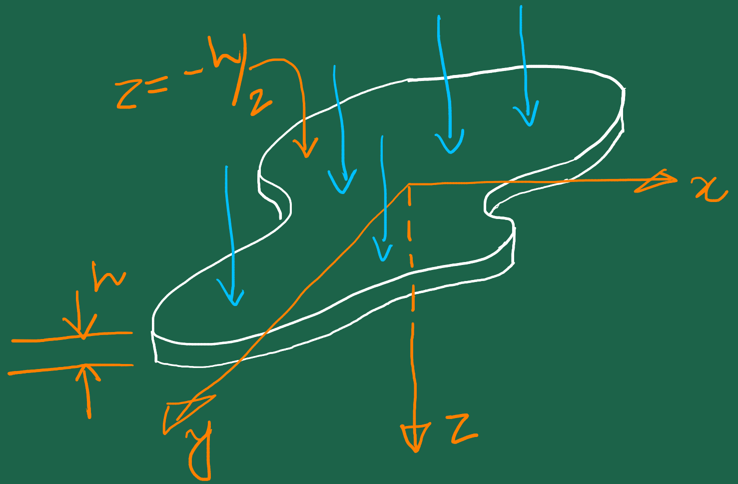
Classical Plate Theory



$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \quad ✓$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \quad ✓$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u_s}{\partial y} - z\frac{\partial^2 w}{\partial y\partial x}\right) + \frac{1}{2}\left(\frac{\partial v_s}{\partial x} - z\frac{\partial^2 w}{\partial x\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x}\right) - z\frac{\partial^2 w}{\partial x\partial y} \quad ✓$$

Kinematical Hypothesis

$$u = u_s - z\frac{\partial w}{\partial x}$$

$$v = v_s - z\frac{\partial w}{\partial y}$$

$$w = w(x,y)$$

$$u_s, v_s \equiv u_s(x,y),\ v_s(x,y)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(\cancel{\frac{\partial v_s}{\partial z}} - \frac{\partial w}{\partial y}\right) + \frac{1}{2}\frac{\partial w}{\partial y} = 0$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\frac{\partial w}{\partial x} + \frac{1}{2}\left(\frac{\partial u_s}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

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$$\cancel{\varepsilon_{zz}}^{\,0} = \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \Rightarrow \underbrace{\sigma_{zz}}_{\neq 0} = \nu(\underbrace{\sigma_{xx}}_{\neq 0} + \underbrace{\sigma_{yy}}_{\neq 0})$$

BUT ... we <u>forcibly assume</u> $\sigma_{zz} = 0$

$$\begin{cases} \varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}})\right] = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\ \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \cancel{\sigma_{zz}})\right] = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \end{cases}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy})$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx})$$

Now, we proceed using the PVW:

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int_A t_i\,\delta u_i\,dV$$

$$LHS = \int_A \int_{-h/2}^{h/2} \left(\sigma_{xx}\,\delta\varepsilon_{xx} + \sigma_{yy}\,\delta\varepsilon_{yy} + 2\,\sigma_{xy}\,\delta\varepsilon_{xy}\right) dz\,dA$$

$$= \int_A \int_{-h/2}^{h/2} \sigma_{xx}\left(\frac{\partial \delta u_s}{\partial x} - z\frac{\partial^2 \delta w}{\partial x^2}\right) dz\,dA + \int_A \int_{-h/2}^{h/2} \sigma_{yy}\left(\frac{\partial \delta v_s}{\partial y} - z\frac{\partial^2 \delta w}{\partial y^2}\right) dz\,dA$$

$$+\,2\int_A \int_{-h/2}^{h/2} \sigma_{xy}\,\frac{1}{2}\left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} - 2z\frac{\partial^2 \delta w}{\partial x\,\partial y}\right) dz\,dA$$

Henceforth, we are going to consider only the bending terms

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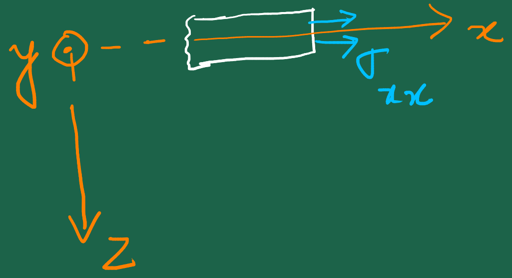
Define :

$$M_x = \int_{-H/2}^{H/2} \sigma_{xx}\, z \, dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_{yy}\, z \, dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, z \, dz$$



$$LHS_{bending} = -\int_A M_x \frac{\partial^2 \delta w}{\partial x^2}\, dA - \int_A M_y \frac{\partial^2 \delta w}{\partial y^2}\, dA - 2\int_A M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y}\, dA$$

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$$-\text{LHS}_{\text{bending}} = +\int_A M_x \frac{\partial^2 \delta w}{\partial x^2}\, dA + \int_A M_y \frac{\partial^2 \delta w}{\partial y^2}\, dA + 2\int_A M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y}\, dA$$

$$= \int_A \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x}\right) dA - \int_A \frac{\partial M_x}{\partial x}\frac{\partial \delta w}{\partial x} dA + \int_A \frac{\partial}{\partial y}\left(M_y \frac{\partial \delta w}{\partial y}\right) dA - \int_A \frac{\partial M_y}{\partial y}\frac{\partial \delta w}{\partial y} dA$$

$$+ \int_A \frac{\partial}{\partial x}\left(M_{xy} \frac{\partial \delta w}{\partial y}\right) dA - \int_A \frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y} + \int_A \frac{\partial}{\partial y}\left(M_{xy} \frac{\partial \delta w}{\partial x}\right) dA - \int \frac{\partial M_{xy}}{\partial y}\frac{\partial \delta w}{\partial x} dA$$

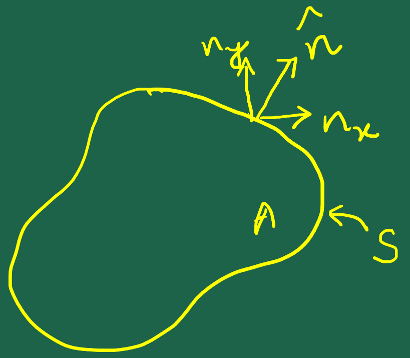
A brief interlude regarding Green's Theorem:

$$\int_A \left\{ \frac{\partial}{\partial x}(P) + \frac{\partial}{\partial y}(Q) \right\} dA = \int_S \left(P n_x + Q n_y\right) ds$$

$$\int_A \nabla \cdot \vec{R}\, dA = \int_S \vec{R} \cdot \hat{n}\, ds$$

$$\vec{R} = P\hat{i} + Q\hat{j}$$

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$



$$-\mathrm{LHS}_{\text{bending}} = \int_A \frac{\partial}{\partial x}\left(M_x \frac{\partial \delta w}{\partial x}\right) dA - \int_A \frac{\partial M_x}{\partial x}\frac{\partial \delta w}{\partial x} dA + \int_A \frac{\partial}{\partial y}\left(M_y \frac{\partial \delta w}{\partial y}\right) dA - \int_A \frac{\partial M_y}{\partial y}\frac{\partial \delta w}{\partial y} dA$$

$$+ \int_A \frac{\partial}{\partial x}\left(M_{xy}\frac{\partial \delta w}{\partial y}\right) dA - \int_A \frac{\partial M_{xy}}{\partial x}\frac{\partial \delta w}{\partial y} dA + \int_A \frac{\partial}{\partial y}\left(M_{xy}\frac{\partial \delta w}{\partial x}\right) dA - \int_A \frac{\partial M_{xy}}{\partial y}\frac{\partial \delta w}{\partial x} dA$$

$$= \oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y\right) dS + \oint_S \left(M_{xy}\frac{\partial \delta w}{\partial y} n_x + M_{xy}\frac{\partial \delta w}{\partial x} n_y\right) dS$$

$$- \int_A \frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}\delta w\right) dA \; (1) + \int_A \frac{\partial^2 M_x}{\partial x^2}\delta w \, dA - \int_A \frac{\partial}{\partial y}\left(\frac{\partial M_y}{\partial y}\delta w\right) dA \; (2) + \int_A \frac{\partial^2 M_y}{\partial y^2}\delta w \, dA$$

$$- \int_A \frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x}\delta w\right) dA \; (3) + \int_A \frac{\partial^2 M_{xy}}{\partial x \partial y}\delta w \, dA - \int_A \frac{\partial}{\partial x}\left(\frac{\partial M_{xy}}{\partial y}\right)\delta w \, dA \; (4) + \int_A \frac{\partial^2 M_{xy}}{\partial x \partial y}\delta w \, dA$$

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$$-\mathrm{LHS}_{\text{bending}} = \oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds + \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$- \int_A \frac{\partial}{\partial x} \left(\frac{\partial M_x}{\partial x} \delta w \right) dA + \int_A \frac{\partial^2 M_x}{\partial x^2} \delta w \, dA \checkmark - \int_A \frac{\partial}{\partial y} \left(\frac{\partial M_y}{\partial y} \delta w \right) dA + \int_A \frac{\partial^2 M_y}{\partial y^2} \delta w \, dA \checkmark$$

(1) (2)

$$- \int_A \frac{\partial}{\partial y} \left(\frac{\partial M_{xy}}{\partial x} \delta w \right) dA + \int_A \frac{\partial^2 M_{xy}}{\partial x \partial y} \delta w \, dA - \int_A \frac{\partial}{\partial x} \left(\frac{\partial M_{xy}}{\partial y} \right) \delta w \, dA + \int_A \frac{\partial^2 M_{xy}}{\partial x \partial y} \delta w \, dA$$

(3) (4)

$$= \oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds + \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$- \oint_S \left(\frac{\partial M_x}{\partial x} \delta w \, n_x + \frac{\partial M_y}{\partial y} \delta w \, n_y \right) ds - \oint_S \left(\frac{\partial M_{xy}}{\partial x} \delta w \, n_y + \frac{\partial M_{xy}}{\partial y} n_x \right) ds$$

$$+ \int_A \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) \delta w \, dA$$

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$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} z\, dz = \int_{-h/2}^{h/2} \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu \varepsilon_{yy}\right) z\, dz$$

$$= \int_{-h/2}^{h/2} \frac{E}{1-\nu^2}\left(\frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial v_s}{\partial y} - \nu z \frac{\partial^2 w}{\partial y^2}\right) z\, dz$$

$$= -\int_{-h/2}^{h/2} \frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) z^2\, dz$$

$$= \frac{-E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\int_{-h/2}^{h/2} z^2\, dz = \frac{-Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$\left[\frac{z^3}{3}\right]_{-h/2}^{h/2} = \frac{1}{3}\,\frac{h^3}{8} \times 2 = \frac{h^3}{12}$$

$$M_y = \int_{-h/2}^{h/2} \sigma_{yy} z\, dz = \frac{-Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z\, dz = \int_{-h/2}^{h/2} 2G\, \varepsilon_{xy} z\, dz = \int_{-h/2}^{h/2} 2\,\frac{E}{2(1+\nu)}\,\frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right) z\, dz = \frac{-E}{(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}\int_{-h/2}^{h/2} z^2\, dz$$

$$= \frac{-Eh^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial x^2}\left(-\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right) + \frac{\partial^2}{\partial y^2}\left(-\frac{Eh^3}{12(1-\nu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right)$$

$$+ 2\frac{\partial^2}{\partial x \partial y}\left(-\frac{Eh^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}\right)$$

$$= \frac{-Eh^3}{12(1-\nu^2)}\left[\frac{\partial^4 w}{\partial x^4} + \nu\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\partial^4 w}{\partial y^2 \partial x^2} + 2(1-\nu)\frac{\partial^4 w}{\partial x^2 \partial y^2}\right]$$

$$= \frac{-Eh^3}{12(1-\nu^2)}\left[\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right]$$

$$= \frac{-Eh^3}{12(1-\nu^2)}\nabla^4 w$$

$$= -D\nabla^4 w \qquad D = \frac{Eh^3}{12(1-\nu^2)} : \text{bending rigidity}$$

$$+\,\mathrm{LHS}_{\text{bending}} = -\oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds - \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$+ \oint_S \left(\frac{\partial M_x}{\partial x} \delta w\, n_x + \frac{\partial M_y}{\partial y} \delta w\, n_y \right) ds + \oint_S \left(\frac{\partial M_{xy}}{\partial x} \delta w\, n_y + \frac{\partial M_{xy}}{\partial y} n_x \right) ds$$

$$+ \int_A \left(+ D \nabla^4 w \right) \delta w \, dA$$

$$\mathrm{RHS}_{\text{bending}} = \int_A t_i \, \delta u_i \, dA = \int q \, \delta w \, dA$$

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$$-\oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds - \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$+ \oint_S \left(\frac{\partial M_x}{\partial x} \delta w \, n_x + \frac{\partial M_y}{\partial y} \delta w \, n_y \right) ds + \oint_S \left(\frac{\partial M_{xy}}{\partial x} \delta w \, n_y + \frac{\partial M_{xy}}{\partial y} n_x \delta w \right) ds$$

$$+ \iint_A \left(+ D \nabla^4 w - q \right) \delta w \, dA = 0$$

Consider the stress eqb. eqn (along x-dir^n)

$$z \frac{\partial \sigma_{xx}}{\partial x} + z \frac{\partial \sigma_{xy}}{\partial y} + z \frac{\partial \sigma_{xz}}{\partial z} = 0$$

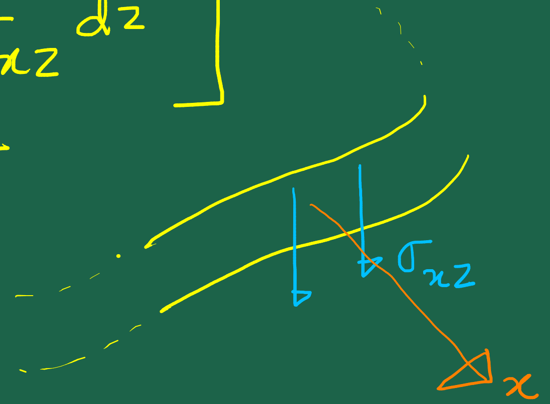
Int. w.r.t. z from $-h/2$ to $h/2$

$$\int_{-h/2}^{h/2} z \frac{\partial \sigma_{xx}}{\partial x} dz + \int_{-h/2}^{h/2} z \frac{\partial \sigma_{xy}}{\partial y} dz + \int_{-h/2}^{h/2} z \frac{\partial \sigma_{xz}}{\partial z} dz = 0$$

$$\Rightarrow \int_{-h/2}^{h/2} \frac{\partial}{\partial x}(z\sigma_{xx})\,dz + \int_{-h/2}^{h/2} \frac{\partial}{\partial y}(z\sigma_{xy})\,dz + \left[z\,\sigma_{xz}\right]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \frac{d(z)}{dz}\sigma_{xz}\,dz = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\underbrace{\int_{-h/2}^{h/2}(z\,\sigma_{xx})\,dz}_{M_x} + \frac{\partial}{\partial y}\underbrace{\int_{-h/2}^{h/2}(z\,\sigma_{xy})\,dz}_{M_{xy}} - \int_{-h/2}^{h/2}\sigma_{xz}\,dz = 0$$

$$\Rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \qquad \left[\text{Define } Q_x = \int_{-h/2}^{h/2}\sigma_{xz}\,dz\right]$$



Next, consider the stress eqb. eqn (along y-dir^n)

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \rightarrow \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y$$

$$\left[\text{Define } Q_y = \int_{-h/2}^{h/2}\sigma_{yz}\,dz\right]$$

$$-\oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds - \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$+ \oint_S \left(\underline{\frac{\partial M_x}{\partial x} \delta w\, n_x} + \frac{\partial M_y}{\partial y} \underline{\underline{\delta w\, n_y}} \right) ds + \oint_S \left(\underline{\underline{\frac{\partial M_{xy}}{\partial x}}} \delta w\, n_y + \underline{\frac{\partial M_{xy}}{\partial y}} n_x \delta w \right) ds$$

$$+ \iint_A \left(+ D \nabla^4 w - q \right) \delta w \, dA = 0 .$$

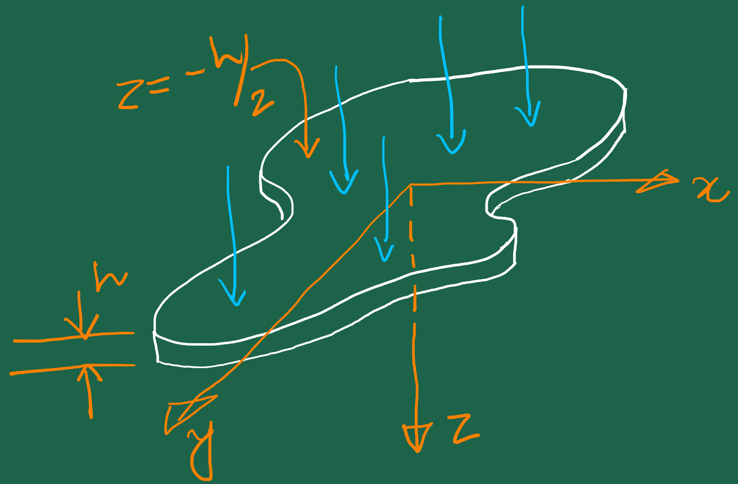
$$\Rightarrow \quad -\oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds - \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

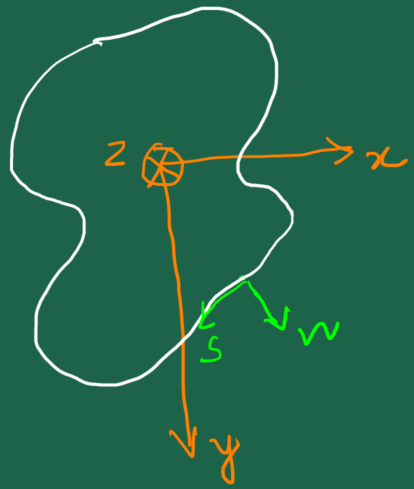
$$+ \oint_S \left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) n_x \delta w \, ds + \oint_S \left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \right) n_y \delta w \, ds$$

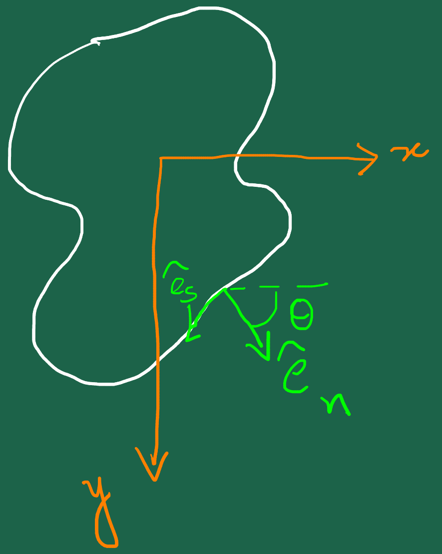
$$+ \int_A \left(D \nabla^4 w - q \right) \delta w \, dA$$

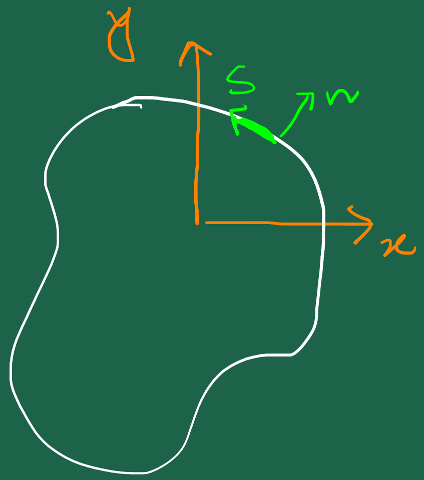
$$-\oint_S \left(M_x \frac{\partial \delta w}{\partial x} n_x + M_y \frac{\partial \delta w}{\partial y} n_y \right) ds - \oint_S \left(M_{xy} \frac{\partial \delta w}{\partial y} n_x + M_{xy} \frac{\partial \delta w}{\partial x} n_y \right) ds$$

$$+ \oint_S Q_x n_x \delta w \, ds + \oint_S Q_y n_y \delta w \, ds + \int_A \left(D \nabla^4 w - q \right) \delta w \, dA = 0$$









$$\hat{e}_n = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \equiv n_x\hat{i} + n_y\hat{j}$$

$$\hat{e}_s = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} = -n_y\hat{i} + n_x\hat{j}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}$$

$$\nabla\phi = \frac{\partial\phi}{\partial s}\hat{e}_s + \frac{\partial\phi}{\partial n}\hat{e}_n = \frac{\partial\phi}{\partial s}\left(-n_y\hat{i} + n_x\hat{j}\right) + \frac{\partial\phi}{\partial n}\left(n_x\hat{i} + n_y\hat{j}\right) = \left\{\frac{\partial\phi}{\partial s}(-n_y) + \frac{\partial\phi}{\partial n}n_x\right\}\hat{i} + \left\{\frac{\partial\phi}{\partial s}n_x + \frac{\partial\phi}{\partial n}n_y\right\}\hat{j}$$

$$\frac{\partial}{\partial x} \equiv -n_y\frac{\partial}{\partial s} + n_x\frac{\partial}{\partial n}$$

$$\frac{\partial}{\partial y} \equiv n_x\frac{\partial}{\partial s} + n_y\frac{\partial}{\partial n}$$

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$$-\oint\left(M_x\frac{\partial\delta w}{\partial x}n_x+M_y\frac{\partial\delta w}{\partial y}n_y\right)ds-\oint\left(M_{xy}\frac{\partial\delta w}{\partial y}n_x+M_{xy}\frac{\partial\delta w}{\partial x}n_y\right)ds$$

$$+\oint Q_x n_x\,\delta w\,ds+\oint Q_y n_y\,\delta w\,ds+\int_A\left(D\nabla^4 w-q\right)\delta w\,dA=0$$

$$\Rightarrow\int\left(D\nabla^4 w-q\right)\delta w\,dA-\oint\left(M_x n_x+M_{xy}n_y\right)\frac{\partial\delta w}{\partial x}ds-\oint\left(M_{xy}n_x+M_y n_y\right)\frac{\partial\delta w}{\partial y}ds$$

$$+\oint\left(Q_x n_x+Q_y n_y\right)\delta w\,ds=0$$

$$\Rightarrow\int\left(D\nabla^4 w-q\right)\delta w\,dA-\oint\left(M_x n_x+M_{xy}n_y\right)\left(-n_y\frac{\partial\delta w}{\partial s}+n_x\frac{\partial\delta w}{\partial n}\right)ds$$

$$-\oint\left(M_{xy}n_x+M_y n_y\right)\left(n_x\frac{\partial\delta w}{\partial s}+n_y\frac{\partial\delta w}{\partial n}\right)ds+\oint\left(Q_x n_x+Q_y n_y\right)\delta w\,ds=0$$

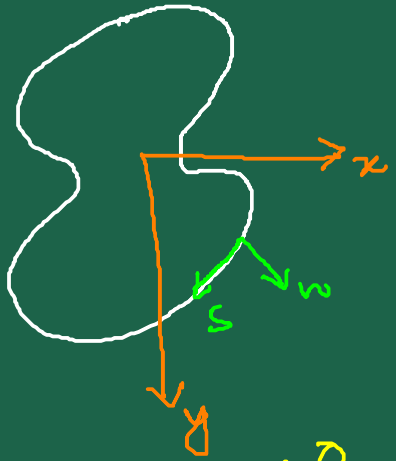
$$\int_A (D\nabla^4 w - q)\,\delta w\, dA - \oint (M_x n_x + M_{xy} n_y)\left(-n_y \frac{\partial \delta w}{\partial s} + n_x \frac{\partial \delta w}{\partial n}\right) dS$$
$$- \oint (M_{xy} n_x + M_y n_y)\left(n_x \frac{\partial \delta w}{\partial s} + n_y \frac{\partial \delta w}{\partial n}\right) dS + \oint (Q_x n_x + Q_y n_y)\,\delta w\, dS = 0$$

$$\Rightarrow \int_A (D\nabla^4 w - q)\,\delta w\, dA - \oint \left(-M_x n_x n_y - M_{xy} n_y^2 + M_{xy} n_x^2 + M_y n_x n_y\right) \frac{\partial \delta w}{\partial s} dS$$
$$- \oint \left(M_x n_x^2 + M_{xy} n_x n_y + M_{xy} n_x n_y + M_y n_y^2\right) \frac{\partial \delta w}{\partial n} dS$$
$$+ \oint (Q_x n_x + Q_y n_y)\,\delta w\, dS = 0$$

$$\Rightarrow \int_A (D\nabla^4 w - q)\,\delta w\, dA - \oint M_{ns} \frac{\partial \delta w}{\partial s} dS - \oint M_n \frac{\partial \delta w}{\partial n} dS + \oint Q_n\,\delta w\, dS = 0$$

where $M_{ns} = -(M_x - M_y)\, n_x n_y + M_{xy}(n_x^2 - n_y^2)$; $M_n = M_x n_x^2 + 2M_{xy} n_x n_y + M_y n_y^2$

$$\int_A (D\nabla^4 w - q)\,\delta w\, dA - \oint M_{ns}\frac{\partial \delta w}{\partial s}\, ds - \oint M_n \frac{\partial \delta w}{\partial n}\, ds + \oint Q_n\, \delta w\, dS = 0$$



$$\oint M_{ns}\frac{\partial \delta w}{\partial s}\, ds = \underbrace{\cancel{\Big[M_{ns}\,\delta w\Big]_{P_1}^{P_1}}}_{0} - \oint \frac{\partial M_{ns}}{\partial s}\,\delta w\, dS$$

$$\therefore \int_A (D\nabla^4 w - q)\,\delta w\, dA + \oint \frac{\partial M_{ns}}{\partial s}\,\delta w\, ds - \oint M_n \frac{\partial \delta w}{\partial n}\, ds + \oint Q_n\, \delta w\, dS = 0$$

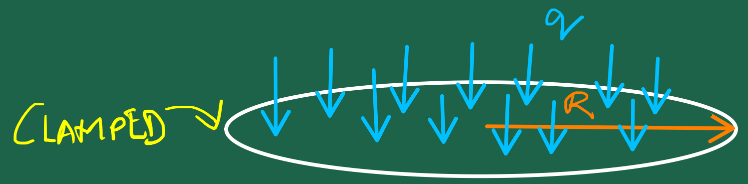
G.D.E. : $\quad D\nabla^4 w - q = 0$

BCs : Either $\dfrac{\partial M_{ns}}{\partial s} + Q_n = 0$ or w is specified

Either $M_n = 0$ or $\dfrac{\partial w}{\partial n}$ is specified

Example

CIRCULAR PLATE : Uniformly distributed load with clamped periphery



Axisymmetric

G.D.E. : $D\nabla^4 w - q = 0$

BCs : Either $\frac{\partial M_{ns}}{\partial s} + Q_n = 0$ or w is specified

Either $M_n = 0$ or $\frac{\partial w}{\partial n}$ is specified

$r \in [0, R]$

$n \to r \quad , \quad s \to r\theta$

At $r = R$: $w = 0 \quad , \quad \frac{\partial w}{\partial n} \equiv \frac{\partial w}{\partial r} = 0$

$D\nabla^4 w = q \quad , \quad \nabla^4 w \equiv \nabla^2(\nabla^2 w)$

$$\nabla^2 w \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)$$

$D\nabla^4 w = q$

$$\Rightarrow D\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = q$$

Set $\frac{q}{D} = 64\beta$

$$w = \beta r^4 + ar^2 \ln r + b\ln r + cr^2 + d$$

Please arrive at this soln form yourself !

At $r = 0$, w should be finite

$\Rightarrow$ b must be 0.

$\boxed{20}$

$M_w \nearrow$ $M_r = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right)$

$w = \beta r^4 + a r^2 \ln r + b \ln r + c r^2 + d$

$\frac{dw}{dr} = 4\beta r^3 + 2ar\ln r + ar + \cancel{\frac{b}{r}}^{\,0} + 2cr$

$\frac{d^2w}{dr^2} = 12\beta r^2 + 2a\ln r + 2a + a - \cancel{\frac{b}{r^2}}^{\,0} + 2c$

$-\frac{M_r}{D} = 12\beta r^2 + 2a\ln r + 3a + 2c + \nu\left(4\beta r^2 + 2a\ln r + a + 2c\right)$

At $r = 0$, M_r should be finite $\Rightarrow$ a must be 0

$\therefore$ $w = \beta r^4 + c r^2 + d$

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$$w = \beta r^4 + c r^2 + d$$

At $r = R$, $w = 0$

$$\Rightarrow \beta R^4 + c R^2 + d = 0 \Rightarrow d = -cR^2 - \beta R^4$$

$$= 2\beta R^4 - \beta R^4 = \beta R^4$$

At $r = R$, $\dfrac{dw}{dr} = 0$

$$\Rightarrow 4\beta R^3 + 2\cancel{a}^{\,0} r \ln r + \cancel{a}^{\,0} r + \cancel{\frac{b}{r}}^{\,0} + 2cR = 0$$

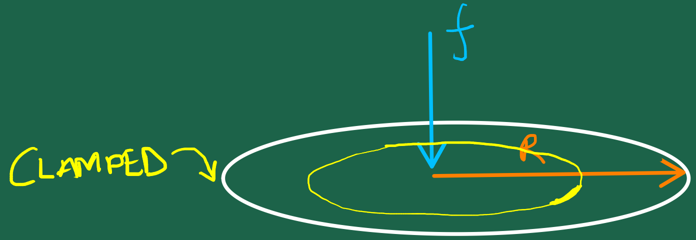
$$\Rightarrow c = -2\beta R^2$$

$$\therefore w = \beta r^4 - 2\beta R^2 r^2 + \beta R^4 = \beta (r^2 - R^2)^2$$

$$= \frac{q}{64D}(r^2 - R^2)^2$$

EXAMPLE

CIRCULAR PLATE : Concentrated load at centre with clamped periphery



$$D\nabla^4 w = f\,\delta_D(r), \quad r \in [0, R]$$

$$\int_{-\infty}^{\infty} \delta_D(x)\,dx = 1$$

$$\int_0^{\infty} \underbrace{\delta_D(r)}_{\frac{1}{m^2}}\ \underbrace{2\pi r\,dr}_{m^2} = 1$$

$$D\nabla^4 w = 0, \quad r \in (0, R]$$

$$\Rightarrow w = ar^2 \ln r + b \ln r + cr^2 + d$$

At $r = R$: $w = 0$ and $\dfrac{dw}{dr} = 0$

As $r \to 0$, w should be finite $\Rightarrow b = 0$

$$Q_r \, 2\pi r = f$$

$$Q_r \equiv Q_n = n_x Q_x + n_y Q_y$$

$$= n_x\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}\right) + n_y\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y}\right)$$

$$= D\frac{d}{dr}\left(\nabla^2 w\right)$$

$$= D\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}$$

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$$w = ar^2 \ln r + \cancel{b}^{\,0} \ln r + cr^2 + d$$

$$\frac{dw}{dr} = 2ar\ln r + 2cr + ar$$

$$r\frac{dw}{dr} = 2ar^2 \ln r + ar^2 + 2cr^2$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) = 4a\cancel{r}\ln r + 2a\cancel{r} + 2a\cancel{r} + 4c\cancel{r}$$

$$\underbrace{D\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}}_{Q_r} = \frac{4}{r}aD$$

$$Q_r\, 2\pi r = f \Rightarrow \frac{4aD}{\cancel{r}}\, 2\pi\cancel{r} = f \Rightarrow a = \frac{f}{8\pi D}$$

$\boxed{25}$

$$w = a r^2 \ln r + c r^2 + d$$

At $r = R$: $w = 0 \Rightarrow aR^2 \ln R + cR^2 + d = 0$

At $r = R$: $\dfrac{dw}{dr} = 0 \Rightarrow 2aR \ln R + 2cR + aR = 0$

$$\Rightarrow c = -aR\ln R - \frac{1}{2}a$$

$$\therefore d = -cR^2 - aR^2 \ln R$$

$$w =$$

Interpretation of effective shear

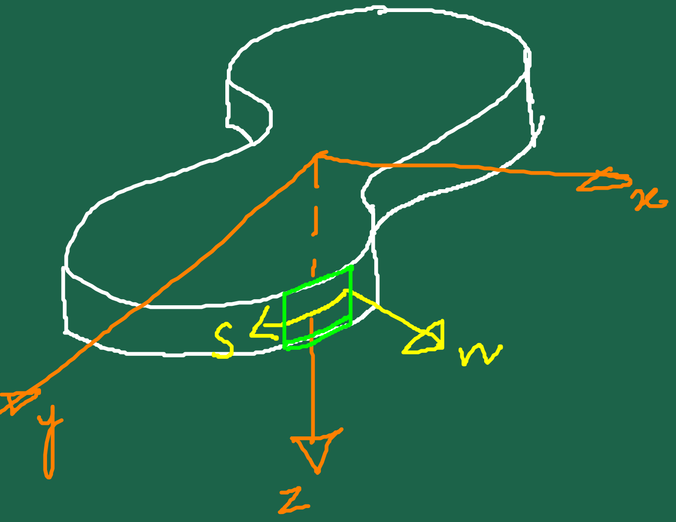
$$Q_{eff} = Q_n + \frac{\partial M_{ns}}{\partial s}$$

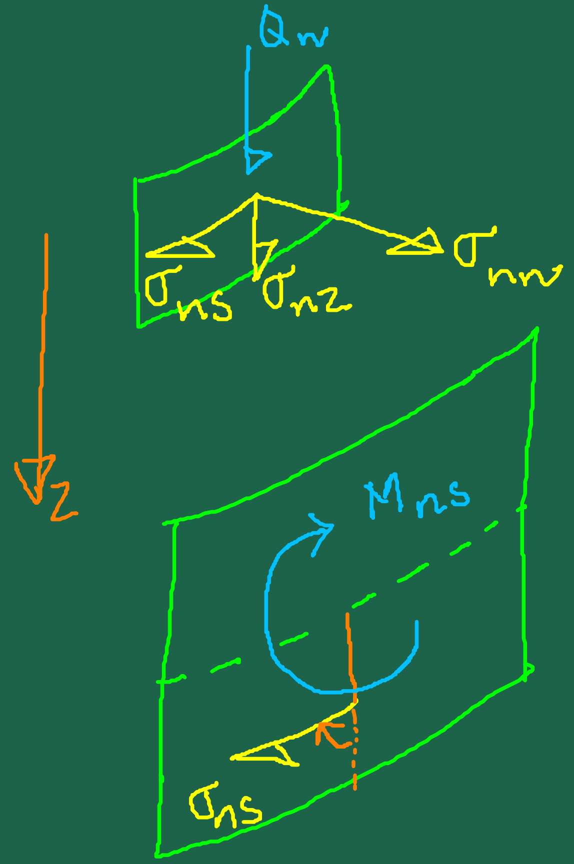
$$Q_n = \int_{-H/2}^{H/2} \sigma_{nz}\, dz$$

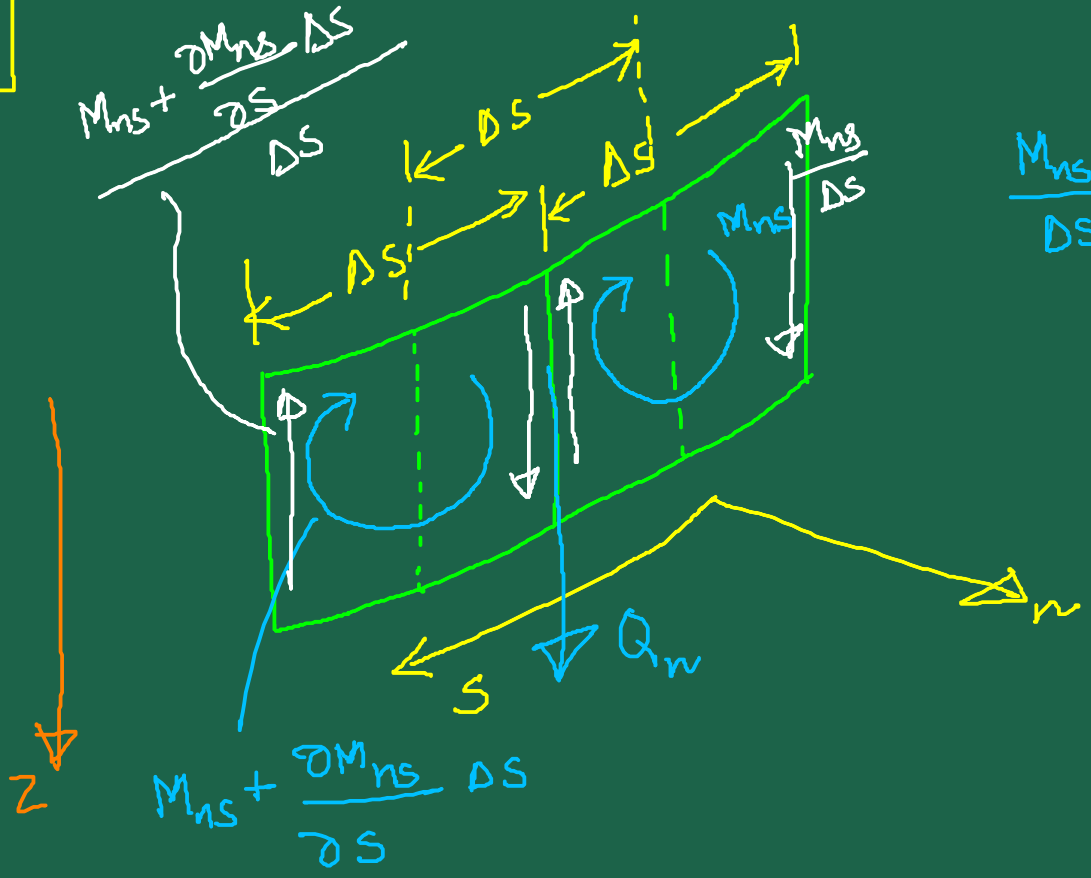
N/m

$$M_{ns} = \int_{-H/2}^{H/2} \sigma_{ns}\, z\, dz$$

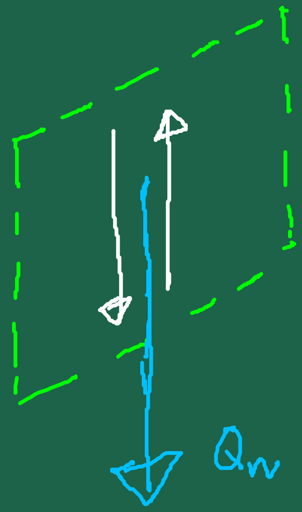
N

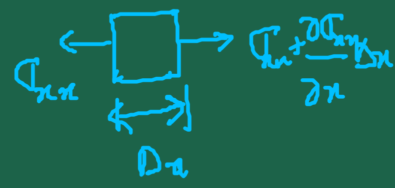






$$\frac{M_{ns}}{\Delta s}$$

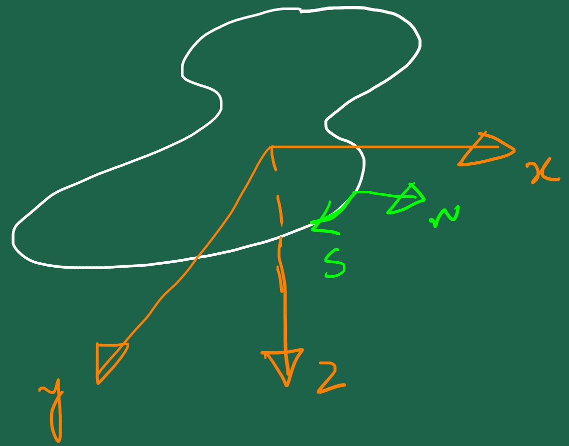


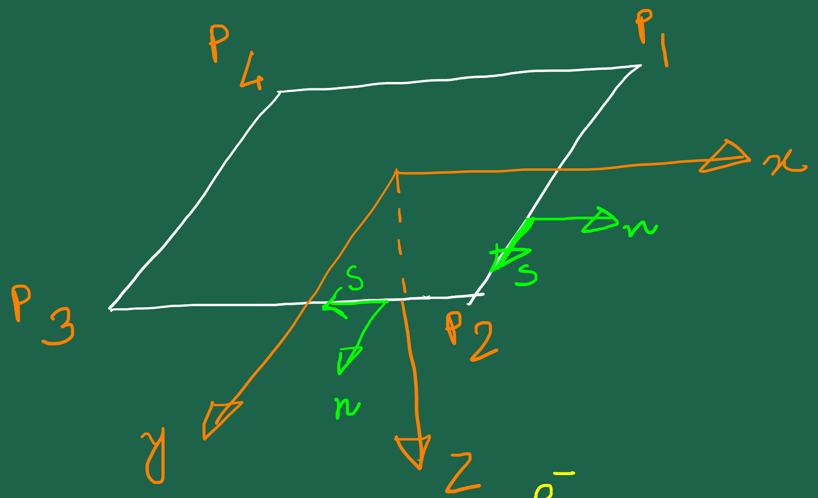


$$Q_n + \frac{M_{ns} + \frac{\partial M_{ns}}{\partial s}\Delta s}{\Delta s} - \frac{M_{ns}}{\Delta s}$$

$$= Q_n + \frac{\partial M_{ns}}{\partial s}$$

Corner points





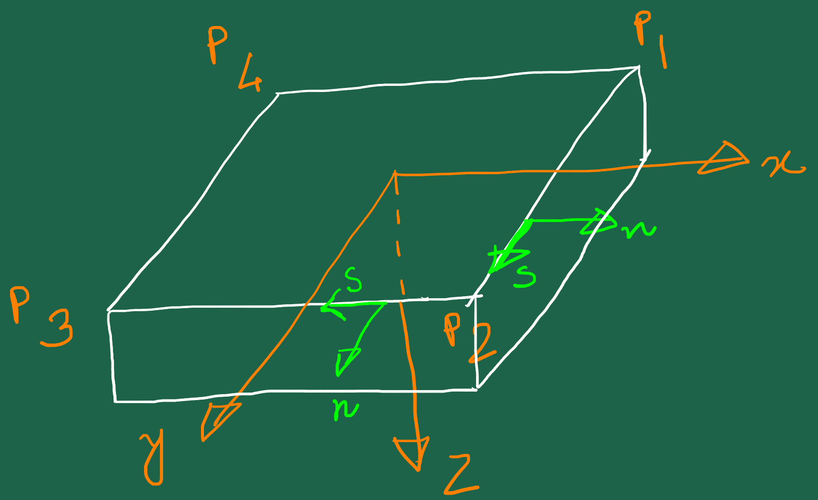
$$\oint \left(M_{ns} \frac{\partial \delta w}{\partial s} \right) ds \rightarrow \int_{P_1^+}^{P_2^-} (\;) \, ds + \int_{P_2^+}^{P_3^-} (\;) \, ds + \int_{P_3^+}^{P_4^-} (\;) \, ds + \int_{P_4^+}^{P_1^-} (\;) \, ds$$

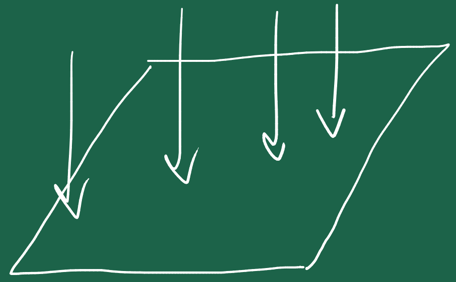
$$\int_{P_1^+}^{P_2^-}(\ )\,ds+\int_{P_2^+}^{P_3^-}(\ )\,ds+\int_{P_3^+}^{P_4^-}(\ )\,ds+\int_{P_4^+}^{P_1^-}(\ )\,ds \qquad (\ )\equiv M_{ns}\frac{\partial \delta w}{\partial s}$$

$$=\Big[M_{ns}\delta w\Big]_{P_1^+}^{P_2^-}-\int_{P_1^+}^{P_2^-}\frac{\partial M_{ns}}{\partial s}\delta w\,ds+\Big[M_{ns}\delta w\Big]_{P_2^+}^{P_3^-}-\int_{P_2^+}^{P_3^-}\frac{\partial M_{ns}}{\partial s}\delta w\,ds+\Big[M_{ns}\delta w\Big]_{P_3^+}^{P_4^-}-\int_{P_3^+}^{P_4^-}\frac{\partial M_{ns}}{\partial s}\delta w\,ds$$

$$+\Big[M_{ns}\delta w\Big]_{P_4^+}^{P_1^-}-\int_{P_4^+}^{P_1^-}\frac{\partial M_{ns}}{\partial s}\delta w\,ds$$

$$=\Big[M_{ns}\delta w\Big]_{P_1^+}^{P_1^-}+\Big[M_{ns}\delta w\Big]_{P_2^+}^{P_2^-}+\Big[M_{ns}\delta w\Big]_{P_3^+}^{P_3^-}+\Big[M_{ns}\delta w\Big]_{P_4^+}^{P_4^-}-\int_{P_1^+}^{P_2^-}+\int_{P_2^+}^{P_3^-}+\int_{P_3^+}^{P_4^-}+\int_{P_4^+}^{P_1^-}$$





At corner P_2 :

face with edge 1-2 : $M_{xy} = M_{ns}$

face with edge 2-3 : $M_{yx} = -M_{ns}$

$$\left[M_{ns}\,\delta w\right]_{P_2^+}^{P_2^-} = M_{ns}\,\delta w\Big|_{P_2^-} - M_{ns}\,\delta w\Big|_{P_2^+}$$

$$= M_{xy}\,\delta w\Big|_{P_2^-} - \left(-M_{yx}\right)\delta w\Big|_{P_2^+} = 2M_{xy}\,\delta w\Big|_{P_2}$$