Energy Methods[*,†]

1 Preliminary Concepts

Configuration The positions of all the material points of a mechanical system are together referred to as the configuration of the system.

Configuration space The set of all configurations that can be taken by a mechanical system is referred to as the configuration space.

Geometric notation Any point in the configuration space of a mechanical system is denoted by $\boldsymbol{X}$. For example, if the mechanical system consists of just one particle, then $\boldsymbol{X}$ can be the position vector of the particle.

Distance in configuration space When a mechanical system goes from one configuration (say, point $\boldsymbol{X}_0$ in configuration space) to another configuration (say, point $\boldsymbol{X}_1$ in configuration space), the "distance" between $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ is taken to be the maximum of the displacement magnitudes of the individual particles making up the system.

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[†] The development of the theory mostly follows the classic "Energy Methods in Applied Mechanics" by Henry L. Langhaar (Dover Publications)

Path in configuration space The configuration X of a mechanical system is said to be a function of a real variable $\bar{t}$ in the interval $a \leq \bar{t} \leq b$ if, to each value of $\bar{t}$ in this interval, there exists a single configuration; thus $X \equiv X(\bar{t})$, $\bar{t} \in [a, b]$. The function so formed represents a path in configuration space.
Note that not every path is admissible because the constraints of the mechanical system have to be respected.
Important: The variable $\bar{t}$ is not necessarily time. If $\bar{t}$ is indeed time, then $X \equiv X(t)$ is called the "motion" of the system.

2 Virtual Displacement

According to the classic book by Goldstein et al.[‡]:

> *A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates ... consistent with the forces and constraints imposed on the system at the given instant t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval* $\mathrm{d}t$*, during which the forces and constraints may be changing.*

In terms of the preliminary concepts, the virtual displacement can be realized by the following:

- Freeze time first.
- Consider paths in configuration space corresponding to some real variable t' (which is strictly not the time t).
- Restrict to paths which are admissible i.e. respect the constraints of the system.

[‡]H. Goldstein, C. P. Poole and J. Safko, *Classical Mechanics*, 3rd ed., Pearson

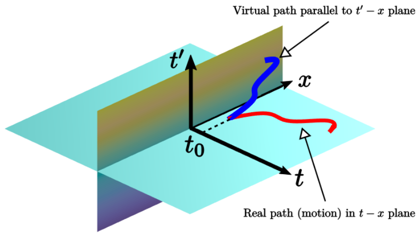

Figure 1: Virtual displacement vs. Real Displacement

These can be understood by considering the simplest of mechanical systems: a single particle travelling along the x-direction as depicted in Fig. 1. The path taken by the particle corresponding to time (t), i.e. the motion of the particle is depicted in red. A small increment in time, dt corresponds to a small real displacement dx. We freeze time at $t = t_0$ and consider the virtual path (depicted in blue) in a plane parallel to the $t' - x$ plane. We note that $x(t' = 0) = x(t = t_0)$.

The virtual displacement of the particle at any t' is given by

$$\begin{aligned}
u &= x(t') - x(t_0') \\
&= x(t_0 + t' - t_0) - x(t_0') \\
&= x(t_0') + (t' - t_0') \left.\frac{\mathrm{d}x}{\mathrm{d}t'}\right|_{t'=t_0'} + \frac{1}{2}(t' - t_0')^2 \left.\frac{\mathrm{d}x}{\mathrm{d}t'}\right|_{t'=t_0'} + \cdots - x(t_0') \\
&= (t' - t_0') \left.\frac{\mathrm{d}x}{\mathrm{d}t'}\right|_{t'=t_0'} + \frac{1}{2}(t' - t_0')^2 \left.\frac{\mathrm{d}x}{\mathrm{d}t'}\right|_{t'=t_0'} + \cdots \\
&= \delta u + \frac{1}{2}\delta^2 u + \cdots \qquad \text{(Written using the } \delta \text{ operator as in the language of calculus of va}
\end{aligned}$$

Note that when $t' \longrightarrow t_0'$, the displacement u is well approximated by just the first term on the R.H.S. i.e. δu only. Of course the virtual displacement corresponding to such a small increment in t' is also small. This virtual infinitesimal displacement (δu) is also referred to as the first variation of the displacement. In fact, the δ operator§ can be read as "the

§NOT to be confused with the Kronecker delta - they are completely separate things!

first variation of".

3 Law of kinetic energy

The work of all forces (external AND internal) that act on a mechanical system equals the increase of kinetic energy of the system.

$$\begin{aligned} W &= \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} \, \mathrm{d}t \\ &= \int_{t_0}^{t_1} m \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} \mathrm{d}t \\ &= \frac{1}{2} \int_{t_0}^{t_1} m \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{v} \cdot \mathbf{v}) \, \mathrm{d}t \\ &= \Delta T. \end{aligned} \tag{1}$$

This law is restricted to inertial reference frames. ΔT changes value from one such reference frame to another. Therefore, W also changes value with change of reference frame.

When a mechanical system begins to move it is gaining kinetic energy. So by the law of kinetic energy, the external and internal forces must be doing net positive work. Therefore, the system does not move spontaneously unless it can experience some arbitrarily small displacement for which the net work of all forces is positive.

Spontaneous movement is impossible if this work done is less or equal to zero for all small displacements. These small displacements are not necessarily realized, so they are virtual displacements. Likewise the work done is virtual work.

4 Fourier's inequality

A motionless mechanical system remains at rest if the virtual work done by all external and internal forces is less than or equal to zero for every small virtual displacement that is consistent with the constraints.

Example: Brick resting on a table. If the brick slides slightly, friction performs negative work. If it tilts slightly, gravity performs negative work. If it executes a small hop no work is done. So the virtual work is less than or equal to zero for each case.

Very important: Fourier's inequality is sufficient but not necessary for equilibrium.

Example: A ball that is just balanced on a dome is in a state of equilibrium. But when rolled slightly gravity performs positive work. So despite the virtual work not being less than or equal to zero, the ball *is* in equilibrium. The equilibrium is not a stable one, but that is a different story! The problem with Fourier's inequaility is that it does not take cognizance of the fact that the virtual work, although positive, is an infinitesimal of higher order than the virtual displacement of the ball. revised to require that the virtual work is stationary in the sense that it is an infinitesimal of higher order than the displacment.

Thus, it would be very desirable to have some condition which is both necessary and sufficient for equilibrium - that brings us to the principle of virtual work.

5 An important aside

Before starting the discussion on the principle of virtual work, note something very important:

In going from one configuration $\boldsymbol{X}_0$ to another configuration $\boldsymbol{X}_1$, there can be a number of different paths and associated with each such path there will be a virtual work. Thus, if we denote the virtual work corresponding to a particular choice of path as W' then there can be many such W'. Each such W' will be *path-dependent*. Now, consider the set of all such W'. Unless the mechanical system under consideration possesses unlimited energy, this set of W' will have a least upper bound, say W. That is, W is the smallest value such that each $W' \leq W$. This least upper bound W is something uniquely identified with the configurations $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$, and it does *not* depend on any particular path between the configurations. Thus, W is *path-independent*; in other words it is simply a function of the two configurations $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$. It is therefore referred to as the work function and denoted completely as $W(\boldsymbol{X}_0, \boldsymbol{X}_1)$.

Now, if $W' \leq 0$ for any path that goes from a configuration $\boldsymbol{X}_0$ to a neighbouring configuration $\boldsymbol{X}$, then it must also be true that $W(\boldsymbol{X}_0, \boldsymbol{X}) \leq 0$. The reverse will also be true. Thus, Fourier's inequality may be recast in terms of the work function as:

> *Fourier's inequality:* The configuration $\boldsymbol{X}_0$ is in equilibrium if $W(\boldsymbol{X}_0, \boldsymbol{X}) \leq 0$ for all configurations $\boldsymbol{X}$ in the neighbourhood of $\boldsymbol{X}_0$.

As already mentioned, this inequality is only a sufficient but not a necessary condition for equilibrium. The necessary condition is given by the principle of virtual work.

6 Principle of virtual work

If the forces do not change discontinuously, then the necessary and sufficient condition for a point $\boldsymbol{X}_0$ in configuration space to be in equilibrium is $\lim_{s \to 0} \frac{W}{s} = 0$, where s is the distance¶ between $\boldsymbol{X}_0$ and another point $\boldsymbol{X}$ in the neighbourhood of $\boldsymbol{X}_0$ and W is the work function associated with the two configuration points $\boldsymbol{X}_0$ and $\boldsymbol{X}$.

If the variational form of W exists, i.e. if W can be expressed as

$$W = \delta W + \frac{1}{2!}\delta^2 W + O(s^3), \tag{2}$$

then $\lim_{s \to 0} \frac{W}{s} = 0$ is equivalent to $\delta W = 0$. This form is the conventionally written form of the principle of virtual work. We are not going to delve into how the variational form is equivalent to the limit form. For the purposes of this course, it will be enough to use the fact that for equilibrium, the first variation of the total work (due to both external and internal forces) must be zero. If the total work is decomposed into the work due to external forces W_e and that due to internal forces W_i then

¶For the meaning of "distance", refer to the preliminary concepts in §1

$\delta W = 0$ gives us

$$-\delta W_i = \delta W_e. \tag{3}$$

7 Deformable Body

The configuration of a deformable body is defined by the displacement vector field. The deformable body is carried from one configuration to another "neighbouring" configuration through $(\delta u_1, \delta u_2, \delta u_3)$ which are the components of $\delta \boldsymbol{u}$.

Now, consider the body to be in equilibrium under the action of traction forces and body forces. Then, by the principle of virtual work, we have

$$\begin{aligned} -\delta W_i = \delta W_e &= \int_V \rho \boldsymbol{b} \cdot \delta \boldsymbol{u} \, \mathrm{d}V + \int_S \mathbf{T} \cdot \delta \boldsymbol{u} \, \mathrm{d}S \\ &\equiv \int_V \rho b_i \delta u_i \, \mathrm{d}V + \int_S T_i \delta u_i \, \mathrm{d}S. \end{aligned}$$

Referring to the chapter "**Analysis of Stress**" under the section "**The state of stress at a point**", we have

$$T_i = \sigma_{ji} n_j,$$

where σ_{ji} is the stress and n_j is the unit normal vector to the plane which is tangent to the surface at the point we are considering the traction. We then have

$$\begin{aligned} \delta W_e &= \int_V \rho b_i \delta u_i \, \mathrm{d}V + \int_S \sigma_{ji} n_j \delta u_i \, \mathrm{d}S \\ &= \int_V \rho b_i \delta u_i \, \mathrm{d}V + \int_V \frac{\partial}{\partial x_j} \left(\sigma_{ji} \delta u_i \right) \, \mathrm{d}V \quad \text{(Using Gauss divergence theorem)} \\ &= \int_V \rho b_i \delta u_i \, \mathrm{d}V + \int_V \left[\frac{\partial \sigma_{ji}}{\partial x_j} \delta u_i + \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \right] \, \mathrm{d}V \\ &= \int_V \left(\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j} \right) \delta u_i \, \mathrm{d}V + \int_V \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \, \mathrm{d}V \end{aligned}$$

Again referring to the chapter "**Analysis of Stress**" under the section "**Cauchy's equation of motion and mechanical equilibrium equations**", we know that for a body in equilibrium we have

$$\rho b_i + \frac{\partial \sigma_{ji}}{\partial x_j} \equiv \rho \boldsymbol{b} + \nabla \cdot \boldsymbol{\sigma} = 0.$$

Therefore,

$$\begin{aligned}
\delta W_e &= \int_V \sigma_{ji} \frac{\partial \delta u_i}{\partial x_j} \, \mathrm{d}V \\
&= \int_V \sigma_{ji} \delta \frac{\partial u_i}{\partial x_j} \, \mathrm{d}V \quad \text{(Interchanging the order of } \delta \text{ and partial derivatives)} \\
&= \int_V \sigma_{ji} \delta \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right] \, \mathrm{d}V \\
&= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, \mathrm{d}V + \frac{1}{2} \int_V \left(\sigma_{ji} \frac{\partial u_i}{\partial x_j} - \sigma_{ji} \frac{\partial u_j}{\partial x_i} \right) \, \mathrm{d}V \\
&= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, \mathrm{d}V + \frac{1}{2} \int_V \left(\sigma_{ji} \frac{\partial u_i}{\partial x_j} - \sigma_{ij} \frac{\partial u_j}{\partial x_i} \right) \, \mathrm{d}V \quad \text{(Using } \sigma_{ji} = \sigma_{ij}\text{)}
\end{aligned}$$

In the second integral we note that the indices are i and j are repeated in both the terms. So they can be easily substituted by something else. In the first term we substitute i by k and j by p. In the second term, we substitute i by p and j by k, to obtain

$$\begin{aligned}
\delta W_e &= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, \mathrm{d}V + \frac{1}{2} \int_V \left(\sigma_{pk} \frac{\partial u_k}{\partial x_p} - \sigma_{pk} \frac{\partial u_k}{\partial x_p} \right) \, \mathrm{d}V \\
&= \int_V \sigma_{ji} \delta \varepsilon_{ij} \, \mathrm{d}V \qquad (4)
\end{aligned}$$

8 First law of thermodynamics

According to the first law of thermodynamics, we have

$$\delta W_e + \delta Q = \delta U + \delta T,$$

where Q is the heat transferred into the body and U is the internal energy of the body.

For an adiabatic process: $Q = 0$.

For a body in equilibrium: $\delta T = 0$.

Therefore, we obtain $\delta W_e = \delta U$. If we define U_0 as the internal energy density per unit volume, then we have

$$\delta W_e = \delta U = \delta \int_V U_0 \, \mathrm{d}V = \int_V \delta U_0 \, \mathrm{d}V. \tag{5}$$

9 Potential energy

A mechanical system is said to be conservative if the virtual work corresponding to a virtual displacement of the system completely around any closed path is zero. For virtual displacements executed with infinitesimal speed, the system is conservative in the static sense. For virtual displacements that are not necessarily executed with infinitesimal speed, the system is conservative in the kinetic sense.

If the virtual work, W_e, associated with the external forces is independent of path and depends only on the terminal configurations then the external forces are said to be conservative. Because of the sole dependence on terminal configurations W_e can be expressed as a point function: $W_e = -V_e(X)$. The initial configuration is not mentioned because it affects only an additive constant. This point function V_e is called the potential energy of the external forces.

Similarly, if the virtual work, W_i, associated with the internal forces is independent of path, then the internal forces are said to be conservative and W_i can be expressed as a point function: $W_i = -V_i(X)$. Here, V_i is called the potential energy of internal forces.

If both internal and external forces are conservative then the total virtual work $W = W_e + W_i$ can be expressed as a point function: $W = -\Pi(X) = -(V_e(X) + V_i(X))$.

10 Strain energy

A mechanical system is said to be elastic if the internal forces are conservative in the kinetic sense. Then $V_i(X)$, the potential energy of internal forces is called the **strain energy.** We can also define a strain energy density (V_{i0}) as the strain energy per unit volume. The strain energy density depends on the strain components: $V_{i0} \equiv V_{i0}(\varepsilon_{ij})$. Thus,

$$\delta V_{i0} = \frac{\partial V_{i0}}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}. \tag{6}$$

The strain energy of a system differs from the internal energy only by an additive constant. Since during any deformation process we track changes of internal energy this additive constant is irrelevant. So, $V_i \equiv U$ and $V_{i0} \equiv U_0$. Because the strain energy differs from the internal energy only by an additive constant, from Eq. (6), we have

$$\delta U_0 = \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij}. \tag{7}$$

Similarly, we rewrite Eq. (5)

$$\delta W_e = \int_V \delta U_0 \, \mathrm{d}V \tag{8}$$

but reinterpret it as a relation between work due to external forces and strain energy.

Combining Eq. (7) and Eq. (8), we have

$$\delta W_e = \int_V \frac{\partial U_0}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} \, \mathrm{d}V. \tag{9}$$

Finally, comparing Eq. (4) with Eq. (9), we obtain

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}. \tag{10}$$

11 Complementary Energy

We have found that

$$\sigma_{ij} = \frac{\partial U_0}{\partial \varepsilon_{ij}}$$

We can find a related expression where on the L.H.S., instead of σ_{ij} we have ε_{ij}, and where on the R.H.S., instead of the derivative w.r.t. ε_{ij}, we have a derivative w.r.t. σ_{ij}. In order to find such an expression, we introduce something called the compelementary energy density

$$U_0' = -U_0 + \sigma_{pq}\varepsilon_{pq}. \tag{11}$$

To Prove: $\varepsilon_{ij} = \dfrac{\partial U_0'}{\partial \sigma_{ij}}$

Proof: Differentiating both sides of Eq. (11), we have

$$\begin{aligned}
\frac{\partial U_0'}{\partial \sigma_{ij}} &= -\frac{\partial U_0}{\partial \sigma_{ij}} + \delta_{pi}\delta_{qj}\varepsilon_{pq} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} \\
&= -\frac{\partial U_0}{\partial \varepsilon_{pq}}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} + \varepsilon_{ij} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} \\
&= -\sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} + \varepsilon_{ij} + \sigma_{pq}\frac{\partial \varepsilon_{pq}}{\partial \sigma_{ij}} \\
&= \varepsilon_{ij}
\end{aligned} \tag{12}$$

The volume integral of the complementary energy density function U_0' is defined as the complementary energy of the body; thus

$$U' = \int_V U_0' \, \mathrm{d}V.$$

12 Generalization of Castigliano's theorem of least work

Despite the name, this theorem does not directly deal with any "work" nor there is anything that directly takes on a "least" or "minimum" value. Instead, it deals with a modified form of the complementary energy (to be represented by Ψ) and expresses an extremum condition.

Theorem: *Consider a body whose boundary S consists of two parts: a part S_1 on which the traction vector is given, and a part S_2 on which the displacement vector is given. Then of all states of stress that satisfy the mechanical equilibrium equations and the boundary conditions on S_1, the* *state of stress which represents the actual equilibrium corresponds to an extremum of the modified complementary energy, thus,*

$$\boxed{\delta\Psi = 0},$$

where the modified complementary energy is given by

$$\boxed{\Psi = U' - \int_{S_2} u_i T_i \, \mathrm{d}S}$$

Proof: Before launching into the proof, we need to show a couple of things:

First, for the stress state that satisfies the mechanical equilibrium equations, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0.$$

Taking the variation of this equation, we have

$$\frac{\partial \delta\sigma_{ij}}{\partial x_j} = 0, \tag{13}$$

where we use the fact that since the body forces ρb_i are given to us, their variations are zero.

Second, considering the definition of the complementary energy density in Eq. (11), and taking the variation, we have

$$\begin{aligned} \delta U_0' &= -\delta U_0 + \delta\sigma_{pq}\varepsilon_{pq} + \sigma_{pq}\delta\varepsilon_{pq} \\ &= -\sigma_{ij}\delta\varepsilon_{ij} + \delta\sigma_{pq}\varepsilon_{pq} + \sigma_{pq}\delta\varepsilon_{pq} \\ &= \delta\sigma_{pq}\varepsilon_{pq} \end{aligned} \tag{14}$$

We now launch into the actual proof:

$$
\begin{aligned}
\delta U' &= \int_V \delta U_0' \, \mathrm{d}V \qquad (15)\\
&= \int_V \varepsilon_{ij} \delta\sigma_{ij} \, \mathrm{d}V \quad \text{(Using Eq. (14))}\\
&= \int_V \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \delta\sigma_{ij} \, \mathrm{d}V\\
&= \int_V \frac{\partial u_i}{\partial x_j} \delta\sigma_{ij} \, \mathrm{d}V \quad \text{(Using } \sigma_{ij} = \sigma_{ji}\text{)}\\
&= \int_V \frac{\partial}{\partial x_j}\left(u_i \delta\sigma_{ij}\right) \, \mathrm{d}V - \int_V u_i \frac{\partial \delta\sigma_{ij}}{\partial x_j} \, \mathrm{d}V\\
&= \int_S u_i \delta\sigma_{ij} n_j \, \mathrm{d}S \quad \text{(Using Gauss divergence theorem in the 1st integral above, and E}\\
&= \int_{S_2} u_i \delta\sigma_{ij} n_j \, \mathrm{d}S + \int_{S_1} u_i \delta\sigma_{ij} n_j \, \mathrm{d}S\\
&= \int_{S_2} u_i \delta\left(\sigma_{ij} n_j\right) \, \mathrm{d}S + \int_{S_1} u_i \delta\left(\sigma_{ij} n_j\right) \, \mathrm{d}S \quad \text{(Geometry does not change so that } \delta n_j = \\
&= \int_{S_1} u_i \delta T_i \, \mathrm{d}S + \int_{S_2} u_i \delta T_i \, \mathrm{d}S\\
&= \int_{S_2} u_i \delta T_i \, \mathrm{d}S \quad \text{(Tractions are specified on } S_1\text{; so } \delta T_i = 0\text{)} \qquad (16)
\end{aligned}
$$

The modified complementary energy is defined as $\Psi = U' - \int_{S_2} u_i T_i \, \mathrm{d}S$. Taking the first variation, we have

$$
\begin{aligned}
\delta\Psi &= \delta U' - \int_{S_2} \left(T_i \delta u_i + u_i \delta T_i\right) \, \mathrm{d}S\\
&= \delta U' - \int_{S_2} u_i \delta T_i \, \mathrm{d}S \quad \text{(Displacements are specified on } S_2\text{; so } \delta u_i = 0\text{)}
\end{aligned}
$$

Finally, using Eq. (16), we can conclude that last line above should be zero. Therefore, we obtain

$$
\boxed{\delta\Psi = 0} \qquad \text{Q.E.D} \qquad (17)
$$

13 Castigliano's theorem on deflections

Theorem: *If an elastic body is mounted in such a way that rigid body displacements are prevented and the body is in equilibrium under the action of surface tractions, body forces, and "point forces", then at the point of application of such a "point force", say* $\mathbf{F} \equiv F_i$, *the displacement component* $\bar{u}_F$ *along the direction of* $\mathbf{F}$ *is given by*

$$\boxed{\bar{u}_F = \frac{\partial U'}{\partial F}}$$

Proof: Let F_1, F_2, ...denote the "point forces". Each of these "point forces" is considered to be distributed on a small spot of surface.

The complementary energy U' depends on $F_1, F_2, \ldots$ as well as on surface tractions and body forces.

The boundary S is divided into S_1 and S_2. On S_2, the displacement vector is 0 because of constraints. Within S_1, consider a small area S_0 on which a concentration of load is applied - it is this concentration of load which is understood to result in the "point force".

Consider a variation of the surface traction that vanishes in the region of S_1 that excludes S_0. Then by Castigliano's theorem of least work, we have

$$\begin{aligned} \delta U' &= \int_S u_i \delta T_i \, \mathrm{d}S \\ &= \int_{S_0} u_i \delta T_i \, \mathrm{d}S \end{aligned}$$

According to the theorem of the mean for integrals if $f(x, y)$ and $\varphi(x, y)$ are continuous real functions in a region A of the (x, y) plane and $\varphi(x, y)$ does not change sign in A then

$$\int_A f(x, y)\varphi(x, y) \, \mathrm{d}A = K \int_A \varphi(x, y) \, \mathrm{d}A,$$

where K is the value of $f(x, y)$ at some point in A.

Since we may suppose T_i not to change within the little region S_0, we can conclude

$$\delta U' = \bar{u}_i \int_{S_0} \delta T_i \, \mathrm{d}S,$$

where $\bar{u}_i$ are the components of the displacement vector u_i at some point within S_0.

Now, for the "point force" vector F_i, we have

$$F_i = \int_{S_0} T_i \, \mathrm{d}S$$

$$\text{or,} \quad \delta F_i = \int_{S_0} \delta T_i \, \mathrm{d}S.$$

Therefore, we have

$$\delta U' = \bar{u}_i \delta F_i.$$

We consider the unit vector along F_i to be N_i so that $F_i = N_i F$, which gives us $\delta F_i = N_i \delta F$. Then, we have

$$\begin{aligned} \delta U' &= \bar{u}_i N_i \delta F \\ &= \bar{u}_F \delta F, \end{aligned}$$

where $\bar{u}_F$ is the component of the displacement vector in the direction of the "point force" F_i.

Furthermore, we have

$$\delta U' = \frac{\partial U'}{\partial F} \delta F.$$

Therefore, we must have

$$\bar{u}_F = \frac{\partial U'}{\partial F}.$$

For Hookean materials, i.e. linear, elastic solid materials, the complementary energy is idential to the strain energy, i.e. $U' = U$, so that for these materials

$$\bar{u}_F = \frac{\partial U}{\partial F} \quad \text{(True only for Hookean materials)}$$

This special case was what was derived by Castigliano himself.