Föppl-von Karman Plate Theory *

We start from the following kinematical hypothesis:

$$u = u_s - z\frac{\partial w}{\partial x}, \tag{1}$$

$$v = v_s - z\frac{\partial w}{\partial y}, \tag{2}$$

$$w \equiv w(x, y). \tag{3}$$

Just like in the case of buckling of beams, we use the nonlinear strain-displacement relations. We also assume the following

$$\frac{\partial w}{\partial x} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y} \gg \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}. \tag{4}$$

$$\begin{aligned}
E_{xx} &\equiv \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
&\approx \frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, &(5)\\
E_{yy} &\equiv \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\
&\approx \frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, &(6)\\
E_{zz} &\equiv \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \\
&= \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}, &(7)\\
E_{xy} &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), &(8)\\
E_{yz} &\equiv \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \\
&\approx \frac{1}{2}\left(-\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} + \left(\frac{\partial u_s}{\partial y} - z\frac{\partial^2 w}{\partial x \partial y}\right)\left(-\frac{\partial w}{\partial x}\right) + \left(\frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)\left(-\frac{\partial w}{\partial y}\right) + 0\right) \\
&\approx 0, &(9)
\end{aligned}$$

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$$
\begin{aligned}
E_{zx} &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right)\\
&\approx \frac{1}{2}\left(-\frac{\partial w}{\partial x}+\frac{\partial w}{\partial x}+\left(-\frac{\partial w}{\partial x}\right)\left(\frac{\partial u_s}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)+\left(-\frac{\partial w}{\partial y}\right)\left(\frac{\partial v_s}{\partial x}-z\frac{\partial^2 w}{\partial x\partial y}\right)+0\right)\\
&\approx 0. \qquad (10)
\end{aligned}
$$

We represent the terms E_{xx}, E_{yy}, and E_{xy} as

$$E_{xx} = E^0_{xx} - z\frac{\partial^2 w}{\partial x^2}, \qquad (11a)$$

$$E_{yy} = E^0_{yy} - z\frac{\partial^2 w}{\partial y^2}, \qquad (11b)$$

$$E_{xy} = E^0_{xy} - z\frac{\partial^2 w}{\partial x\partial y}, \qquad (11c)$$

where

$$E^0_{xx} = \frac{\partial u_s}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad (12a)$$

$$E^0_{yy} = \frac{\partial v_s}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad (12b)$$

$$E^0_{xy} = \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \qquad (12c)$$

where, notably, each of E^0_{xx}, E^0_{yy}, and E^0_{xy} is independent of z.

Considering the virtual work equation:

$$\int_V \sigma_{ij}\delta E_{ij}\ \mathrm{d}V = \int_A t_i\delta u_i\ \mathrm{d}A,$$

and again forcibly assuming $\sigma_{zz} = 0$ (just as in the classical plate theory), we have from the left hand side

$$
\begin{aligned}
\mathrm{LHS} &= \int_V \sigma_{ij}\delta E_{ij}\ \mathrm{d}V\\
&= \int_V \left[\sigma_{xx}\delta E_{xx} + \sigma_{yy}\delta E_{yy} + 2\sigma_{xy}\delta E_{xy}\right]\ \mathrm{d}V\\
&= \int_A\int_{-h/2}^{h/2} \sigma_{xx}\delta\left(E^0_{xx} - z\frac{\partial^2 w}{\partial x^2}\right) + \sigma_{yy}\delta\left(E^0_{yy} - z\frac{\partial^2 w}{\partial y^2}\right) + 2\sigma_{xy}\delta\left(E^0_{xy} - z\frac{\partial^2 w}{\partial x\partial y}\right)\ \mathrm{d}z\ \mathrm{d}A\\
&= \underbrace{\int_A \left[\left(N_x\delta E^0_{xx} + N_y\delta E^0_{yy} + 2N_{xy}\delta E^0_{xy}\right)\right]\ \mathrm{d}A}_{\mathrm{LHS}_1} + \underbrace{\int_A \left[-\left(M_x\frac{\partial^2 w}{\partial x^2} + M_y\frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x\partial y}\right)\right]\ \mathrm{d}A}_{\mathrm{LHS}_2}.
\end{aligned}
$$

To write the last step, we have used the following definitions:

$$\int_{-h/2}^{h/2} \sigma_{xx}\ \mathrm{d}z = N_x; \qquad \int_{-h/2}^{h/2} z\sigma_{xx}\ \mathrm{d}z = M_x;$$

$$\int_{-h/2}^{h/2} \sigma_{yy}\ \mathrm{d}z = N_y; \quad \int_{-h/2}^{h/2} z\sigma_{yy}\ \mathrm{d}z = M_y;$$
$$\int_{-h/2}^{h/2} \sigma_{xy}\ \mathrm{d}z = N_{xy}; \quad \int_{-h/2}^{h/2} z\sigma_{xy}\ \mathrm{d}z = M_{xy}.$$

We now consider the two integrals LHS_1 and LHS_2 separately.

$$\begin{aligned}
\text{LHS}_1 &= \int_A \left[N_x \delta E^0_{xx} + N_y E^0_{yy} + 2N_{xy}E^0_{xy}\right]\ \mathrm{d}A \\
&= \int_A \Bigg[N_x\left(\frac{\partial \delta u_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + N_y\left(\frac{\partial \delta v_s}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y}\right) \\
&\qquad + 2N_{xy}\frac{1}{2}\left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x}\right)\Bigg]\ \mathrm{d}A \\
&= \int_A \Bigg[N_x\frac{\partial \delta u_s}{\partial x} + N_y\frac{\partial \delta v_s}{\partial y} + N_{xy}\frac{\partial \delta u_s}{\partial y} + N_{xy}\frac{\partial \delta v_s}{\partial x} \\
&\qquad + \left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\frac{\partial \delta w}{\partial x} + \left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\frac{\partial \delta w}{\partial y}\Bigg]\ \mathrm{d}A \\
&= \int_A \Bigg[\underbrace{\frac{\partial}{\partial x}(N_x\delta u_s)}_{①} - \frac{\partial N_x}{\partial x}\delta u_s + \underbrace{\frac{\partial}{\partial y}(N_y\delta v_s)}_{④} - \frac{\partial N_y}{\partial y}\delta v_s \\
&\qquad + \underbrace{\frac{\partial}{\partial y}(N_{xy}\delta u_s)}_{⑤} - \frac{\partial N_{xy}}{\partial y}\delta u_s + \underbrace{\frac{\partial}{\partial x}(N_{xy}\delta v_s)}_{②} - \frac{\partial N_{xy}}{\partial x}\delta v_s \\
&\qquad + \underbrace{\frac{\partial}{\partial x}\left\{\left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w\right\}}_{③} - \frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w \\
&\qquad + \underbrace{\frac{\partial}{\partial y}\left\{\left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\delta w\right\}}_{⑥} - \frac{\partial}{\partial y}\left(N_y\frac{\partial w}{\partial y} + N_{xy}\frac{\partial w}{\partial x}\right)\delta w\Bigg]\ \mathrm{d}A
\end{aligned}$$

We consider the terms ①, ②, ③ together and the terms ④, ⑤, ⑥ together, use the Green's theorem alongwith the relations $n_x = \dfrac{\mathrm{d}y}{\mathrm{d}s}$ and $n_y = -\dfrac{\mathrm{d}x}{\mathrm{d}s}$. We also rearrange the remaining terms. Thus, we obtain

$$\text{LHS}_1 = \oint \left[\left\{N_x\delta u_s + N_{xy}\delta v_s + \left(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}\right)\delta w\right\}\ \mathrm{d}y\right.$$

$$
\begin{aligned}
&- \left\{ N_y \delta v_s + N_{xy} \delta u_s + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\} \,\mathrm{d}x \Bigg] \\
&- \int_A \Bigg[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_s + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v_s \\
&+ \left\{ \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \right\} \delta w \Bigg] \,\mathrm{d}A
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{LHS}_2 = - \oint \Bigg[&\left(M_x \frac{\partial \delta w}{\partial x} + M_{xy} \frac{\partial \delta w}{\partial y} - \frac{\partial M_x}{\partial x} \delta w - \frac{\partial M_{xy}}{\partial y} \delta w \right) \mathrm{d}y \\
&- \left(M_y \frac{\partial \delta w}{\partial y} + M_{xy} \frac{\partial \delta w}{\partial x} - \frac{\partial M_y}{\partial y} \delta w - \frac{\partial M_{xy}}{\partial x} \delta w \right) \mathrm{d}x \Bigg] \\
&- \int_A \left[\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right] \delta w \,\mathrm{d}A
\end{aligned}
$$

For the right hand side of the virtual work equation we have

$$
\mathrm{RHS} \equiv \int_A t_i \delta u_i \,\mathrm{d}A = \int_A q \delta w \,\mathrm{d}A.
$$

Substituting the expressions of LHS_1, LHS_2, and the RHS into the virtual work equation, collecting the coefficients of δu_s, δv_s, and δw_s in the area integrals, and setting them individually to zero we have the following governing equations:

$$
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{13a}
$$

$$
\frac{\partial N_{xy}}{\partial y} + \frac{\partial N_y}{\partial y} = 0, \tag{13b}
$$

$$
- \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) - \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = q \tag{13c}
$$

Now, we expand (13c) and use (13a) and (13b) in it. We also use the the following

$$
- \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = D \nabla^4 w,
$$

to obtain the following:

$$
- \left(N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) + D \nabla^4 w - q = 0. \tag{14}
$$

We set N_x, N_y, and N_{xy} in terms of a scalar function F as

$$
N_x = \frac{\partial^2 F}{\partial x^2}, \tag{15a}
$$

$$N_y = \frac{\partial^2 F}{\partial y^2}, \tag{15b}$$

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{15c}$$

and substitute in (14) to obtain

$$\boxed{D\nabla^4 w = q + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.} \tag{16}$$

This is the first of the Föppl-von Karman equations. In this equation, both w and F are unknown. So we need another equation to solve for w and F.

We know that the first of the compatibility equations is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \tag{17}$$

where ε_{xx}, ε_{yy}, and ε_{xy} are infinitesimal strain tensor components. These are related to E_{xx}, E_{yy}, and E_{xy} as

$$\begin{aligned} E_{xx} &= \varepsilon_{xx} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ E_{yy} &= \varepsilon_{yy} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ E_{xy} &= \varepsilon_{xy} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned}$$

Therefore, from (17), we have

$$\frac{\partial^2 E_{xx}}{\partial y^2} - \frac{\partial^2}{\partial y^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{\partial^2 E_{yy}}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} = 2\frac{\partial^2 E_{xy}}{\partial x \partial y} - 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right),$$

or, $$\frac{\partial^2 E_{xx}}{\partial y^2} + \frac{\partial^2 E_{yy}}{\partial x^2} - 2\frac{\partial^2 E_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right).$$

Using (11), expanding the terms and simplifying, we obtain the following

$$\frac{\partial^2 E^0_{xx}}{\partial y^2} + \frac{\partial^2 E^0_{yy}}{\partial x^2} - 2\frac{\partial^2 E^0_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{18}$$

Now,

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_{xx}\, \mathrm{d}z, \\ &= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2} \left(E_{xx} + \nu E_{yy}\right)\, \mathrm{d}z, \end{aligned}$$

$$
\begin{aligned}
&= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left(E_{xx}^0 - z\frac{\partial^2 w}{\partial x^2} + \nu E_{yy}^0 - \nu z\frac{\partial^2 w}{\partial y^2}\right)\,\mathrm{d}z,\\
&= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left(E_{xx}^0 + \nu E_{yy}^0\right)\,\mathrm{d}z,\\
&= \frac{Eh}{1-\nu^2}\left(E_{xx}^0 + \nu E_{yy}^0\right),\\
&= C\left(E_{xx}^0 + \nu E_{yy}^0\right).
\end{aligned}
$$

Setting $C = \dfrac{Eh}{1-\nu^2}$, and expanding similarly for N_y and N_{xy}, we obtain

$$N_x = C\left(E_{xx}^0 + \nu E_{yy}^0\right), \tag{19a}$$

$$N_y = C\left(E_{yy}^0 + \nu E_{xx}^0\right), \tag{19b}$$

$$N_{xy} = C(1-\nu)E_{xy}^0. \tag{19c}$$

Inverting the relations (19a), (19b), and (19c), we have

$$E_{xx}^0 = \frac{1}{Eh}\left(N_x - \nu N_y\right), \tag{20a}$$

$$E_{yy}^0 = \frac{1}{Eh}\left(N_y - \nu N_x\right), \tag{20b}$$

$$E_{xy}^0 = \frac{(1+\nu)}{Eh}N_{xy}. \tag{20c}$$

Substituting these expressions of E_{xx}^0, E_{yy}^0, and E_{xy}^0 in (14), we have

$$\frac{1}{Eh}\left[\frac{\partial^2 N_x}{\partial y^2} - \nu\frac{\partial^2 Ny}{\partial y^2} + \frac{\partial^2 Ny}{\partial x^2} - \nu\frac{\partial^2 N_x}{\partial x^2} - 2(1+\nu)\frac{\partial^2 N_{xy}}{\partial x\partial y}\right] = \left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.$$

Finally, using the relations from Eqs (15a), (15b), and (15c), we have

$$\boxed{\nabla^4 F = Eh\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right].} \tag{21}$$

This equation is the second of the Föppl-von Karman equations.