Problem Sheet 2: Beam Theory

1. The following steps will show that the flexure formula from the engineering theory of beams is exact provided the bending moment M is a constant or a linear function of x along the beam (considered prismatic with a constant second moment of area I). In the x-component of the stress equilibrium equations, use the flexure formula $\sigma_{xx} = My/I$ and the expressions of the shear stresses from the linear elastic isotropic constitutive law: $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$ (here, λ and G are constants) to obtain:

$$\frac{y}{I}\frac{\partial M}{\partial x} + G\left\{\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} - \frac{\partial^2 u_x}{\partial x^2}\right) + \frac{\partial}{\partial x}(\nabla \cdot \mathbf{u})\right\} = 0, \tag{a}$$

where $\mathbf{u} = [u_x \;\; u_y \;\; u_z]^\mathsf{T}$ is the displacement field. Again using the flexure formula and the same constitutive law, show that

$$\nabla \cdot \mathbf{u} = \frac{My}{I\lambda} - \frac{2G}{\lambda}\frac{\partial u_x}{\partial x}. \tag{b}$$

Now, combine relations (a) and (b) to obtain:

$$\left(\frac{y}{GI} + \frac{y}{I\lambda}\right)\frac{\partial M}{\partial x} - \left(1 + \frac{2G}{\lambda}\right)\frac{\partial^2 u_x}{\partial x^2} = -\left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right). \tag{c}$$

Now use the fact that due to the kinematic simplifications in the engineering theory of beams, the only non-zero strain is ε_{xx} together with the flexure formula and the aforementioned constitutive law to show that

$$u_x = \frac{y}{(\lambda + 2G)I}\int M \,\mathrm{d}x + \chi(y,z), \tag{d}$$

where χ is an arbitrary function of y and z. Using relation (d) in (c), show that the following equation holds

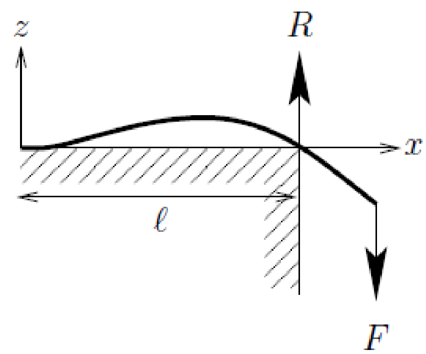
$$\frac{1}{GI}\frac{\partial M}{\partial x} = -\frac{1}{y}\left(\frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial z^2}\right).$$

From this conclude with proper justification that $M = C_1 x + C_2$, where C_1 and C_2 are constants.

2. Consider the buckling of a slender column under various end conditions as follows:
   (a) The end at $x = 0$ is clamped and the other end at $x = L$ is free. Find the critical buckling load.
   (b) The end at $x = 0$ is clamped and the other end at $x = L$ is pinned. Show that the condition for finding the critical load is $\tan(kL) = kL$, where $k = \sqrt{\frac{P}{EI}}$. Determine the critical loads either graphically or numerically using Jupyter Notebook.
   (c) Both ends at $x = 0$ and $x = L$ are clamped. Show that two conditions are obtained for finding the critical load:

$$\text{Either} \quad \sin\frac{kL}{2} = 0 \quad \text{or} \quad \tan\frac{kL}{2} = \frac{kL}{2}.$$

3. Consider a beam of length L, clamped at a distance $l < L$ from the edge of a horizontal table. A weight F hung from the other end deflects the beam as shown in the figure.



Assume that the weight of the beam itself is negligible compared with F. Based on the principle of virtual work show that the beam deflection $w(x)$ will be governed by the following governing differential equation:

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = \frac{R}{EI}\delta_{\mathrm{D}}(x - l),$$

where δ_{D} is the Dirac delta-function and R is the reaction force from the edge of the table on the beam. What are the boundary conditions? The boundary conditions must be systematically deduced from the variational formulation based on the virtual work principle.