Energy Methods

Preliminary Concepts

Configuration: The positions of all the material points of a mechanical system are together referred to as the configuration of the system.

Configuration Space: The set of all configurations that can be taken by a mechanical system is referred to as the configuration space.

Geometric Notation: Any point in the configuration space of a mechanical system is denoted by X. For example, if the mechanical system consists of just one particle, then X can be the position vector of the particle.

Distance in Configuration Space: When a mechanical system goes from one configuration, say X_0 in configuration space, to another configuration, say X_1 in configuration space, the "distance" between X_0 and X_1 is taken to be the maximum of the displacement magnitudes of the individual particles making up the system.

Path in Configuration Space:

$$X \equiv X(\bar{t}), \quad a \le \bar{t} \le b$$

This fn. represents a path in configuration space.

Note: Every path may <u>not</u> be admissible because of the constraints

Variable $\bar{t}$ is <u>not</u> necessarily the time.

If it is indeed the time then the $X(\bar{t})$ is called the <u>motion</u> of the system.

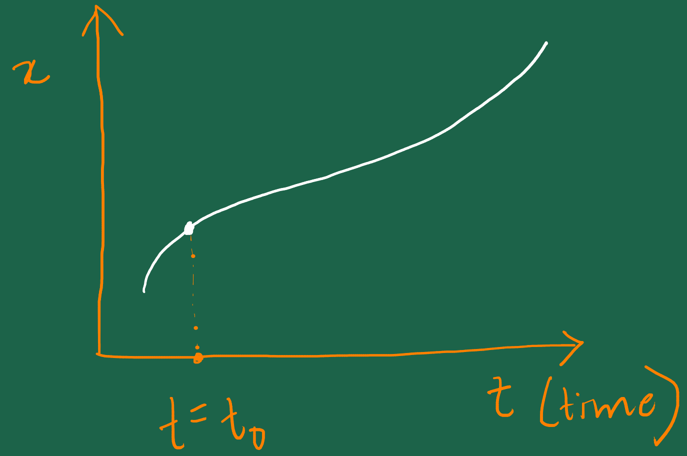
Virtual Displacement

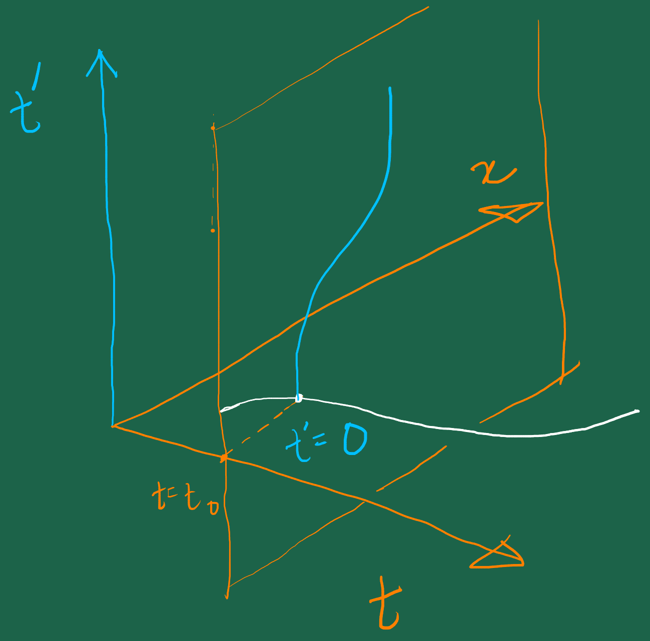
A virtual infinitesimal displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates ... consistent with the forces and constraints imposed on the system at the given instant t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

Freeze time

Consider paths in configuration corresponding to some variable t' (NOT time)

Restrict to admissible paths





$x(t=t_0)$
$= x(t'=0)$

Virtual displacement of the particle at any t'

$$u = x(t') - x(t'=0)$$

$$= x(0 + t' - 0) - x(0)$$

$$= x(0) + (t'-0)\left.\frac{dx}{dt'}\right|_{t'=0} + \frac{1}{2}(t'-0)^2 \left.\frac{d^2x}{dt'^2}\right|_{t'=0} + \cdots - x(0)$$

$$= \delta u + \frac{1}{2}\tilde{\delta} u + \cdots$$

When t' is very small, u is well approximated by δu

$|$ dy δy

Law of Kinetic Energy

Very Imp. : Work done (due to both ext. & int. forces) $= \Delta K$ ↑

$$W = \int_{t_0}^{t_1} \vec{F} \cdot \vec{v}\, dt \qquad \left[\vec{v} = \frac{d\vec{r}}{dt} \right]$$

$$= \int_{t_0}^{t_1} m \frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

$$= \frac{1}{2} \int_{t_0}^{t_1} m \frac{d}{dt} \left(\vec{v} \cdot \vec{v} \right) dt$$

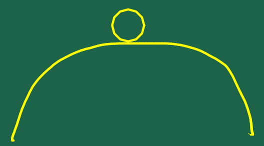
$$= \frac{1}{2} \int_{t_0}^{t_1} m\, d\left(|\vec{v}|^2 \right) = \Delta K$$

Law of KE is restricted to inertial frames of reference.

Mechanical sys. begins to move → $\Delta K \uparrow$ → W (ext + int) $\uparrow$

Fourier's inequality

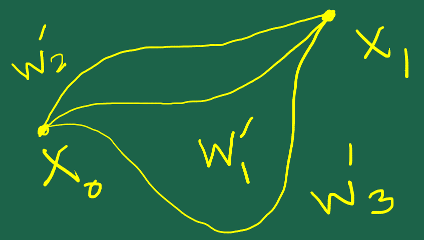
Motionless mech. sys. remains at rest if the virtual work done by all ext + int. forces ≤ 0



Only a sufficient condition! Not necessary

We would like to find some condition that is both necessary + sufficient

Work Function



W' are path-dependent

Suppose ... $W_3' < W_1' < W_2'$ ....

If we consider these values to be part of a set and we consider mechanical sys which do not possess infinite energy

then this set will have a least upper bound W

No matter which W' we choose, $W' < W$

W is path-independent

$W \equiv W(X_0, X_1)$

Fourier's inequality $\rightarrow$ $W \leq 0$

$W' < W \leq 0$

Principle of Virtual Work (PVW)

$$W = \delta W + \frac{1}{2}\delta^2 W + \cdots$$

$\boxed{\text{For equilibrium: } \delta W = 0}$ $\rightarrow$ PVW

$$\delta W = 0$$

$$\Rightarrow \delta W_i + \delta W_e = 0$$

$$\Rightarrow \delta W_i = -\delta W_e$$

DEFORMABLE BODY

Configuration of a deformable body $\rightarrow$ $\delta\vec{u} \equiv \delta u_i$

Consider a deformable body in eqb. under the action of traction & body forces

$$-\delta W_i = \delta W_e = \int_V \rho\vec{b}\cdot\delta\vec{u}\, dV + \int_S \vec{T}\cdot\delta u\, dS$$

$$= \int_V \rho b_i\, \delta u_i\, dV + \int_S T_i\, \delta u_i\, dS$$

Use $T_i = \sigma_{ji} n_j$

$$\therefore \quad \delta W_e = \int \rho b_i \delta u_i\, dV + \int_S \sigma_{ji} n_j \delta u_i\, dS$$

$$\delta W_e = \int_V \rho b_i \delta u_i \, dV + \int_S (\sigma_{ji} \delta u_i) n_j \, dS$$

$$= \int_V \rho b_i \delta u_i \, dV + \int_V \frac{\partial}{\partial x_j} (\sigma_{ji} \delta u_i) \, dV$$

$$= \int_V \rho b_i \delta u_i \, dV + \int_V \left(\frac{\partial \sigma_{ji}}{\partial x_j} \delta u_i + \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \right) dV$$

$$= \int_V \left(\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right) \delta u_i \, dV + \int_V \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \, dV$$

$$\int_S v_j n_j \, dS = \int_S \vec{v} \cdot \hat{n} \, dS = \int_V \nabla \cdot \vec{v} \, dV = \int_V \frac{\partial v_j}{\partial x_j} \, dV$$

We know that $\rho b_i + \dfrac{\partial \sigma_{ji}}{\partial x_j} = 0$

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$$\therefore \quad \delta W_e = \int_V \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \, dV$$

$$= \int_V \sigma_{ji} \, \delta\left(\frac{\partial u_i}{\partial x_j}\right) dV$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$= \int_V \sigma_{ji} \, \delta \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dV$$

$$= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, dV + \frac{1}{2} \int_V \sigma_{ji} \, \delta \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dV$$

$$= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, dV + \frac{1}{2}\int_V \sigma_{ji} \, \delta\left(\frac{\partial u_i}{\partial x_j}\right) dV - \frac{1}{2}\int \sigma_{ij} \, \delta\left(\frac{\partial u_j}{\partial x_i}\right) dV \quad [\sigma_{ij} = \sigma_{ji}]$$

$$= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, dV + \frac{1}{2}\int_V \sigma_{pk} \, \delta\left(\frac{\partial u_k}{\partial x_p}\right) dV - \frac{1}{2}\int \sigma_{pk} \, \delta\left(\frac{\partial u_k}{\partial x_p}\right) dV$$

$$\therefore \boxed{\delta W_e = \int_V \sigma_{ij}\, \delta \varepsilon_{ij}\, dV} \rightarrow \text{Very Important relation}$$

$$\boxed{-\delta W_i = \delta W_e = \int_V \sigma_{ij}\, \delta \varepsilon_{ij}\, dV}$$

First Law of Thermodynamics

$$W_e + \underset{\substack{\uparrow \\ \text{heat} \\ \text{transferred} \\ \text{to the body}}}{Q} = \underset{\substack{\uparrow \\ \text{change} \\ \text{in internal energy}}}{\Delta U} + \Delta K$$

$$\delta W_e + \delta Q = \underset{\delta U}{\delta(\Delta U)} + \delta(\Delta K)$$

Adiabatic process : $\delta Q = 0$

For a body in static eqb. : $\Delta K = 0 \rightarrow \delta(\Delta K) = 0$

$$\therefore \delta W_e = \delta U$$

$$= \delta \int_V \underset{\substack{\uparrow \\ \text{int. energy / vol.}}}{U_0}\, dV = \int \delta U_0 \, dV$$

POTENTIAL ENERGY

Mech. sys. → conservative ← if the virtual work assoc. with any closed path is zero

conservative → static sense → virtual displ. exec. at infinitesimal speed

conservative → kinetic sense → virtual displ. are not necessarily exec. at infinitesimal speed

W_e → ind. of path → dep. on terminal configurations, $W_e = -V_e(x)$
(ext. forces are conservative)

pt. fn; pot. energy of ext. forces

W_i → ind. of path → dep. on terminal configurations, $W_i = -V_i(x)$
(int. forces are conservative

pt fn., pot energy of int. forces

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If <u>both</u> int. and ext. forces are conservative,

total virtual work : $W_e + W_i = W = -\Pi(x)$

$= -\left(V_e(x) + V_i(x)\right)$

STRAIN ENERGY

Mechanical system → elastic ← $\underbrace{\text{int. forces}}$ are conservative in the kinetic sense

↓

V_i (P.E.) → Strain Energy

Define a strain energy density V_{i0} s.t. $V_i = \int_V V_{i0}\, dV$

Strain energy density will be a fn of the strain components

$$V_{i0} \equiv V_{i0}(\varepsilon_{ij})$$

$$\delta V_{i0} = \frac{\partial V_{i0}}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}$$

NOTE: The strain energy of a system differs from the internal energy only by an additive constant

$$\delta V_{i0} \equiv \delta U_0$$

$$\delta V_{i0} = \frac{\partial V_{i0}}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \rightarrow \delta U_{i0} = \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}$$

Earlier $\rightarrow$ $\delta W_e = \int_V \delta U_0 \, dV$

$$\therefore \delta W_e = \int_V \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \, dV \quad - (*_1)$$

Earlier $\rightarrow$ $\delta W_e = \int_V \sigma_{ij} \delta \varepsilon_{ij} \, dV \hookrightarrow (*_2)$

Comparing $(*_1)$ and $(*_2)$, we obtain

$$\boxed{\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}} \rightarrow \varepsilon_{ij} \overset{?}{=} \frac{\partial \boxed{?}}{\partial \sigma_{ij}}$$

Complementary Energy

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}} \quad \rightarrow \quad \varepsilon_{ij} \stackrel{?}{=} \frac{\partial \boxed{?}}{\partial \sigma_{ij}}$$

Complementary energy density : $U_0' := -U_0 + \sigma_{pq}\varepsilon_{pq}$

To prove : $\varepsilon_{ij} = \dfrac{\partial U_0'}{\partial \sigma_{ij}}$

Proof : $U_0' = -U_0 + \sigma_{pq}\varepsilon_{pq}$

$$\Rightarrow \frac{\partial U_0'}{\partial \sigma_{ij}} = -\frac{\partial U_0}{\partial \sigma_{ij}} + \frac{\partial \sigma_{pq}}{\partial \sigma_{ij}}\varepsilon_{pq} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}$$

$$= -\frac{\partial U_0}{\partial \varepsilon_{pq}}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} + \delta_{pi}\delta_{qj}\varepsilon_{pq} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}$$

$$= -\sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} + \varepsilon_{ij} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}}$$

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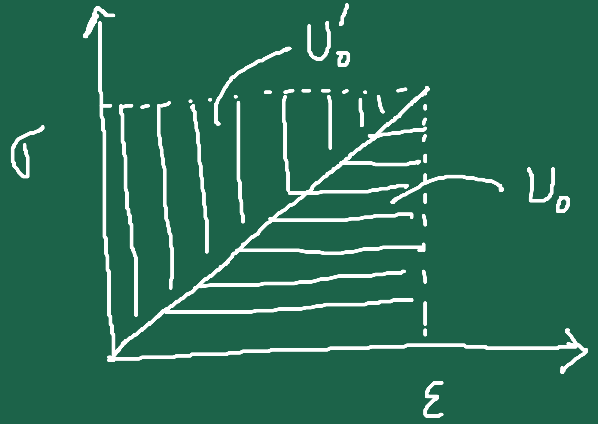
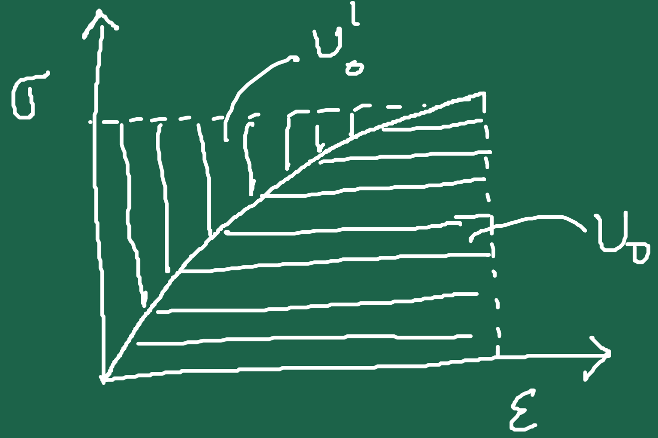
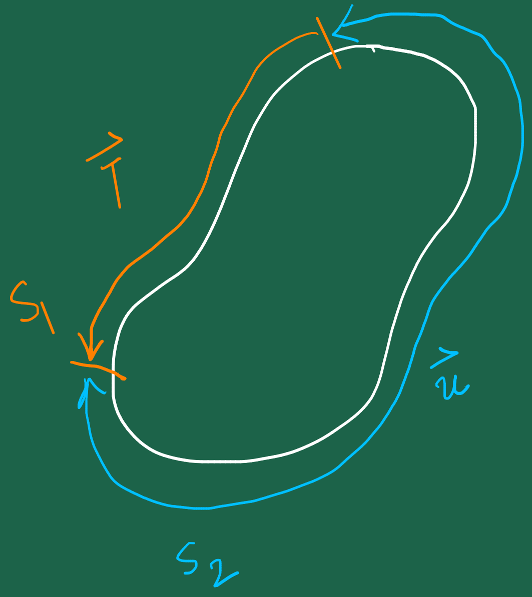
$$\boxed{\frac{\partial U_0'}{\partial \sigma_{ij}} = \varepsilon_{ij}}$$

Hence, proved





Generalization of Castigliano's Theorem of Least Work



$$\delta\Psi = 0$$

Ψ : Modified complementary energy

$$\Psi := U' - \int_{S_2} u_i T_i \, dS$$

Before the proof, a couple of things to note

$\dfrac{\partial\sigma_{ij}}{\partial x_j} + \rho b_i = 0$

$$\Rightarrow \frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0 \quad \text{— } (\#_1)$$

$U_0' = -U_0 + \sigma_{pq}\varepsilon_{pq}$

$$\Rightarrow \delta U_0' = -\delta U_0 + \delta\sigma_{pq}\varepsilon_{pq} + \sigma_{pq}\delta\varepsilon_{pq}$$

$$= -\sigma_{pq}\delta\varepsilon_{pq} + \delta\sigma_{pq}\varepsilon_{pq} + \sigma_{pq}\delta\varepsilon_{pq}$$

$$= \delta\sigma_{pq}\varepsilon_{pq} \quad \text{— } (\#_2)$$

$$\delta U' = \int_V \delta U_0' \, dV$$

$$= \int_V \varepsilon_{ij} \, \delta\sigma_{ij} \, dV \qquad (\text{Using } \#_2)$$

$$= \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \delta\sigma_{ij} \, dV$$

$$= \int_V \left(\frac{1}{2}\frac{\partial u_i}{\partial x_j}\delta\sigma_{ij} + \frac{1}{2}\frac{\partial u_j}{\partial x_i}\delta\sigma_{ji}\right) dV \qquad \left[\because \sigma_{ij} = \sigma_{ji}\right]$$

$$= \int_V \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij} \, dV$$

$$= \int_V \frac{\partial}{\partial x_j}\left(u_i \, \delta\sigma_{ij}\right) dV - \int_V u_i \underbrace{\frac{\partial \delta\sigma_{ij}}{\partial x_j}}_{=0 \ (\text{Using } \#_1)} dV$$

$$\delta U' = \int_V \frac{\partial}{\partial x_j}\left(u_i\,\delta\sigma_{ij}\right) dV$$

$$= \int_S u_i\,\delta\sigma_{ij}\,n_j\,dS$$

$$= \int_S u_i\,\delta\left(\sigma_{ij}\,n_j\right) dS \qquad [\delta n_j = 0;\ \text{geometry does not change}]$$

$$= \int_S u_i\,\delta T_i\,dS \qquad [\text{Using } T_i = \sigma_{ij}\,n_j]$$

$$= \int_{S_1} u_i\,\delta T_i\,dS + \int_{S_2} u_i\,\delta T_i\,dS$$

$$= \int_{S_2} u_i\,\delta T_i\,dS \qquad [\text{Using the fact that since } T_i \text{ is speified over } S_1,\ \delta T_i = 0 \text{ over } S_1]$$

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$$\delta U' = \int_{S_2} u_i \, \delta T_i \, dS$$

$$= \int_{S_2} \delta (u_i T_i) \, dS$$

[Using the fact that u_i is specified over S_2, so $\delta u_i = 0$ over S_2]

$$= \delta \int_{S_2} u_i T_i \, dS$$

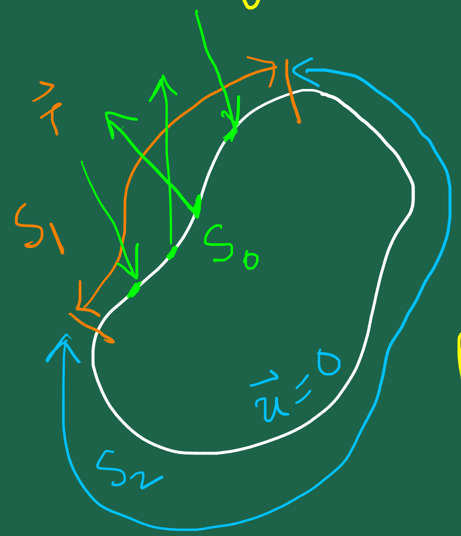
$$\therefore \quad \delta U' - \delta \int_{S_2} u_i T_i \, dS = 0$$

$$\Rightarrow \delta \left(U' - \int_{S_2} u_i T_i \, dS \right) = 0$$

$$\Rightarrow \delta \underline{\Psi} = 0$$

Hence, proved

Castigliano's Theorem on Deflections



$$\bar{u}_F = \frac{\partial U'}{\partial F}$$

Pre-Theorem : Theorem of the mean for integrals

$f(x,y)$ and $g(x,y)$

$$\int_A f(x,y)\, g(x,y)\, dA = K \int_A g(x,y)\, dA$$

where K is some value of $f(x,y)$ within A provided $g(x,y)$ does not change in sign in A.

$$\delta U' = \int_S u_i \, \delta T_i \, dS$$

$$= \int_{S_0} u_i \, \delta T_i \, dS + \int_{S_1} u_i \underbrace{\delta T_i}_{=0} dS + \int_{S_2} u_i \underbrace{\delta T_i}_{=0} dS$$

$$= \int_{S_0} u_i \, \delta T_i \, dS$$

Over the little region S_0, we do not expect δT_i to change sign

$\therefore \quad \delta U' = \bar{u}_i \int_{S_0} \delta T_i \, dS$, where $\bar{u}_i$ are the components of the displacement vector u_i at some point within S_0.

For this point force F_i,

$$F_i = \int_{S_0} T_i \, dS$$

$$\Rightarrow \delta F_i = \delta \int_{S_0} T_i \, dS = \int_{S_0} \delta T_i \, dS$$

$\vec{v} = |\vec{v}| \hat{n}$

$$\therefore \quad \delta U' = \bar{u}_i \, \delta F_i$$

Now, $F_i = F N_i \rightarrow$

$$\delta U' = \bar{u}_i N_i \, \delta F$$

$$= \bar{u}_F \, \delta F \quad \text{— (\#3)}$$

$$\frac{\partial U'}{\partial \varepsilon_{pq}} V$$

$$\delta U' = \frac{\partial U'}{\partial F} \delta F \quad \text{— (\#4)}$$

Comparing (#3) and (#4)

$$\boxed{\bar{u}_F = \frac{\partial U'}{\partial F}}$$

Hence, proved

For linear, elastic materials $U' \equiv U$

$$\bar{u}_F = \frac{\partial U}{\partial F}$$

(Not true for non-linear material behaviour!)

Prelude to Problems

$$\bar{u}_F = \frac{\partial U'}{\partial F}$$

For linear elastic materials: $U' = U$

$$U = \int_V U_0 \, dV$$

$$U_0 = \left(\frac{1}{2}\lambda + G\right) I_e^2 - 2G \, II_e$$

$$= \frac{1}{2E}\left[I_1^2 - 2(1+\nu) I_2\right]$$

To be discussed later

For thin rods under bending, σ_{11} is dominant.

$$U_0 = \frac{1}{2E}\sigma_{11}^2$$

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$$U = \int_V U_0 \, dV = \int_V \frac{\sigma^2}{2E} \, dV \qquad \left[\sigma = \frac{My}{I}\right]$$

$$= \frac{1}{2E} \int_0^L \int_A \frac{M^2 y^2}{I^2} \, dA \, dx$$

$$= \frac{1}{2E} \int_0^L \frac{M^2}{I^2} \underbrace{\int_A y^2 \, dA}_{I} \, dx$$

$$= \frac{1}{2E} \int_0^L \frac{M^2}{I} \, dx = \int_0^L \frac{M^2}{2EI} \, dx$$

Regarding the form of the strain energy density

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}$$

U_0 : dependent on deformations

$$U_0 \equiv U_0(\varepsilon_{ij})$$

ε_{ij} → strain components are <u>dependent</u> on the choice of coordinate sys.

U_0 → physical quantity → <u>independent</u> of the choice of coordinate sys.

* Apparent discrepancy

$$U_0 \equiv U_0(I_e, II_e, III_e)$$, I_e, II_e, III_e : strain invariants

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Now, for linear elastic materials:

If U_0 just depended on III_e even linearly, then $\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}$ would lead to a dependence of σ_{ij} on quadratic terms involving the strain components because III_e involves cubic terms.

$$U_0 \equiv U_0(\mathrm{I}_e, \mathrm{II}_e)$$

If U_0 depended on I_e linearly then $\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}$ would produce a constant term.

↳ Represents a non-zero value of stress even in the absence of deformations

Although possible physically, we will not consider such situations.

So, U_0 should not depend on I_e linearly.

U_0 should depend on terms quadratic in the strain components.

Ensured if U_0 depends on I_e^2 and II_e (linearly)

$$U_0 = A I_e^2 + B II_e$$

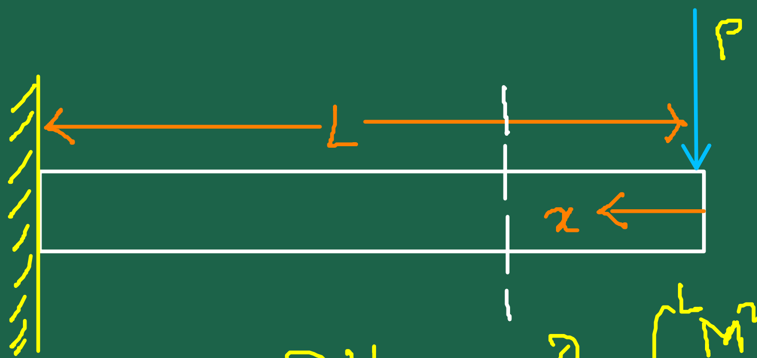
$$\sigma_{ij} = \frac{\partial}{\partial \varepsilon_{ij}} \left(A I_e^2 + B II_e \right)$$

$\vdots \qquad \vdots$

$\sigma_{ij} \rightarrow$ linearly dependent on ε_{ij} components

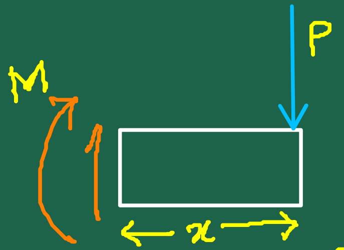
$\rightarrow$ Compare the above with $\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2G \varepsilon_{ij}$

Problems



$$\bar{u}_p = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx$$

$$M = -Px$$



$$\bar{u}_p = \frac{\partial}{\partial P}\int_0^L \frac{P^2x^2}{2EI}\,dx = \int_0^L \frac{Px^2}{EI}\,dx = \left[\frac{Px^3}{3EI}\right]_0^L = \frac{PL^3}{3EI}$$

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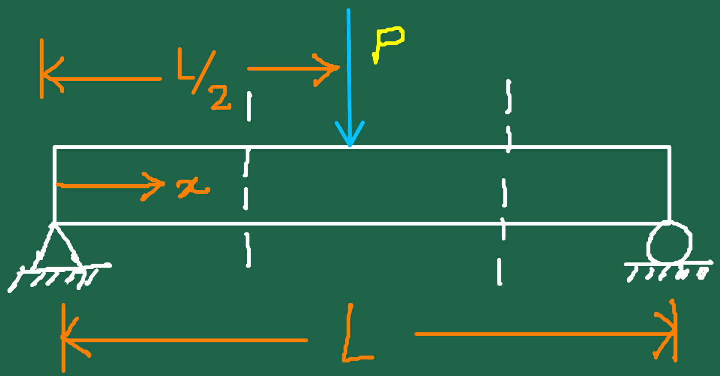
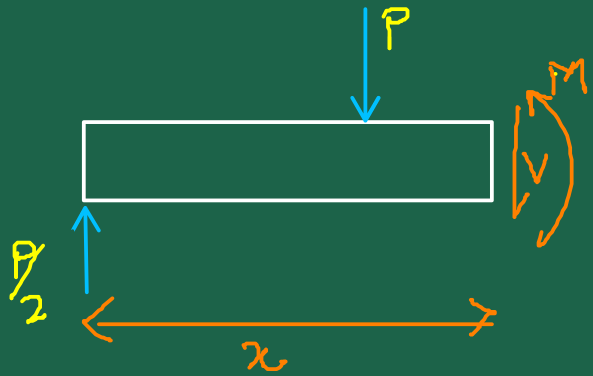
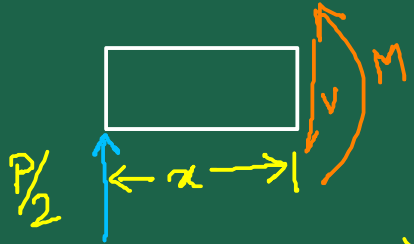




$$M - \frac{P}{2}x + P\left(x - \frac{L}{2}\right) = 0 \quad , \text{if } x > \frac{L}{2}$$

$$M - \frac{P}{2}x + P\left(x - \frac{L}{2}\right)H\left(x - \frac{L}{2}\right) = 0$$



$$M - \frac{P}{2}x = 0 \Rightarrow M = \frac{P}{2}x \quad , \text{if } x < \frac{L}{2}$$

$$\bar{u}_P = \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx$$

$$= \frac{\partial}{\partial P}\, 2\int_0^{L/2} \frac{M^2}{2EI}\,dx = \frac{\partial}{\partial P}\int_0^{L/2} \frac{P^2x^2}{4EI}\,dx = \int_0^{L/2}\frac{Px^2}{2EI}\,dx = \left[\frac{Px^3}{6EI}\right]_0^{L/2}$$

$$= \frac{PL^3}{48EI}$$

$$M - \frac{P}{2}x + \underbrace{P\left(x - \frac{L}{2}\right) H\left(x - \frac{L}{2}\right)} = 0$$

$$M - \frac{P}{2}x + P\left\langle x - \frac{L}{2}\right\rangle^{1} = 0$$

$\langle \ \rangle$ Macaulay bracket

Singularity function

$\langle x - a \rangle^{n}$

$$\bar{u}_p = \frac{\partial}{\partial P}\int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,dx$$

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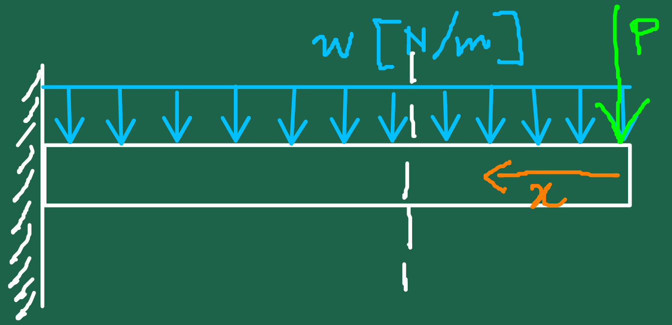
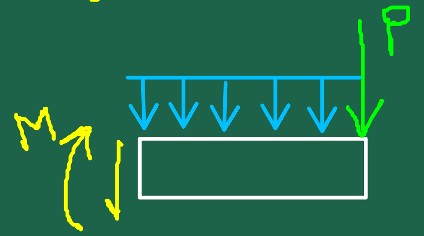


Deflection at the free end?

Introduce a fictitious point force at the free end



$$M + Px + wx\frac{x}{2} = 0 \Rightarrow M = -Px - w\frac{x^2}{2}$$

$$d = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\,dx \Bigg|_{P=0}$$

$$= \int_0^L \frac{\left(-Px - w\,x^2/2\right)}{EI}(-x)\,dx \Bigg|_{P=0} = \int_0^L \frac{wx^3}{2EI}\,dx = \frac{wL^4}{8EI}$$

Corollary to Castigliano's theorem on deflections

Just like $\bar{u}_F = \dfrac{\partial U'}{\partial F}$, we have for slope ($\theta$)

$$\theta = \frac{\partial U'}{\partial M_0}$$

For Hookean materials: $\theta = \dfrac{\partial U}{\partial M_0}$ $\left[\text{just like } \bar{u}_F = \dfrac{\partial U}{\partial F}\right]$

For beam problems:

$$\bar{u}_F = \frac{\partial}{\partial F}\int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}\,dx$$

$$\theta = \frac{\partial}{\partial M_0}\int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx$$

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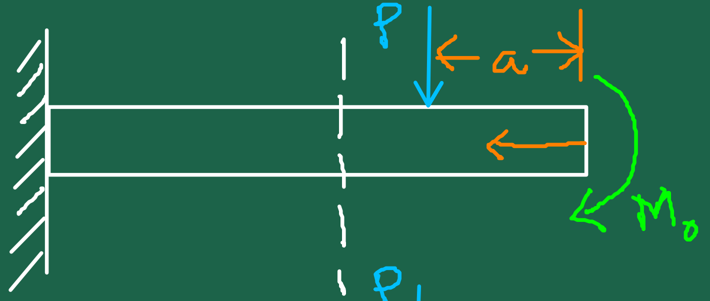
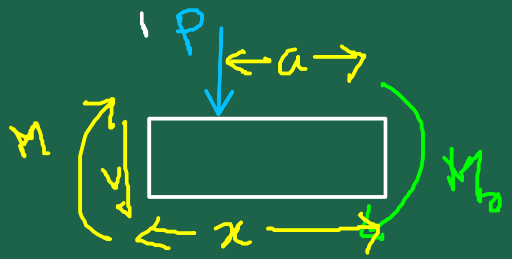


Rotation (slope) at the free end?



$$M + M_0 + P(x-a)H(x-a) = 0 \Rightarrow M = -M_0 - P(x-a)H(x-a)$$

$$\theta = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx \Big|_{M_0=0}$$

$$= \int_0^L \frac{-M_0 - P(x-a)H(x-a)}{EI}(-1)\,dx \Big|_{M_0=0}$$

$$= \int_a^L \frac{P(x-a)}{EI}\,dx = \frac{P}{EI}\left[\frac{x^2}{2} - ax\right]_a^L = \frac{P}{2EI}(L-a)^2$$

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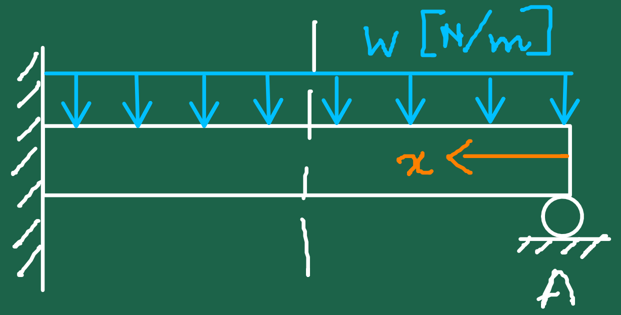
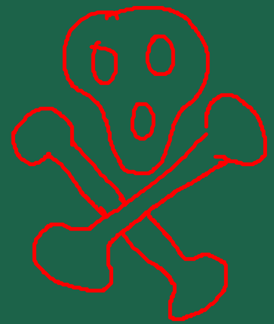
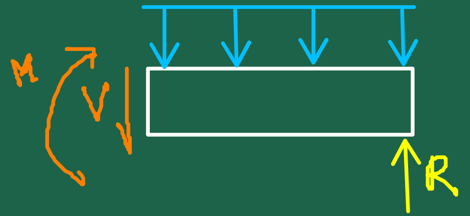


Reaction force at end A?



Do not use this method for statically determinate problems!



$$M - Rx + wx\frac{x}{2} = 0 \Rightarrow M = Rx - w\frac{x^2}{2}$$

$$\cancel{\delta_R} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R}\,dx$$

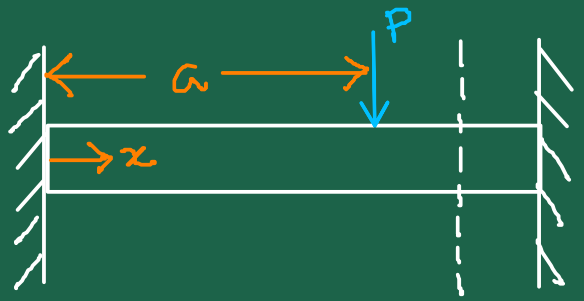
$$\Rightarrow 0 = \int_0^L \frac{\left(Rx - w\frac{x^2}{2}\right)}{EI}(x)\,dx$$

$$\Rightarrow 0 = \int_0^L \frac{\left(Rx^2 - \frac{wx^3}{2}\right)}{EI}\,dx$$

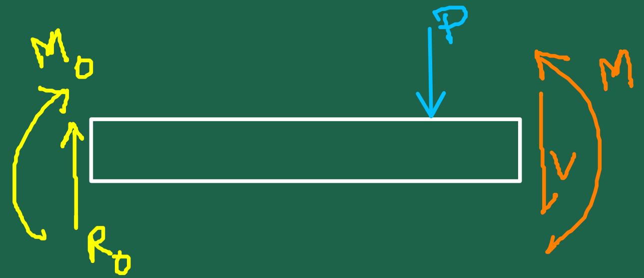
$$\Rightarrow 0 = \left[R\frac{x^3}{3} - w\frac{x^4}{8}\right]_0^L$$

$$\Rightarrow 0 = R\frac{L^3}{3} - w\frac{L^4}{8}$$

$$\Rightarrow R = \frac{3wL}{8}$$



Determine reactions (both forces and moments)



$$M - M_0 - R_0 x + P(x-a)H(x-a) = 0$$

$$0 \leftarrow \cancel{\delta}_{R_0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_0}\,dx$$

$$0 \leftarrow \cancel{\theta}_{M_0} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_0}\,dx$$

$\rightarrow$

$$R_0 = \frac{P(L-a)^2(L+2a)}{L^3}$$

$$M_0 = -\frac{P(L-a)^2 a}{L^2}$$

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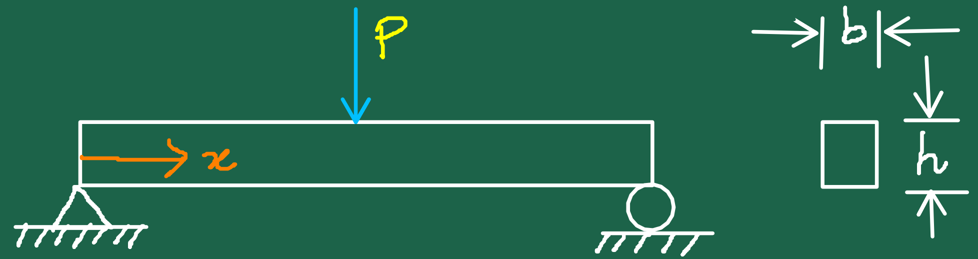

$$\sigma = K\varepsilon^{1/3}$$

For linear material behaviour $\quad d = \dfrac{\partial U}{\partial P} = \dfrac{\partial}{\partial P}\displaystyle\int_0^L \dfrac{M^2}{2EI}\,dx = \int_0^L \dfrac{M}{EI}\dfrac{\partial M}{\partial P}\,dx \rightarrow \times$ Not valid here because of non-linear material behaviour

$$M = \int_{-h/2}^{h/2} y\,\sigma_{xx}\,b\,dy$$

$$= \int_{-h/2}^{h/2} y\,K\varepsilon^{1/3}\,b\,dy \qquad \varepsilon = \frac{\Delta l}{l} = \frac{y}{R} \leftarrow \text{radius of curvature}$$

$$\cancel{\left[MR = EI\right]}$$

$$= \int_{-h/2}^{h/2} y\,K\left(\frac{y}{R}\right)^{1/3} b\,dy = \frac{Kb}{R^{1/3}}\int_{-h/2}^{h/2} y^{4/3}\,dy = \frac{Kb}{R^{1/3}}\left[\frac{y^{7/3}}{7/3}\right]_{-h/2}^{h/2} = \frac{3Kb}{7R^{1/3}}\left[\left(\frac{h}{2}\right)^{7/3} - \left(-\frac{h}{2}\right)^{7/3}\right]$$

$$= \frac{6Kb}{7R^{1/3}}\left(\frac{h}{2}\right)^{7/3}$$

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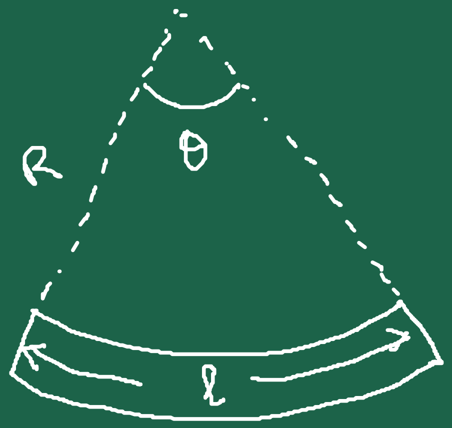
$$M = \frac{6kb}{7R^{1/3}}\left(\frac{h}{2}\right)^{7/3} = \frac{C'}{R^{1/3}} \Rightarrow M^3 = \frac{C}{R}$$



$$dU' = \frac{\partial U'}{\partial M}\,dM = \theta\,dM$$

$$= \frac{l}{R}\,dM$$

$$U' = \int_0^M \frac{l}{R}\,dM = l\underbrace{\int_0^M \frac{dM}{R}}_{f(M)}$$

$$\therefore \quad U' = l\,f(M)$$

$$U' = \int_0^L f(M)\,dx$$

$$\left[\, U = \cancel{\int_0^L \frac{M^2}{2EI}\,dx} \,\right]$$

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For deflection,

$$d = \frac{\partial U'}{\partial P} = \frac{\partial}{\partial P}\int_0^L f\, dx$$

$$= \int_0^L \frac{\partial f}{\partial M}\frac{\partial M}{\partial P}\, dx$$

Our $f(M) = \int_0^M \frac{dM}{R} \Rightarrow \frac{\partial f}{\partial M} = \frac{1}{R}$

$$\therefore d = \int_0^L \frac{1}{R}\frac{\partial M}{\partial P}\, dx$$

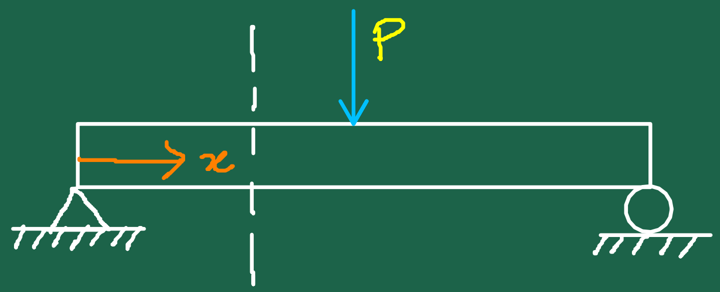
But we have earlier obtained $M^3 = \frac{C}{R}$

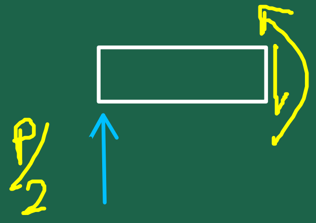
$$U = \int_0^L \frac{M^2}{2EI}\, dx$$

$$d = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}\, dx$$

$$d = \int_0^L \frac{M^3}{C} \frac{\partial M}{\partial P} \, dx$$

$$= 2 \int_0^{L/2} \frac{M^3}{C} \frac{\partial M}{\partial P} \, dx$$





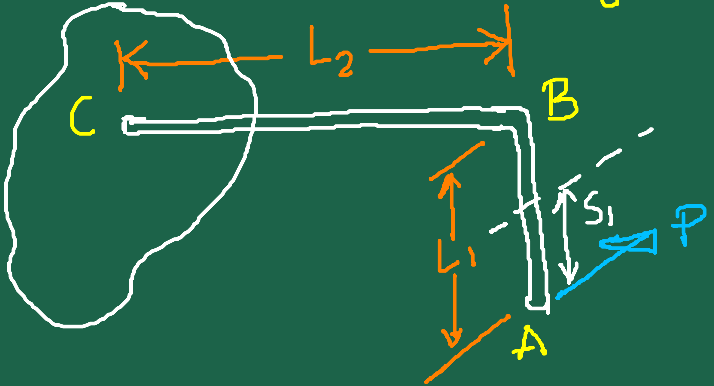
$$M - \frac{P}{2} x = 0$$

$$\Rightarrow M = \frac{P}{2} x$$

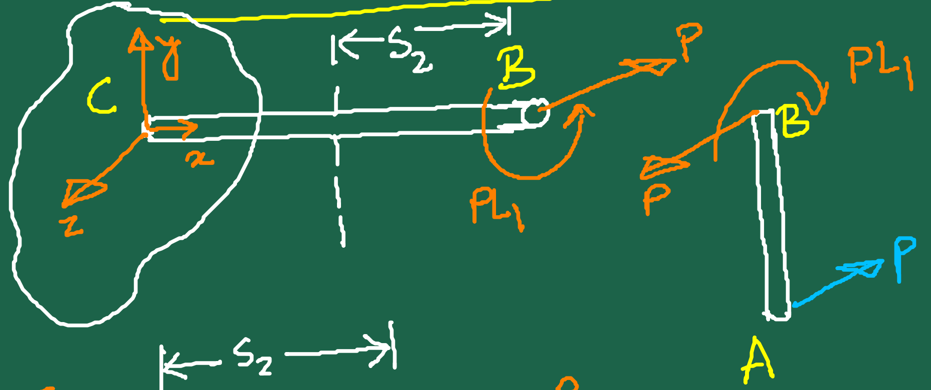
Strain energy:

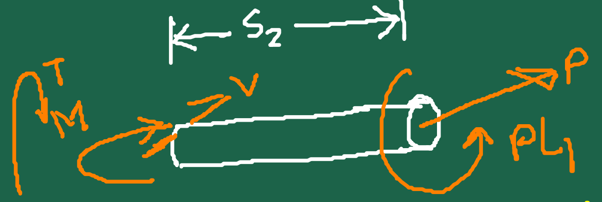
Bending: $U = \int_0^L \frac{M^2}{2EI}\,dx$; Axial Stretching $= \int_0^L \frac{P^2}{2AE}\,dx$; Torsion $= \int_0^L \frac{T^2}{2GJ}\,dx$

Deflection at A?



For the part BC:

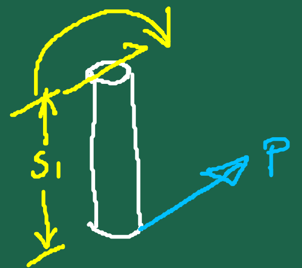




$M - PS_2 = 0 \Rightarrow M = PS_2$, $V = -P$

$T = PL_1$

For the part AB:



$M - PS_1 = 0$

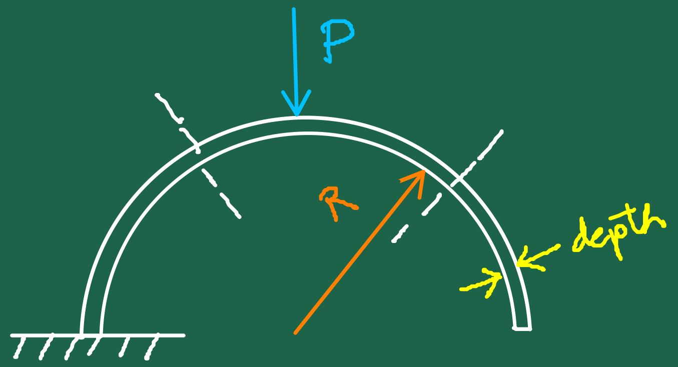
$\Rightarrow M = PS_1$

Total strain energy:

$$U = \underbrace{\int_0^{L_1} \frac{M^2}{2EI}\, ds_1}_{\text{for AB}} + \underbrace{\int_0^{L_2} \left[\frac{M^2}{2EI} + \frac{T^2}{2GJ}\right] ds_2}_{\text{for BC}}$$

$$d_A = \frac{\partial U}{\partial P} = \int_0^{L_1} \frac{M}{EI}\frac{\partial M}{\partial P}\, ds_1 + \int_0^{L_2} \frac{M}{EI}\frac{\partial M}{\partial P}\, ds_2 + \int_0^{L_2} \frac{T}{GJ}\frac{\partial T}{\partial P}\, ds_2$$

$$= \frac{PL_1^3}{3EI} + \frac{PL_2^3}{3EI} + \frac{PL_1^2 L_2}{GJ}$$

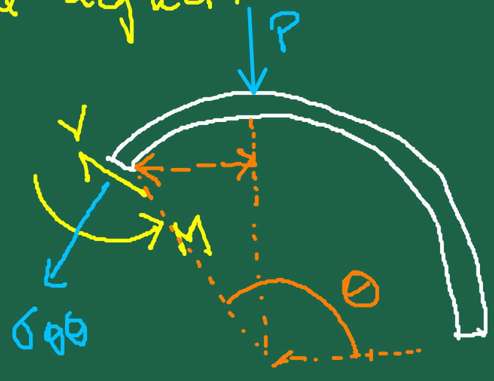


$R > 2\,(\text{depth})$

↳ Beam is thin

Deflection (both vertical & horizontal) at the pt. of application of P.

Vertical deflection :



$$M - PR\sin\left(\theta - \frac{\pi}{2}\right) H\left(\theta - \frac{\pi}{2}\right) = 0$$

$$d_v = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\int_0^{\pi} \frac{M^2}{2EI} R\,d\theta$$

$$= \int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial P} R\,d\theta$$

$$= \frac{\pi P R^3}{4EI}$$

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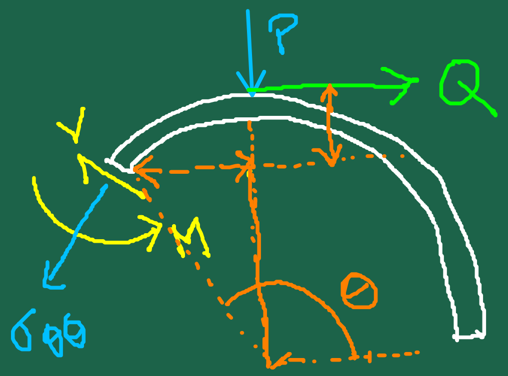

$$M - PR\sin\left(\theta - \frac{\pi}{2}\right) H\left(\theta - \frac{\pi}{2}\right) - Q\left[R - R\cos\left(\theta-\frac{\pi}{2}\right) H\left(\theta - \frac{\pi}{2}\right)\right] = 0$$

Horizontal deflection:

$$d_H = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = \left.\frac{\partial}{\partial Q}\int_0^{\pi} \frac{M^2}{2EI} R\,d\theta\right|_{Q=0}$$

$$= \left.\int_0^{\pi} \frac{M}{EI}\frac{\partial M}{\partial Q} R\,d\theta\right|_{Q=0}$$

$$= \frac{PR^3}{2EI}$$