2D Elasticity

Plane Stress

\# Domain bounded by two parallel planes

\# Distance between these two planes is very small compared to other dimensions

\# The two planes are at $z = \pm h$

\# The two planes are stress-free: $\sigma_{zz} = \sigma_{zx} = \sigma_{yz} = 0$

\# There is very little variation in σ_{zz}, σ_{zx}, σ_{yz} through the thickness; in fact these will be approximated to be zero

\# Because of the very small thickness, the non-zero components will have very little variation in the z-direction

$$\sigma_{xx} \equiv \sigma_{xx}(x,y) \; ; \quad \sigma_{yy} \equiv \sigma_{yy}(x,y) \; ; \quad \sigma_{xy} \equiv \sigma_{xy}(x,y)$$

\# In order to have these stresses to be independent of z,we must not have any body forces or tractions in the z-direction

\# Non-zero body forces or tractions must be independent of z

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From constitutive law

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}})\right] \quad — ①$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \cancel{\sigma_{zz}})\right] \quad — ②$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\cancel{\sigma_{zz}} - \nu(\sigma_{xx} + \sigma_{yy})\right] \quad — ③$$

$$\varepsilon_{xy} = \frac{1}{2G}\sigma_{xy} = \frac{1+\nu}{E}\sigma_{xy} \quad — ④$$

$$\varepsilon_{yz} = \frac{1}{2G}\cancel{\sigma_{yz}} = 0$$

$$\varepsilon_{zx} = \frac{1}{2G}\cancel{\sigma_{zx}} = 0$$

All strains are independent of z

Stress eqb. eqns

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\cancel{\sigma_{zx}}}{\partial z} = 0$$

$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\cancel{\sigma_{yz}}}{\partial z} = 0$$

$$\frac{\partial\cancel{\sigma_{zx}}}{\partial x} + \frac{\partial\cancel{\sigma_{yz}}}{\partial y} + \frac{\partial\cancel{\sigma_{zz}}}{\partial z} = 0 \rightarrow 0=0$$

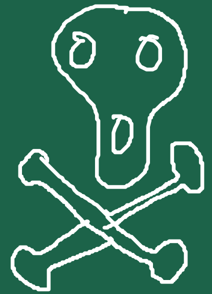
$$\left.\begin{aligned} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\}$$

Airy Stress Function

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

→ Subs. in stress eqb. eqns → identically satisfies them!

But we must be careful in choosing ϕ. For some arbitrary choice of ϕ it may turn out that the strains are incompatible!



→ This is ensured by actually including in the mathematical framework of the Airy Stress Function, the compatibility eqns.

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad ✓$$

$$\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z^2}} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2\frac{\cancel{\partial^2 \varepsilon_{yz}}}{\partial y \partial z} \Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 0 \quad ✓$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \cancel{\frac{\partial^2 \varepsilon_{xx}}{\partial z^2}} = 2\frac{\cancel{\partial^2 \varepsilon_{zx}}}{\partial z \partial x} \Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = 0 \quad ✓$$

$$\cancel{\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z}} = \frac{\partial}{\partial x}\left(-\frac{\cancel{\partial \varepsilon_{yz}}}{\partial x} + \frac{\cancel{\partial \varepsilon_{zx}}}{\partial y} + \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}}\right) \Rightarrow 0 = 0$$

$$\cancel{\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x}} = \frac{\partial}{\partial y}\left(\frac{\cancel{\partial \varepsilon_{yz}}}{\partial x} - \frac{\cancel{\partial \varepsilon_{zx}}}{\partial y} + \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}}\right) \Rightarrow 0 = 0$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\cancel{\partial \varepsilon_{yz}}}{\partial x} + \frac{\cancel{\partial \varepsilon_{zx}}}{\partial y} - \cancel{\frac{\partial \varepsilon_{xy}}{\partial z}}\right) \Rightarrow \frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = 0 \quad ✓$$

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$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad \text{— } (5)$$

Using (1), (2) & (4) in (5):

$$\frac{1}{E}\frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu \sigma_{yy}) + \frac{1}{E}\frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu \sigma_{xx}) = 2\frac{1+\nu}{E}\frac{\partial^2}{\partial x \partial y}(\sigma_{xy})$$

$$\Rightarrow \left(-\nu\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{xx} + \left(\frac{\partial^2}{\partial x^2} - \nu\frac{\partial^2}{\partial y^2}\right)\sigma_{yy} = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow -(1+\nu)\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{xx} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{yy} - (1+\nu)\frac{\partial^2 \sigma_{yy}}{\partial y^2} = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow -(1+\nu)\left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right) + \nabla^2(\sigma_{xx} + \sigma_{yy}) = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad \text{— } (6)$$

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$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial F_x}{\partial x} \quad — (7)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \Rightarrow \frac{\partial^2 \sigma_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = -\frac{\partial F_y}{\partial y} \quad — (8)$$

(7) + (8)

$$\Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y} \quad — (9)$$

Use (9) in (6)

$$-(1+\nu)\left(-2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} - \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y}\right) + \nabla^2(\sigma_{xx} + \sigma_{yy}) = 2(1+\nu)\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow \nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad — (10)$$

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We consider body forces that are conservative in nature

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y} \qquad \left[\vec{F} = -\nabla V\right]$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial V}{\partial x} = 0 \Rightarrow \frac{\partial}{\partial x}(\sigma_{xx} - V) + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (11)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial}{\partial y}(\sigma_{yy} - V) = 0 \quad (12)$$

$$\sigma_{xx} - V = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} - V = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

$$(10) \rightarrow \nabla^2(\sigma_{xx} + \sigma_{yy}) = (1+\nu)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

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$$\nabla^2\left(V+\frac{\partial^2\phi}{\partial y^2}+V+\frac{\partial^2\phi}{\partial x^2}\right)=(1+\nu)\nabla^2 V$$

$$\Rightarrow\quad 2\nabla^2 V+\nabla^2\left(\nabla^2\phi\right)=(1+\nu)\nabla^2 V$$

$$\Rightarrow\quad \nabla^4\phi=-(1-\nu)\nabla^2 V \quad — \quad 13$$

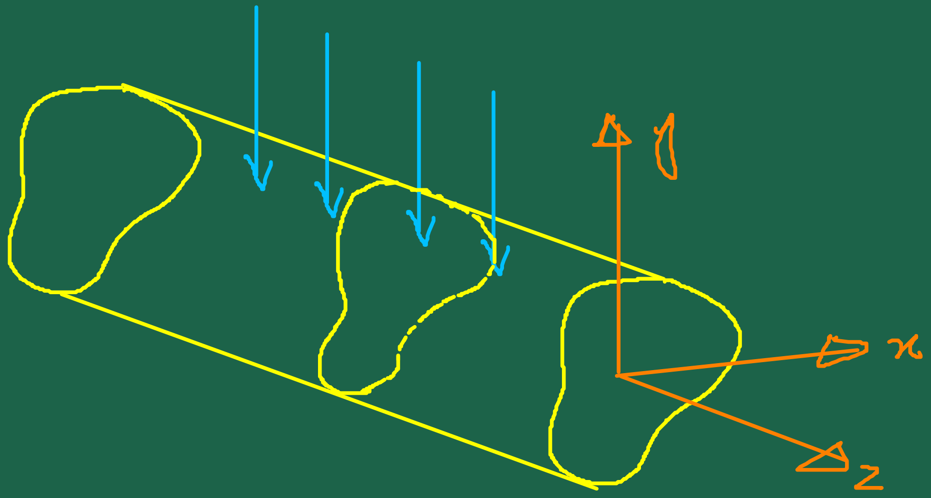
∇^4 : Biharmonic operator

If body forces vanish or $\nabla^2 V=0$, then

$$\nabla^4\phi=0 \rightarrow \text{Biharmonic equation}$$

$\hookrightarrow$ 14

Plane strain



If body forces and tractions are uniform along z and do not have any z-component, then $u \equiv u(x,y)$, $v \equiv v(x,y)$, $w \equiv 0$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \leftarrow$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right) = 0$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial y}} + \cancel{\frac{\partial v}{\partial z}}\right) = 0$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \Rightarrow \varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \nu\sigma_{xx} + \nu\sigma_{yy}\right)\right]$$

$$= \frac{1}{E}\left[(1-\nu^2)\sigma_{xx} - \nu(1+\nu)\sigma_{yy}\right]$$

$$= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \quad — (1)$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] \quad — (2)$$

$$\cancelto{0}{\varepsilon_{zz}} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\Rightarrow \sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right) \quad — (3)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy} \quad — (4)$$

$$\cancelto{0}{\varepsilon_{yz}} = \frac{1+\nu}{E}\sigma_{yz} \Rightarrow \sigma_{yz} = 0$$

$$\cancelto{0}{\varepsilon_{zx}} = \frac{1+\nu}{E}\sigma_{zx} \Rightarrow \sigma_{zx} = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \cancel{\sigma_{zx}}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \cancel{\sigma_{yz}}}{\partial z} = 0$$

$$\frac{\partial \cancel{\sigma_{zx}}}{\partial x} + \frac{\partial \cancel{\sigma_{xz}}}{\partial y} + \cancel{\frac{\partial \sigma_{zz}}{\partial z}} = 0$$

Compatibility eqns

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \quad — (5)$$

Use (1), (2), (4) in (5)

$$\Rightarrow \frac{1+\nu}{E}\frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] + \frac{1+\nu}{E}\frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_{yy} - \nu\sigma_{xx}\right] = 2\frac{1+\nu}{E}\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow (1-\nu)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{xx} - \nu\frac{\partial^2 \sigma_{yy}}{\partial y^2} + (1-\nu)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{yy} - \nu\frac{\partial^2 \sigma_{xx}}{\partial x^2}$$

$$- (1-\nu)\frac{\partial^2 \sigma_{xx}}{\partial x^2} - (1-\nu)\frac{\partial^2 \sigma_{yy}}{\partial y^2} = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow (1-\nu)\nabla^2(\sigma_{xx} + \sigma_{yy}) - \left(\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2}\right) = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \quad — (6)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial F_x}{\partial x} \quad \text{— (7)}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \Rightarrow \frac{\partial^2 \sigma_{xy}}{\partial y \partial x} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = -\frac{\partial F_y}{\partial y} \quad \text{— (8)}$$

(7) + (8)

$$\Rightarrow \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad \text{— (9)}$$

Use (9) in (6)

$$(1-\nu)\nabla^2(\sigma_{xx} + \sigma_{yy}) - \left(-2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} - \frac{\partial F_x}{\partial x} - \frac{\partial F_y}{\partial y}\right) = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}$$

$$\Rightarrow (1-\nu)\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad \text{— (10)}$$

Consider the stress eqb. eqns again but with conservative body forces

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial V}{\partial x} = 0 \Rightarrow \frac{\partial}{\partial x}\left(\sigma_{xx} - V\right) + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial V}{\partial y} = 0 \Rightarrow \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial}{\partial y}\left(\sigma_{yy} - V\right) = 0$$

$$\sigma_{xx} - V = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{yy} - V = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

Use in (10):

$$(1-\nu)\nabla^2\left(V + \frac{\partial^2 \phi}{\partial y^2} + V + \frac{\partial^2 \phi}{\partial x^2}\right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

$$= \nabla^2 V$$

$$\Rightarrow 2(1-\nu)\nabla^2 V + (1-\nu)\nabla^2\left(\nabla^2 \varphi\right) = \nabla^2 V$$

$$\Rightarrow \quad (1-\nu)\nabla^4\phi = (1-2+2\nu)\nabla^2 V$$

$$\Rightarrow \quad \nabla^4\phi = -\frac{1-2\nu}{1-\nu}\nabla^2 V \rightarrow \text{Plane Strain}$$

Earlier for Plane Stress, we had obtained

$$\nabla^4\phi = -(1-\nu)\nabla^2 V$$

When body forces vanish or $\nabla^2 V = 0$,

$$\boxed{\nabla^4\phi = 0} \rightarrow \text{True for Plane Strain as it was for Plane Stress}$$

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Polynomial solutions to the biharmonic equation

$$\nabla^4\phi = 0 \rightarrow \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\phi = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}x^m y^n = A_{00} + A_{10}x + A_{01}y + A_{11}xy + A_{20}x^2 + A_{02}y^2 + \cdots$$

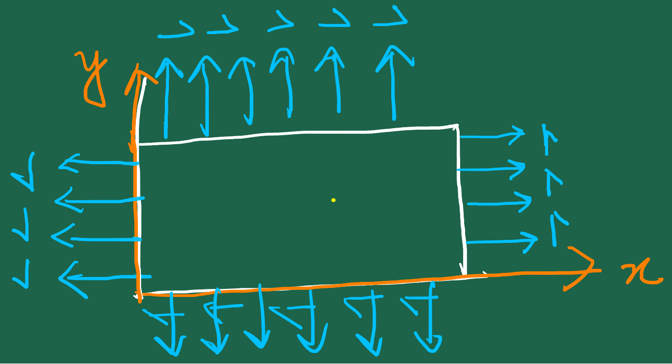
$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}, \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}, \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

Terms with $m+n \leq 1 \rightarrow$ do not contribute stresses

Second-degree terms will produce a const. stress field

$$\phi = A_{11}xy + A_{20}x^2 + A_{02}y^2$$

$$\sigma_{xx} = 2A_{02}, \quad \sigma_{yy} = 2A_{20}, \quad \sigma_{xy} = -A_{11}$$



Third degree terms will produce linear stress field

$$\phi = A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3$$

$$\sigma_{xx} = 2A_{12}x + 6A_{03}y$$

$$\sigma_{yy} = 6A_{30}x + 2A_{21}x$$

$$\sigma_{xy} = -2A_{21}x - 2A_{12}y$$

Consider the simple special case

$$A_{12} = A_{21} = A_{30} = 0$$

$$\sigma_{xx} = 6A_{03}y$$

$$\sigma_{yy} = 0$$

$$\sigma_{xy} = 0$$

Terms with $m+n \leq 3$ identically satisfy the biharmonic eqn

For $m+n > 3$, special conditions involving the coefficients will have to be found.

Principle of Superposition

$$\nabla^4 \phi = 0$$

$$\phi_{I} \qquad \phi_{II}$$

$$\phi_{soln} = \phi_{I} + \phi_{II}$$

Works because of the linear nature of the biharmonic eqn.

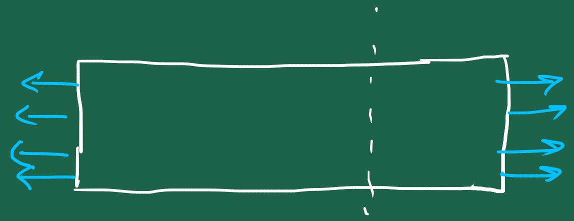
$\phi \dfrac{\partial^2 \phi}{\partial x^2} \rightarrow$ Non-lin.

$\dfrac{\partial^4 \phi}{\partial x^4} \rightarrow$ Linear

$\dfrac{\partial^4 \phi}{\partial x^2 \partial y^2} \rightarrow$ Linear

$\dfrac{\partial^2 \phi}{\partial x^2} \dfrac{\partial^2 \phi}{\partial y^2} \rightarrow$ Non-Lin

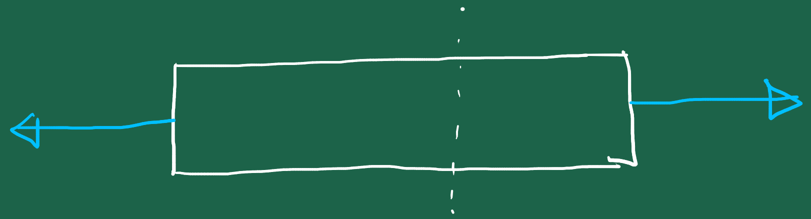
St. Venant's Principle



σ_{xx}

$$\phi = A_{02}\, y^2$$

$$\sigma_{xx} = 2A_{02}, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0$$



T

$$T = \int_A \sigma_{xx}\, dA$$

For statically equivalent loadings the stress field is going to be the same as long as we stay from the ends.