vec{\Omega} + \rho\vec{b} = 0$$

$$\Rightarrow \frac{G}{1-2\nu}\nabla\nabla^2\varphi + \frac{G}{2(1-\nu)}\nabla\times\nabla^2\vec{\Omega} + \frac{\rho\vec{b}}{2(1-\nu)} = 0$$

$$\Rightarrow \nabla^2\underbrace{\left[\frac{G}{1-2\nu}\nabla\varphi + \frac{G}{2(1-\nu)}\nabla\times\vec{\Omega}\right]}_{-\vec{B}} = -\frac{\rho\vec{b}}{2(1-\nu)}$$

set $\vec{B} = -\frac{G}{1-2\nu}\nabla\varphi - \frac{G}{2(1-\nu)}\nabla\times\vec{\Omega}$ — ①

$$\therefore \boxed{\nabla^2\vec{B} = \frac{\rho\vec{b}}{2(1-\nu)}} \quad — ②$$

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$$\nabla^2(\vec{x}\cdot\vec{B}) = \vec{x}\cdot\nabla^2\vec{B} + 2\nabla\cdot\vec{B}$$

$$\Rightarrow \vec{x}\cdot\nabla^2\vec{B} = \nabla^2(\vec{x}\cdot\vec{B}) - 2\nabla\cdot\vec{B} \quad \text{--- } (3)$$

$$\vec{x}\cdot\nabla^2\vec{B} = \frac{\vec{x}\cdot(\rho\vec{b})}{2(1-\nu)} \quad (4) \ (\text{Using } (2))$$

$$\nabla\cdot\vec{B} = -\frac{G}{1-2\nu}\nabla^2\phi \quad (5) \ (\text{Using } (1) \ \& \ \nabla\cdot\nabla\times\vec{\Omega}=0)$$

Sub. (4) and (5) in (3)

$$\frac{\vec{x}\cdot(\rho\vec{b})}{2(1-\nu)} = \nabla^2(\vec{x}\cdot\vec{B}) + \frac{2G\nabla^2\phi}{1-2\nu}$$

$$\Rightarrow \frac{\vec{x}\cdot(\rho\vec{b})}{2(1-\nu)} = \nabla^2\Big[\underbrace{\vec{x}\cdot\vec{B} + \frac{2G\phi}{1-2\nu}}_{-\beta}\Big]$$

$$\text{Set } \beta = -\vec{x}\cdot\vec{B} - \frac{2G\phi}{1-2\nu} \quad — ⑥$$

$$\boxed{\nabla^2 \beta = -\frac{\vec{x}\cdot(\rho\vec{b})}{2(1-\nu)}} \quad — ⑦$$

Both eqns ② and ⑦ are Poisson eqns (satisfying the initial motivation!)

$$\vec{u} = \nabla\phi + \nabla\times\vec{\Omega} \quad \text{(Helmholtz displacement potential)}$$

Going back to Eq. (1)

$$\vec{B} = -\frac{G\nabla\phi}{1-2\nu} - G\frac{\nabla\times\vec{\Omega}}{2(1-\nu)}$$

$$\Rightarrow \vec{B} = -G\frac{\nabla\phi}{1-2\nu} - \frac{G(\vec{u}-\nabla\phi)}{2(1-\nu)}$$

$$\Rightarrow \vec{B} = +G\nabla\phi\left[-\frac{1}{1-2\nu} + \frac{1}{2(1-\nu)}\right] - \frac{G\vec{u}}{2(1-\nu)}$$

$$\Rightarrow \vec{u} = \nabla\phi\left[-\frac{2(1-\nu)}{1-2\nu} + 1\right] - \frac{2(1-\nu)}{G}\vec{B}$$

$$\Rightarrow \vec{u} = \nabla\phi\left[\frac{-2+\cancel{2\nu}+1-\cancel{2\nu}}{1-2\nu}\right] - \frac{2(1-\nu)}{G}\vec{B}$$

$$\Rightarrow \vec{u} = \frac{-1}{1-2\nu}\nabla\phi - \frac{2(1-\nu)}{G}\vec{B} \quad \text{--- } (8)$$

Going back to Eq (6)

$$\beta = -\vec{x}\cdot\vec{B} - \frac{2G\phi}{1-2\nu}$$

$$\Rightarrow \nabla\beta = -\nabla(\vec{x}\cdot\vec{B}) - \frac{2G\nabla\phi}{1-2\nu}$$

$$\Rightarrow -\frac{\nabla\phi}{1-2\nu} = \frac{1}{2G}\left[\nabla\beta + \nabla(\vec{x}\cdot\vec{B})\right] \quad \text{--- } (9)$$

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Sub. (9) in (8)

$$\vec{u} = \frac{1}{2G}\nabla\left[\beta + \vec{x}\cdot\vec{B}\right] - \frac{2(1-\nu)}{G}\vec{B}$$

$$\boxed{\vec{u} = \frac{1}{2G}\left[\nabla\left(\beta + \vec{x}\cdot\vec{B}\right) - 4(1-\nu)\vec{B}\right]} \quad \text{--- (10)}$$

→ Papkovich-Neuber Representation

where $\vec{B}$ and β are given by Eqns (2) and (7).

3D Elasticity ... contd.

Proof for the vector identity:

$$\nabla^2(\vec{x}\cdot\vec{B}) = \vec{x}\cdot\nabla^2\vec{B} + 2\,\nabla\cdot\vec{B}$$

$$\nabla^2(\vec{x}\cdot\vec{B}) = \nabla\cdot\nabla(\vec{x}\cdot\vec{B})$$

$$\nabla(\vec{x}\cdot\vec{B}) = \frac{\partial}{\partial x_i}(x_K B_K) = \frac{\partial x_K}{\partial x_i}B_K + x_K\frac{\partial B_K}{\partial x_i} = \delta_{Ki}B_K + x_K\frac{\partial B_K}{\partial x_i}$$

$$= B_i + x_K\frac{\partial B_K}{\partial x_i}$$

$$\nabla\cdot\nabla(\vec{x}\cdot\vec{B}) = \frac{\partial}{\partial x_i}\left(B_i + x_K\frac{\partial B_K}{\partial x_i}\right)$$

$$= \frac{\partial B_i}{\partial x_i} + \frac{\partial x_K}{\partial x_i}\frac{\partial B_K}{\partial x_i} + x_K\frac{\partial^2 B_K}{\partial x_i^2}$$

$$= \frac{\partial B_i}{\partial x_i} + \delta_{Ki}\frac{\partial B_K}{\partial x_i} + x_K\frac{\partial^2 B_K}{\partial x_i^2} = \frac{\partial B_i}{\partial x_i} + \frac{\partial B_i}{\partial x_i} + x_K\frac{\partial^2 B_K}{\partial x_i^2}$$

$$\therefore \nabla^2(\vec{x}\cdot\vec{B}) = 2\frac{\partial B_i}{\partial x_i} + x_k\frac{\partial^2 B_k}{\partial x_i^2}$$

$$\Rightarrow \boxed{\nabla^2(\vec{x}\cdot\vec{B}) = 2\,\nabla\cdot\vec{B} + \vec{x}\cdot\nabla^2\vec{B}}$$

Hence, proved

$$\frac{\partial^2 B_k}{\partial x_i^2} = \nabla^2\vec{B}$$

$$\frac{\partial^2}{\partial x_i\,\partial x_i}(B_k)$$