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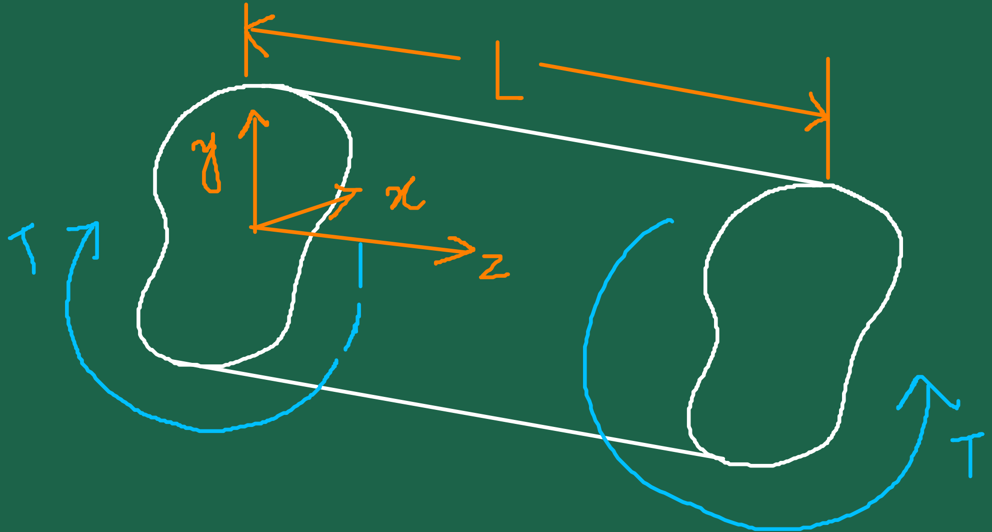
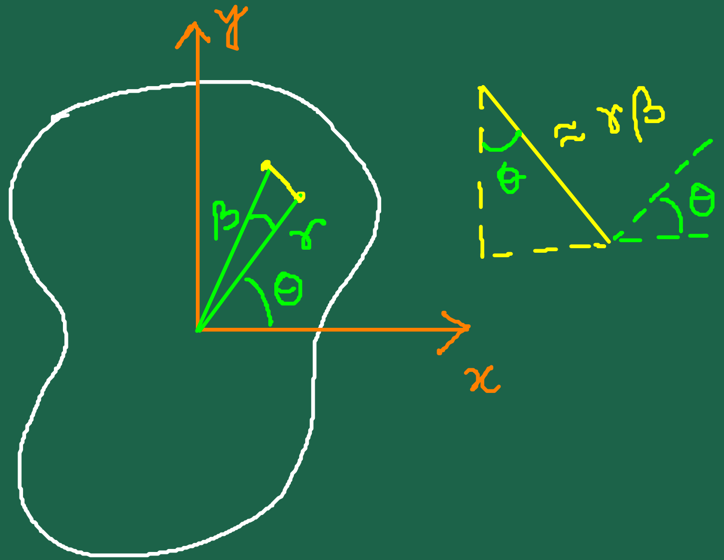
Torsion





$$u = -r\beta\sin\theta = -y\beta = -\alpha yz$$

$$v = r\beta\cos\theta = x\beta = \alpha xz$$

$$w = f(x,y)$$

Rate of twist : $\alpha = \dfrac{d\beta}{dz}$

At $z=0$, $\beta = 0$

At any general z : $\beta = \alpha z$

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$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(-\alpha z + \alpha z\right) = 0$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\alpha x + \frac{\partial \kappa}{\partial y}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)$$

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Use the VWE : $\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int t_i\,\delta u_i\,dA$

$$LHS = \int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV$$

$$= \int \left(2\,\sigma_{yz}\,\delta\varepsilon_{yz} + 2\,\sigma_{zx}\,\delta\varepsilon_{zx}\right) dV$$

$$= 4GL\int_A \left(\varepsilon_{yz}\,\delta\varepsilon_{yz} + \varepsilon_{zx}\,\delta\varepsilon_{zx}\right) dA$$

$$= 4GL\int_A \left[\frac{1}{2}\left(\alpha x + \frac{\partial\kappa}{\partial y}\right)\frac{1}{2}\left(x\,\delta\alpha + \frac{\partial\delta\kappa}{\partial y}\right) + \frac{1}{2}\left(-\alpha y + \frac{\partial\kappa}{\partial x}\right)\frac{1}{2}\left(-y\,\delta\alpha + \frac{\partial\delta\kappa}{\partial x}\right)\right] dA$$

$$= GL\int_A \left\{\left(\alpha x + \frac{\partial\kappa}{\partial y}\right)x + \left(-\alpha y + \frac{\partial\kappa}{\partial x}\right)(-y)\right\}\delta\alpha\,dA + GL\int_A \left\{\left(\alpha x + \frac{\partial\kappa}{\partial y}\right)\frac{\partial\delta\kappa}{\partial y} + \left(-\alpha y + \frac{\partial\kappa}{\partial x}\right)\frac{\partial\delta\kappa}{\partial x}\right\} dA$$

$$= GL\int_A \left\{\left(\alpha x + \frac{\partial \kappa}{\partial y}\right)x + \left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)(-y)\right\}\delta\alpha\, dA + GL\iint_A \left\{\left(\alpha x + \frac{\partial\kappa}{\partial y}\right)\frac{\partial \delta\kappa}{\partial y} + \left(-\alpha y + \frac{\partial \kappa}{\partial x}\right)\frac{\partial\delta\kappa}{\partial x}\right\} dA$$

$$= GL\iint_A\left\{\alpha\left(x^2+y^2\right) + x\frac{\partial\kappa}{\partial y} - y\frac{\partial\kappa}{\partial x}\right\}\delta\alpha\,dA + GL\iint_A\left[\frac{\partial}{\partial y}\left\{\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\delta\kappa\right\} - \frac{\partial}{\partial y}\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\delta\kappa\right.$$

$$\left. + \frac{\partial}{\partial x}\left\{\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)\delta\kappa\right\} - \frac{\partial}{\partial x}\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)\delta\kappa\right] dA$$

$$= GL\iint_A\left\{\alpha\left(x^2+y^2\right) + x\frac{\partial\kappa}{\partial y} - y\frac{\partial\kappa}{\partial x}\right\}\delta\alpha\,dA + GL\iint\left[\frac{\partial}{\partial x}\left\{\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)\delta\kappa\right\} + \frac{\partial}{\partial y}\left\{\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\delta\kappa\right\}\right]dA$$

$$- GL\int_A\left[\frac{\partial}{\partial x}\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\right]\delta\kappa\,dA$$

$$\text{RHS} = \int_A t_i\, \delta x_i\, dA = T\,\delta\beta = T L\,\delta\alpha$$

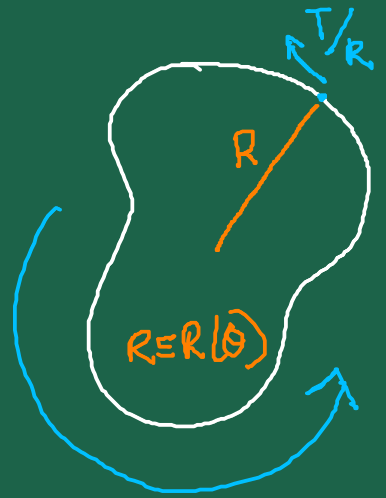
$$= T\,\delta\beta\Big|_L - T\,\delta\beta\Big|_0$$

$$= \Big[T\,\delta\beta\Big]_0^L = \int_0^L T\,\frac{d\delta\beta}{dz}\,dz$$

$$= \int_0^L T\,\delta\,\frac{d\beta}{dz}\,dz$$

$$= \int_0^L T\,\delta\alpha\, dz$$

$$= T\,\delta\alpha \int_0^L dz = T L\,\delta\alpha$$



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$$GL\iint_A\left\{\alpha(x^2+y^2)+x\frac{\partial\kappa}{\partial y}-y\frac{\partial\kappa}{\partial x}\right\}\delta\alpha\, dA+GL\iint_A\left[\frac{\partial}{\partial x}\left\{\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)\delta\kappa\right\}+\frac{\partial}{\partial y}\left\{\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\delta\kappa\right\}\right]dA$$

$$-GL\iint_A\left[\frac{\partial}{\partial x}\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\right]\delta\kappa\, dA = TL\,\delta\alpha$$

$$GL\iint_A\left\{\alpha(x^2+y^2)+x\frac{\partial\kappa}{\partial y}-y\frac{\partial\kappa}{\partial x}\right\}\delta\alpha\, dA+GL\oint\left\{\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)\delta\kappa\, n_x+\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\delta\kappa\, n_y\right\}dS$$

$$-GL\iint_A\left[\frac{\partial}{\partial x}\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)\right]\delta\kappa\, dA = TL\,\delta\alpha$$

Therefore,

$$GL\iint_A\left\{\alpha(x^2+y^2)+\left(x\frac{\partial\kappa}{\partial y}-y\frac{\partial\kappa}{\partial x}\right)\right\}dA = TL$$

$$\frac{\partial}{\partial x}\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)+\frac{\partial}{\partial y}\left(\alpha x+\frac{\partial\kappa}{\partial y}\right) \Rightarrow \frac{\partial^2\kappa}{\partial x^2}+\frac{\partial^2\kappa}{\partial y^2}=0$$

$$\left(-\alpha y+\frac{\partial\kappa}{\partial x}\right)n_x+\left(\alpha x+\frac{\partial\kappa}{\partial y}\right)n_y = 0$$ on the boundary of the c/s

$$W = K(x,y) = \alpha\,\phi(x,y)$$

warping fn.

$$\alpha = 0$$

$$W = 0$$

$$W = K(x,y) = \alpha\,\phi(x,y)$$

$$G\iint_A \left\{\alpha(x^2+y^2) + \left(x\frac{\partial K}{\partial y} - y\frac{\partial K}{\partial x}\right)\right\} dA = \frac{T}{\cancel{L}} \Rightarrow G\iint_A \left[\alpha(x^2+y^2) + \alpha\left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right)\right] dA = T$$

$$\Rightarrow \boxed{T = G\alpha J}$$

where $J = \iint_A \left\{(x^2+y^2) + \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x}\right)\right\} dA$

$$\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} = 0 \Rightarrow \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

$\left(-\alpha y + \frac{\partial K}{\partial x}\right)n_x + \left(\alpha x + \frac{\partial K}{\partial y}\right)n_y = 0$ on the boundary of the c/s

$\Rightarrow \boxed{\cancel{\alpha}\left(-y + \frac{\partial \phi}{\partial x}\right)n_x + \cancel{\alpha}\left(x + \frac{\partial \phi}{\partial y}\right)n_y = 0}$ " " " " " "

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Simplest case: $\phi = \text{const.}$

$-y n_x + x n_y = 0$ on the boundary

$n_x = \frac{dy}{ds}$

$n_y = -\frac{dx}{ds}$

$\Rightarrow -y\frac{dy}{ds} + x\left(-\frac{dx}{ds}\right) = 0$ " " "

$\Rightarrow \frac{1}{2}\frac{d}{ds}(x^2+y^2) = 0$ " " "

$\Rightarrow x^2+y^2 = \text{const.}$ on the boundary

The boundary is nothing but a circle!

$J = \int_A (x^2+y^2)\, dA \rightarrow$ polar moment of inertia (from UG classes!)

$T = G\alpha J \Rightarrow \alpha = \frac{T}{GJ} \Rightarrow \frac{\beta}{L} = \frac{T}{GJ} \Rightarrow \beta = \frac{TL}{GJ}$

$[\beta = \alpha L]$

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$\phi = Axy \rightarrow$ Ellipse

$\phi = A(y^3 - 3x^2y) \rightarrow$ Equilateral triangle

Prandtl Stress Function

$$\sigma_{xz} = 2G\alpha \frac{\partial \psi}{\partial y}, \quad \sigma_{yz} = -2G\alpha \frac{\partial \psi}{\partial x}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \qquad (\sigma_{zz} = 0)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\alpha x + \alpha \frac{\partial \phi}{\partial y}\right) \Rightarrow \sigma_{yz} = 2G\,\varepsilon_{yz} = G\alpha\left(x + \frac{\partial \phi}{\partial y}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(-\alpha y + \alpha \frac{\partial \phi}{\partial x}\right) \Rightarrow \sigma_{zx} = 2G\,\varepsilon_{zx} = G\alpha\left(-y + \frac{\partial \phi}{\partial x}\right)$$

$$\therefore \quad -2G\alpha \frac{\partial \psi}{\partial x} = G\alpha\left(x + \frac{\partial \phi}{\partial y}\right) \Rightarrow -2\frac{\partial \psi}{\partial x} = x + \frac{\partial \phi}{\partial y}$$

$$2G\alpha \frac{\partial \psi}{\partial y} = G\alpha\left(-y + \frac{\partial \phi}{\partial x}\right) \Rightarrow 2\frac{\partial \psi}{\partial y} = -y + \frac{\partial \phi}{\partial x}$$

$$-2\frac{\partial\psi}{\partial x} = x + \frac{\partial\phi}{\partial y}$$

$$2\frac{\partial\psi}{\partial y} = -y + \frac{\partial\phi}{\partial x}$$

$$\Rightarrow -2\frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial^2\psi}{\partial y^2} = 2$$

$$\Rightarrow \boxed{\nabla^2\psi = -1}$$

No traction boundary condition on the outer surface of the shaft

$$\sigma_{xz} \cdot n_x + \sigma_{yz} n_y = 0$$

$$\Rightarrow 2G\alpha\frac{\partial\psi}{\partial y}\frac{dy}{ds} - 2G\alpha\frac{\partial\psi}{\partial x}\left(-\frac{dx}{ds}\right) = 0$$

$$\Rightarrow \frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx = 0$$

$$\Rightarrow d\psi = 0 \Rightarrow \boxed{\psi = \text{const on the periphery}}$$

$$GJ = G\int_A \left\{ (x^2+y^2) + \left(x\frac{\partial \phi}{\partial y} - y\frac{\partial \phi}{\partial x} \right) \right\} dA$$

$$= G\int_A \left\{ (x^2+y^2) + x\left(-2\frac{\partial \psi}{\partial x} - x\right) - y\left(2\frac{\partial \psi}{\partial y} + y\right) \right\} dA$$

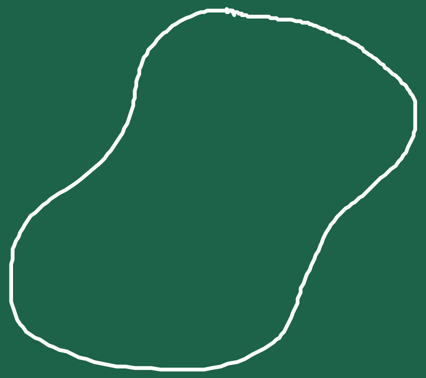
$$= G\int_A \left\{ \cancel{x^2} + \cancel{y^2} - 2x\frac{\partial \psi}{\partial x} - \cancel{x^2} - 2y\frac{\partial \psi}{\partial y} - \cancel{y^2} \right\} dA$$

$$= -2G\int_A \left(x\frac{\partial \psi}{\partial x} + y\frac{\partial \psi}{\partial y} \right) dA$$

$$= -2G\int_A \left\{ \frac{\partial}{\partial x}(x\psi) - \psi + \frac{\partial}{\partial y}(y\psi) - \psi \right\} dA$$

$$\boxed{GJ = -2G\oint (x\psi n_x + y\psi n_y)\, ds + 4G\int_A \psi \, dA}$$

Simply-connected region



Multiply-connected region

Consider the ψ to be 0 on the periphery

$$\nabla^2 \psi = -1$$

$\psi = 0$ on the boundary

$$GJ = -2G\oint (x n_x + y n_y)\psi \, ds + 4G\int_A \psi \, dA$$

$$= 0 + 4G\int_A \psi \, dA$$

Example: Circular c/s

Take $\psi = K\left(R^2 - x^2 - y^2\right)$

$$\nabla^2 \psi = -1$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -1$$

$$\Rightarrow -2K - 2K = -1$$

$$\Rightarrow 4K = 1$$

$$\Rightarrow K = \frac{1}{4}$$

$$\therefore \psi = \frac{1}{4}\left(R^2 - x^2 - y^2\right)$$

$$GJ = 4G \int_A \psi \, dA \qquad dA = r\,d\theta\,dr$$

$$= 4G \int_A \frac{1}{4}\left(R^2 - x^2 - y^2\right) dA$$

$$= 4G \int \frac{1}{4}\left(R^2 - r^2\right) dA$$

$$= 4G \int_0^R \frac{1}{4}\left(R^2 - r^2\right) 2\pi r \, dr$$

$$= G\left[2\pi R^2 \int_0^R r\,dr - 2\pi \int_0^R r^3\,dr\right]$$

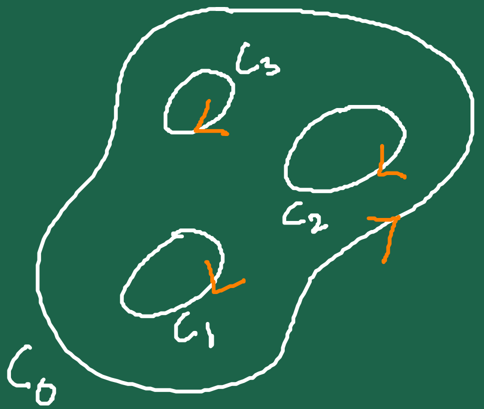
$$= G\left[\cancel{2}\pi R^2 \frac{R^2}{\cancel{2}} - \cancel{2}\pi \frac{R^4}{\cancel{4}\,2}\right]$$

$$= \frac{1}{2} G \pi R^4$$

Multiply-connected region

$$\nabla^2 \psi = -1$$

ψ = const. on the periphery



$C_0 : \psi = 1 \to 0$

$C_1 : \psi = 3 \to 2$

$C_2 : \psi = 2.5 \to 1.5$

$C_3 : \psi = 4 \to 3$

On the outer periphery, ψ is taken as 0.

On the other peripheries (inner contours), ψ is a const. on each of them, but the constant values themselves are not unknown.

$$GJ = -2G \oint (x n_x + y n_y)\,\psi \, ds + 4G \int_A \psi \, dA$$

$$= -2G \Big[\underbrace{\oint_{C_0} \{ \;\} ds}_{=0} + \oint_{C_1} \{ \;\} ds + \oint_{C_2} \{ \;\} ds \ldots \Big] + 4G \int_A \psi \, dA$$

$$= -2G \sum_{i=1}^{N} \psi_i \oint_{C_i} (x n_x + y n_y)\, ds \quad + 4G \int_A \psi \, dA$$

$$GJ = -2G\sum_{i=1}^{N} \psi_i \oint_{C_i} (x n_x + y n_y)\, dS + 4G\int_A \psi \, dA$$

$$\oint_{C_i} (x n_x + y n_y)\, dS$$ → integration over each of the inner peripheries has to be done in a clockwise sense

$$= \int_{A_i} \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)\right] dA$$

$$= \int_{A_i} (1 + 1)\, dA$$

$$= 2A_i \rightarrow -2A_i$$ (because of the clockwise sense of the integration)

$$\boxed{GJ = \sum_{i=1}^{N} 4G\,\psi_i A_i + 4G\int_A \psi \, dA}$$