Föppl-von Karman Plate Theory

Kinematical hypothesis:

$$u = u_s - z\frac{\partial w}{\partial x}$$

$$v = v_s - z\frac{\partial w}{\partial y}$$

$$w = w(x,y)$$

0.1   0.01

$$\frac{\partial w}{\partial x} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} \gg \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$$

0.1   0.01

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\}$$

$$\approx \frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$E_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$\approx \frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$E_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

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$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$\approx \frac{1}{2}\left(\frac{\partial u_s}{\partial y} - z\frac{\partial^2 w}{\partial y\partial x} + \frac{\partial v_s}{\partial x} - z\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right)$$

$$= \frac{1}{2}\left(\cancel{-\frac{\partial w}{\partial y}} + \cancel{\frac{\partial w}{\partial y}} + \underline{\left(\frac{\partial u_s}{\partial y} - z\frac{\partial^2 w}{\partial y\partial x}\right)}\underline{\left(-\frac{\partial w}{\partial x}\right)} + \underbrace{\left(\frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)}_{0.01}\underbrace{\left(-\frac{\partial w}{\partial y}\right)}_{0.1} + 0\right)$$

$$\approx 0$$

$u = u_s - z\frac{\partial w}{\partial x}$

$$\frac{\partial u}{\partial y}\frac{\partial u}{\partial z} = \frac{\partial u}{\partial y}\left(-\frac{\partial w}{\partial x}\right)$$

$$0.01 \times 0.1 = 0.001$$

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$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{\partial w}{\partial x}} - \cancel{\frac{\partial w}{\partial x}} + \left(-\frac{\partial w}{\partial x}\right)\left(\frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \left(-\frac{\partial w}{\partial y}\right)\left(\frac{\partial v_s}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}\right) + 0\right)$$

$$\approx 0$$

$$\varepsilon_{xx} = \varepsilon^0_{xx} - z\frac{\partial^2 w}{\partial x^2} \quad ; \quad \varepsilon^0_{xx} = \frac{\partial u_s}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\varepsilon_{yy} = \varepsilon^0_{yy} - z\frac{\partial^2 w}{\partial y^2} \quad ; \quad \varepsilon^0_{yy} = \frac{\partial v_s}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$\varepsilon_{xy} = \varepsilon^0_{xy} - z\frac{\partial^2 w}{\partial x \partial y} \quad ; \quad \varepsilon^0_{xy} = \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)$$

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Forcibly assume that $\sigma_{zz} = 0$

Appl. of VWE: $\int_V \sigma_{ij}\,\delta\epsilon_{ij}\,dV = \int_A t_i\,\delta u_i\,dA$

$$LHS = \int_V \sigma_{ij}\,\delta\epsilon_{ij}\,dV$$

$$= \int_V \left(\sigma_{xx}\,\delta\epsilon_{xx} + \sigma_{yy}\,\delta\epsilon_{yy} + 2\,\sigma_{xy}\,\delta\epsilon_{xy}\right) dV$$

$$= \int_A \int_{-h/2}^{h/2} \left[\sigma_{xx}\,\delta\left(\epsilon^0_{xx} - z\frac{\partial^2 w}{\partial x^2}\right) + \sigma_{yy}\,\delta\left(\epsilon^0_{yy} - z\frac{\partial^2 w}{\partial y^2}\right) + 2\,\sigma_{xy}\,\delta\left(\epsilon^0_{xy} - z\frac{\partial^2 w}{\partial x \partial y}\right)\right] dz\,dA$$

$$= \underbrace{\int_A \left[N_x\,\delta\epsilon^0_{xx} + N_y\,\delta\epsilon^0_{yy} + 2\,N_{xy}\,\delta\epsilon^0_{xy}\right] dA}_{LHS_1} + \underbrace{\int -\left[M_x\frac{\partial^2 w}{\partial x^2} + M_y\frac{\partial^2 w}{\partial y^2} + 2\,M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right] dA}_{LHS_2}$$

$$LHS_1 = \int_A \left[N_x \delta \epsilon^0_{xx} + N_y \delta \epsilon^0_{yy} + 2 N_{xy} \delta \epsilon^0_{xy} \right] dA$$

$$= \int_A \left[N_x \left(\frac{\partial \delta u_s}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + N_y \left(\frac{\partial \delta v_s}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) + 2 N_{xy} \frac{1}{2} \left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \right) \right] dA$$

$$= \int_A \left[N_x \frac{\partial \delta u_s}{\partial x} + N_y \frac{\partial \delta v_s}{\partial y} + N_{xy} \frac{\partial \delta u_s}{\partial y} + N_{xy} \frac{\partial \delta v_s}{\partial x} + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \frac{\partial \delta w}{\partial y} \right] dA$$

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$$= \int_A \Bigg[N_x \frac{\partial \delta u_s}{\partial x} + N_y \frac{\partial \delta v_s}{\partial y} + N_{xy} \frac{\partial \delta u_s}{\partial y} + N_{xy} \frac{\partial \delta v_s}{\partial x} + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \frac{\partial \delta w}{\partial y} \Bigg] dA$$

$$= \int_A \Bigg[\frac{\partial}{\partial x}\left(N_x \delta u_s\right) - \frac{\partial N_x}{\partial x} \delta u_s + \frac{\partial}{\partial y}\left(N_y \delta v_s\right) - \frac{\partial N_y}{\partial y} \delta v_s + \frac{\partial}{\partial y}\left(N_{xy} \delta u_s\right) - \frac{\partial N_{xy}}{\partial y} \delta u_s$$

$$+ \frac{\partial}{\partial x}\left(N_{xy} \delta v_s\right) - \frac{\partial N_{xy}}{\partial x} \delta v_s + \frac{\partial}{\partial x}\left\{ \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right\} - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w$$

$$+ \frac{\partial}{\partial y}\left\{ \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\} - \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \Bigg] dA$$

$$\int_A \left[\frac{\partial}{\partial x}(\ \ ) + \frac{\partial}{\partial y}(\ \ ) \right] dA = \oint \left[(\ \ ) n_x + (\ \ ) n_y \right] ds \qquad n_x = \frac{dy}{ds} \ ; \ n_y = -\frac{dx}{ds}$$

$$= \oint (\ \ )\, dy - \oint (\ \ )\, dx$$

$$
= \iint_A \Bigg[\frac{\partial}{\partial x}(N_x \delta u_s) - \frac{\partial N_x}{\partial x}\delta u_s + \frac{\partial}{\partial y}(N_y \delta v_s) - \frac{\partial N_y}{\partial y}\delta v_s + \frac{\partial}{\partial y}(N_{xy}\delta u_s) - \frac{\partial N_{xy}}{\partial y}\delta u_s
$$

$$
+ \frac{\partial}{\partial x}(N_{xy}\delta v_s) - \frac{\partial N_{xy}}{\partial x}\delta v_s + \frac{\partial}{\partial x}\left\{\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w\right\} - \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w
$$

$$
+ \frac{\partial}{\partial y}\left\{\left(N_y \frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\delta w\right\} - \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\delta w \Bigg] dA
$$

$$
LHS_1 = \oint \left\{ N_x \delta u_s + N_{xy}\delta v_s + \left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w \right\} dy - \oint \left\{ N_y \delta v_s + N_{xy}\delta u_s + \left(N_y \frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\delta w \right\} dx
$$

$$
- \iint_A \Bigg[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right)\delta u_s + \left(\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x}\right)\delta v_s
$$

$$
+ \left\{ \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y \frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right) \right\}\delta w \Bigg] dA
$$

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$$LHS_2 = \int -\left[M_x \frac{\partial^2 \delta w}{\partial x^2} + M_y \frac{\partial^2 \delta w}{\partial y^2} + 2 M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y}\right] dA$$

$$= -\oint \left[\left(M_x \frac{\partial \delta w}{\partial x} + M_{xy} \frac{\partial \delta w}{\partial y} - \frac{\partial M_x}{\partial x} \delta w - \frac{\partial M_{xy}}{\partial y} \delta w\right) dy \right.$$

$$\left. - \left(M_y \frac{\partial \delta w}{\partial y} + M_{xy} \frac{\partial \delta w}{\partial x} - \frac{\partial M_y}{\partial y} \delta w - \frac{\partial M_{xy}}{\partial x} \delta w\right) dx \right]$$

$$- \int_A \left[\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y}\right] \delta w \, dA$$

$$RHS = \int_A t_i \delta u_i \, dA = \int_A q \, \delta w \, dA$$

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$$LHS_1 + LHS_2 = RHS$$

GDE :

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$-\left\{\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right) + \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y}\right)\right\} = q$$

$$\Rightarrow -\left\{\underline{\frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x}} + N_x\frac{\partial^2 w}{\partial x^2} + \underline{\frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y}} + N_{xy}\frac{\partial^2 w}{\partial x \partial y} + \underline{\frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y}} + N_y\frac{\partial^2 w}{\partial y^2} + \underline{\frac{\partial N_{xy}}{\partial y}\frac{\partial w}{\partial x}} + N_{xy}\frac{\partial^2 w}{\partial y \partial x}\right.$$

$$\left. + \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y}\right\} = q$$

$$-\left\{ \frac{\partial N_x}{\partial x}\frac{\partial w}{\partial x} + N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial N_{xy}}{\partial x}\frac{\partial w}{\partial y} + N_{xy}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial N_y}{\partial y}\frac{\partial w}{\partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial N_{xy}}{\partial y}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} \right\} = q$$

$$\Rightarrow -\left\{ N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} \right\} = q$$

Use $\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = -D\nabla^4 w$

$$\therefore -\left\{ N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right\} + D\nabla^4 w = q$$

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We write N_x, N_y, N_{xy} in terms of a scalar function F

$$N_x = \frac{\partial^2 F}{\partial y^2}, \quad N_y = \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

$$\therefore \quad N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} - D \nabla^4 w = q$$

$$\Rightarrow \boxed{-\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D \nabla^4 w = q}$$

$\hookrightarrow$ 1st Föppl-von Kármán Eqn

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$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

$$E_{xx} = \varepsilon_{xx} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$E_{yy} = \varepsilon_{yy} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

$$E_{xy} = \varepsilon_{xy} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

$$\frac{\partial^2 E_{xx}}{\partial y^2} - \frac{\partial^2}{\partial y^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{\partial^2 E_{yy}}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} = 2\frac{\partial^2 E_{xy}}{\partial x \partial y} - 2\frac{\partial^2}{\partial x \partial y}\left\{\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}$$

$$\Rightarrow \frac{\partial^2 E^0_{xx}}{\partial y^2} + \frac{\partial^2 E^0_{yy}}{\partial x^2} - 2\frac{\partial^2 E^0_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

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$$N_x = \int_{-h/2}^{h/2} \sigma_{xx}\, dz$$

$$= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(\varepsilon_{xx} + \nu \varepsilon_{yy} \right) dz$$

$$= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(\varepsilon^0_{xx} - z \frac{\partial^2 w}{\partial x^2} + \nu \varepsilon^0_{yy} - \nu z \frac{\partial^2 w}{\partial y^2} \right) dz$$

$$= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(\varepsilon^0_{xx} + \nu \varepsilon^0_{yy} \right) dz$$

$$= \underbrace{\frac{Eh}{1-\nu^2}}_{C} \left(\varepsilon^0_{xx} + \nu \varepsilon^0_{yy} \right)$$

$$= C \left(\varepsilon^0_{xx} + \nu \varepsilon^0_{yy} \right)$$

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$$N_x = C\left(\varepsilon^0_{xx} + \nu \varepsilon^0_{yy}\right)$$

$$N_y = C\left(\varepsilon^0_{yy} + \nu \varepsilon^0_{xx}\right)$$

$$N_{xy} = C(1-\nu)\,\varepsilon^0_{xy}$$

Invert these relations

$$\varepsilon^0_{xx} = \frac{1}{Eh}\left(N_x - \nu N_y\right)$$

$$\varepsilon^0_{yy} = \frac{1}{Eh}\left(N_y - \nu N_x\right)$$

$$\varepsilon^0_{xy} = \frac{(1+\nu)}{Eh} N_{xy}$$

Subs in the following

$$\frac{\partial^2 \varepsilon^0_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon^0_{yy}}{\partial x^2} - 2\frac{\partial^2 \varepsilon^0_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

$$\frac{1}{Eh}\left[\frac{\partial^2 N_x}{\partial y^2} - \nu\frac{\partial^2 N_y}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} - \nu\frac{\partial^2 N_x}{\partial x^2} - 2(1+\nu)\frac{\partial^2 N_{xy}}{\partial x \partial y}\right]$$

$$= \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}$$

Subs. $N_x = \frac{\partial^2 F}{\partial y^2}$, $N_y = \frac{\partial^2 F}{\partial x^2}$, $N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$

$$\boxed{\nabla^4 F = Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]}$$

↳ 2nd Föppl-von Kármán Eqn.