## **2D Elasticity – I** \*

## 1 Plane stress

The salient features of a plane stress situation are:

- Domain bounded by two parallel planes
- Distance between two planes is small compared to other dimensions
- When referred to a rectangular Cartesian coordinate system, it is usual to take the two planes to be represented by  $z = \pm h$  such that the *very small* distance between the two planes is 2h, and the x y plane (denoting the mid-surface) is used to describe the problem.
- The two planes z = h and z = -h are stress-free so that  $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$  on each face.
- It is expected that there will be very little variation in  $\sigma_{zz}$ ,  $\sigma_{xz}$ , and  $\sigma_{yz}$  through the thickness; hence, they are approximately taken as zero throughout the entire domain.
- Again because of the very small thickness, the non-zero stress components will have little variation with *z*; they are approximately taken to be *independent* of *z*. Thus,

$$\sigma_{xx} \equiv \sigma_{xx}(x, y), \sigma_{yy} \equiv \sigma_{yy}(x, y), \sigma_{xy} \equiv \sigma_{xy}(x, y).$$

• There can be no body forces in the *z*-direction; this follows immediately from the *z*-component of the mechanical equilibrium equation.

From constitutive equation of a linear, elastic, isotropic solid, we have

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for the plane stress situation:

$$\varepsilon_{xx} = \frac{1}{E} \left( \sigma_{xx} - \nu \sigma_{yy} \right), \tag{1a}$$

$$\varepsilon_{yy} = \frac{1}{E} \left( \sigma_{yy} - \nu \sigma_{xx} \right), \tag{1b}$$

$$\varepsilon_{zz} = -\frac{\nu}{E} \left( \sigma_{xx} + \sigma_{yy} \right), \qquad (1c)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy},\tag{1d}$$

$$\varepsilon_{\gamma z} = 0,$$
 (1e)

$$\varepsilon_{zx} = 0. \tag{1f}$$

Adding (1a) and (1b) and substituting in (1c), we have

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu} \left( \varepsilon_{xx} + \varepsilon_{yy} \right). \tag{2}$$

## NOTE:

- Even though  $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ , we still have  $\varepsilon_{zz} \neq 0$ .
- All strains are independent of *z*.

Now, consider the compatibility equations<sup>†</sup>:

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y},$$
(3a)

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = 2 \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z},$$
(3b)

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} = 2 \frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x},$$
(3c)

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{xy}}{\partial z} \right), \qquad (3d)$$

$$\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y} \left( -\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{yy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} \right), \qquad (3e)$$

$$\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z} \left( -\frac{\partial \varepsilon_{yy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} \right).$$
(3f)

<sup>&</sup>lt;sup>†</sup>Refer to Problem Sheet 1 on "Kinematics".

The only compatibility equation that is considered is (3a). The other three compatibility equations involving  $\varepsilon_{zz}$  are neglected.

Now, using (1a) and (1b) in (3a), we have, after rearranging

$$\left(-\nu\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\sigma_{xx} + \left(\frac{\partial^2}{\partial x^2} - \nu\frac{\partial^2}{\partial y^2}\right)\sigma_{yy} = 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}.$$
 (4)

Adding and subtracting  $\frac{\partial^2 \sigma_{xx}}{\partial x^2}$  and  $\frac{\partial^2 \sigma_{yy}}{\partial y^2}$ , we have

$$-(1+\nu)\left(\frac{\partial^2\sigma_{xx}}{\partial x^2}+\frac{\partial^2\sigma_{yy}}{\partial y^2}\right)+\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y}.$$
 (5)

Consider the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0, \tag{6a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0, \tag{6b}$$

Differentiating the first of the preceding equations with respect to x and the second with respect to y, we have

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial F_x}{\partial x},$$
(7a)

$$\frac{\partial^2 \sigma_{xy}}{\partial \gamma \partial x} + \frac{\partial^2 \sigma_{yy}}{\partial \gamma^2} = -\frac{\partial F_y}{\partial \gamma}.$$
 (7b)

Adding these two equations, we obtain

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right).$$
(8)

Substituting (8) in (5), we have

$$\nabla^2 \left( \sigma_{xx} + \sigma_{yy} \right) = -(1+\nu) \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right). \tag{9}$$

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We consider the equilibrium equations agains but assume the body forces to be conservative in nature so that they are expressed as the gradient of a scalar potential, i.e.  $F_x = -\frac{\partial V}{\partial x}$  and  $F_y = -\frac{\partial V}{\partial y}$ , so that

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial V}{\partial x} = 0, \qquad (10a)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial V}{\partial y} = 0.$$
(10b)

We define the following:

$$\sigma_{xx} - V = \frac{\partial^2 \varphi}{\partial \gamma^2},\tag{11a}$$

$$\sigma_{yy} - V = \frac{\partial^2 \varphi}{\partial x^2},\tag{11b}$$

$$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y},\tag{11c}$$

so that (10a) and (10b) are satisfied identically.

Using the expressions of  $\sigma_{xx}$  and  $\sigma_{yy}$  in Eq. (9), we have

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(V + \frac{\partial^2 \varphi}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \left(V + \frac{\partial^2 \varphi}{\partial x^2}\right) = (1+\nu) \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$
  
or,  $2\nabla^2 V + \nabla^2 (\nabla^2 \varphi) = (1+\nu) \nabla^2 V$ ,  
or,  $\nabla^4 \varphi = -(1-\nu) \nabla^2 V$  (12)

Note that  $\nabla^4$  is referred to as the biharmonic operator.

If the body forces vanish or if  $\nabla^2 V = 0$  then we have

$$\nabla^4 \varphi = 0. \tag{13}$$

This equation is referred to as the biharmonic equation, and its solutions are referred to as the biharmonic solutions.

## 2 Plane Strain

Consider an infinitely long cylindrical (prismatic) body

• If body forces and tractions on the lateral boundaries are independent of the longitudinal axis variable, *z*, and have no *z*-component then the deformation field can be taken as

$$u \equiv u(x, y), \quad v \equiv v(x, y)$$

This deformation is referred to as a state of plane strain in the x - y plane.

• All cross sections will have identical displacements and thus the 3D problem is reduced to a 2D problem.

Considering the preceding points, we have

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right),$$
$$\varepsilon_{zz} = \varepsilon_{zx} = \varepsilon_{yz} = 0.$$

The only compatibility equation that does not reduce to a 0 = 0 form is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}.$$
 (14)

Considering the constitutive equation for a linear, elastic, isotropic material we have

$$\varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{yy} + \sigma_{zz} \right) \right], \tag{15a}$$

$$\varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu \left( \sigma_{zz} + \sigma_{xx} \right) \right], \tag{15b}$$

$$\varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu \left( \sigma_{xx} + \sigma_{yy} \right) \right] \implies \sigma_{zz} = \nu \left( \sigma_{xx} + \sigma_{yy} \right), \quad (15c)$$

$$\varepsilon_{xy} = \frac{1+\nu}{E} \sigma_{xy},\tag{15d}$$

$$\sigma_{yz} = \sigma_{zx} = 0$$
 (Since  $\varepsilon_{yz} = \varepsilon_{zx} = 0$ ) (15e)

Note that unlike the plane stress case, in the plane strain case we have  $\sigma_{zz} \neq 0$ .

Substituting the expression of  $\sigma_{zz}$  in the expressions for  $\varepsilon_{xx}$  and  $\varepsilon_{yy}$ , we have

$$\varepsilon_{xx} = \frac{1+\nu}{E} \left[ (1-\nu)\sigma_{xx} - \nu\sigma_{yy} \right], \qquad (16a)$$

$$\varepsilon_{yy} = \frac{1+\nu}{E} \left[ (1-\nu)\sigma_{yy} - \nu\sigma_{xx} \right].$$
(16b)

We next use these expressions for  $\varepsilon_{xx}$  and  $\varepsilon_{yy}$  (from Eq. (16)(a) and (b)) as well as for  $\varepsilon_{xy}$  (from Eq. (15)(c)) in the compatibility equation, Eq. (14), to obtain

$$\frac{1+\nu}{E}\frac{\partial^2}{\partial y^2}\left[(1-\nu)\sigma_{xx}-\nu\sigma_{yy}\right] + \frac{1+\nu}{E}\frac{\partial^2}{\partial x^2}\left[(1-\nu)\sigma_{yy}-\nu\sigma_{xx}\right] = 2\frac{1+\nu}{E}\frac{\partial^2\sigma_{xy}}{\partial x\partial y}$$
(17)

or, 
$$(1-\nu)\nabla^2 \left(\sigma_{xx} + \sigma_{yy}\right) - \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y}.$$
 (18)

Considering the equilibrium equations, we have

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0 \implies \frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial f_x}{\partial x} = 0, \quad (19a)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \implies \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial f_y}{\partial y} = 0.$$
(19b)

Partially differentiating the first of the preceding equations with respect to *x* and the second with respect to *y*, and adding them, we obtain

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right). \tag{20}$$

Using Eq. (20) in Eq. (18), we obtain

$$(1-\nu)\nabla^2\left(\sigma_{xx}+\sigma_{yy}\right)=-\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right).$$
(21)

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Proceeding similarly as in the plane stress case we consider a conservative body force per unit volume, we set

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} + V, \qquad (22a)$$

$$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} + V, \qquad (22b)$$

$$\sigma_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y},\tag{22c}$$

where V is a scalar potential such that  $f_x = -\frac{\partial V}{\partial x}$  and  $f_y = -\frac{\partial V}{\partial y}$ .

Using these expressions in Eq. (21), we have

$$(1-\nu)\nabla^{2}\left(\frac{\partial^{2}\varphi}{\partial y^{2}}+V+\frac{\partial^{2}\varphi}{\partial x^{2}}+V\right)=\nabla^{2}V,$$
  
or, 
$$\nabla^{4}\varphi=-\frac{1-2\nu}{1-\nu}\nabla^{2}V.$$
 (23)

Again, as in the plane stress case, if the body forces vanish or if  $\nabla^2 V = 0$  then we have

$$\nabla^4 \varphi = 0. \tag{24}$$

Summary for plane stress and plane strain cases:

Plane stress: 
$$\nabla^4 \varphi = -(1-\nu)\nabla^2 V$$
, (25)

Plane strain: 
$$\nabla^4 \varphi = -\frac{1-2\nu}{1-\nu} \nabla^2 V.$$
 (26)