Calculus of Variations

$\#\quad y = 3x^2 , \quad x \in [x_1 , x_2]$

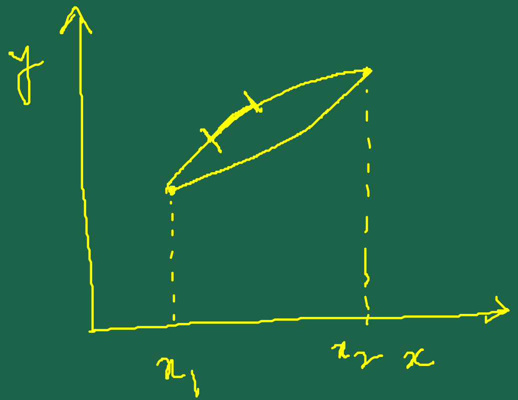
$\#\quad I = \int_{x_1}^{x_2} f(x)\, dx$

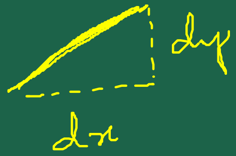
* Find a curve between 2 pts such that the length of the curve is minimum.

$$L = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2}$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

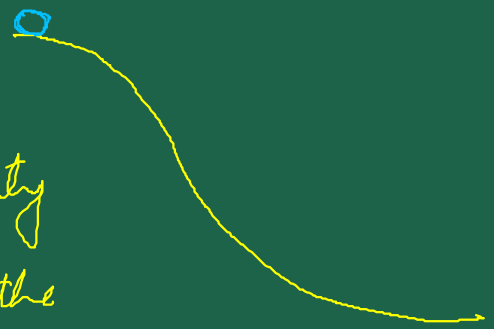
$$= \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, dx$$





$I \rightarrow y'$

* Find the shape of the frictionless path such that the particle travelling down it under the action of gravity takes the minimum time to cover the path



$$T = \int_{x_1}^{x_2} \frac{\sqrt{dx^2 + dy^2}}{V} \quad ; \quad V = \sqrt{2gy}$$

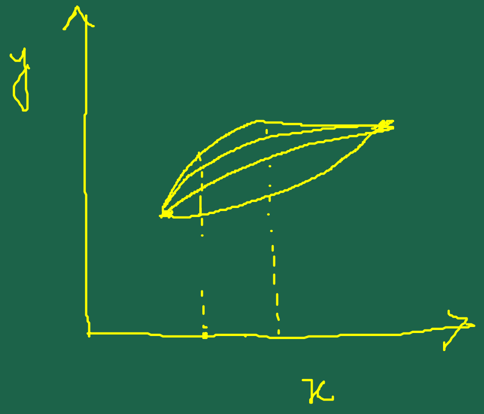
$$= \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{2gy}} \, dx \qquad I \to y, y'$$

$$I \equiv I(x, y, y', \ldots, y^{(n)})$$

Extremization of I

Restrict to $I \equiv I(x, y, y')$

Temptation!! $\rightarrow$ $\dfrac{dI}{dy}$ ✗



δy is for a fixed x

dy is always accompanied by dx

Extremization of functions

$y = f(x)$; suppose $f(x)$ is min. at $x = a$

$$y = f(x) = f(x-a+a) = f(a) + \frac{x-a}{\lfloor 1} f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \dots$$

$f(x) - f(a) > 0 \leftarrow$ This is what we need to ensure

Depending on where x is located $x-a$ can be +ve or −ve

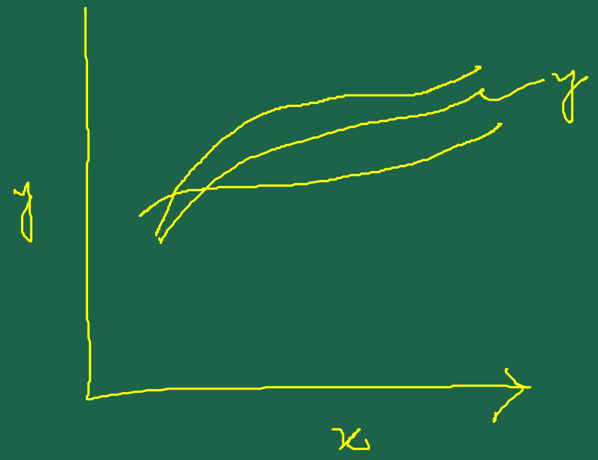
We want to ensure that this variable sign of $x-a$ does not influence $f(x) - f(a)$. We ensure that by requiring that $\underbrace{f'(a) = 0}_{\text{Necessary condition}}$.

Hence, to ensure that $f(x) - f(a) > 0$, we just require that $f''(a) > 0$

We are going to exploit this idea to extremization of the integrations.

Extremization of functionals

$I \equiv I(x, y, y') \rightarrow$ A fn. of functions



$$y \rightarrow y + \varepsilon\eta$$

$$\bar{I} \equiv \bar{I}(x, y+\varepsilon\eta, y'+\varepsilon\eta')$$

For minimization, we need to ensure that

$$\bar{I} - I > 0$$

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$

$$\bar{I} = \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\, dx$$

$\boxed{6}$

$$\bar{I} - I = \int_{x_1}^{x_2} F(x, y+\varepsilon\eta, y'+\varepsilon\eta')\,dx - \int_{x_1}^{x_2} F(x, y, y')\,dx$$

$$= \int_{x_1}^{x_2} \Big[F(x,y,y') + \frac{\varepsilon\eta}{\lfloor 1}\frac{\partial F}{\partial y} + \frac{\varepsilon\eta'}{\lfloor 1}\frac{\partial F}{\partial y'} + \frac{\varepsilon^2\eta^2}{\lfloor 2}\frac{\partial^2 F}{\partial y^2} + \frac{\varepsilon^2\eta'^2}{\lfloor 2}\frac{\partial^2 F}{\partial y'^2}$$

$$+ 2\frac{\varepsilon^2\eta\eta'}{\lfloor 2}\frac{\partial^2 F}{\partial y\partial y'} + O(\varepsilon^3)\Big]dx - \int_{x_1}^{x_2} F(x,y,y')\,dx$$

$$= \int_{x_1}^{x_2}\Big[\varepsilon\Big(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'}\Big) + \frac{1}{2}\varepsilon^2\Big(\eta^2\frac{\partial^2 F}{\partial y^2} + \eta'^2\frac{\partial^2 F}{\partial y'^2} + 2\eta\eta'\frac{\partial^2 F}{\partial y\partial y'}\Big)\Big]dx$$

$$+ O(\varepsilon^3)$$

$$= \int_{x_1}^{x_2}\Big[\varepsilon\Big(\eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'}\Big)F + \frac{1}{2}\varepsilon^2\Big(\eta\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial y'}\Big)^2 F\Big]dx + O(\varepsilon^3)$$

$$\bar{I} - I = \delta I + \frac{1}{2}\delta^2 I + O(\varepsilon^3)$$

$\varepsilon \to 0$

$\because$ δI is dependent on ε and its sign is also given by ε so we have to ensure that its influence on $\bar{I} - I$ is switched off. So, we require that $\delta I = 0$

$$\delta I = 0 \Rightarrow \int_{x_1}^{x_2} \left(\varepsilon \eta \frac{\partial F}{\partial y} + \varepsilon \eta' \frac{\partial F}{\partial y'} \right) dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \varepsilon \eta \frac{\partial F}{\partial y}\, dx + \left[\varepsilon \eta \frac{\partial F}{\partial y'} \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\varepsilon \frac{\partial F}{\partial y'} \right) \eta \, dx = 0$$

Case I: Values of the curve are specified at the boundaries.

$y + \varepsilon\eta$ should be such that $\eta = 0$ at x_1 and x_2

$$\therefore \left[\varepsilon\eta \frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} = 0 - 0 = 0$$

$$\therefore \int_{x_1}^{x_2} \left[\varepsilon\eta \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\varepsilon \frac{\partial F}{\partial y'}\right)\eta\right] dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \eta \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] dx = 0$$

$\because$ η is arbitrary, so the only way that the 0 value of the integration is possible is if $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \longrightarrow \text{Euler-Lagrange eqn}$$

Case 2 : y is not specified at the boundaries

$$\int_{x_1}^{x_2} \varepsilon \eta \frac{\partial F}{\partial y}\, dx + \left[\varepsilon \eta \frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\varepsilon \frac{\partial F}{\partial y'}\right) \eta \, dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} \varepsilon \eta \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] dx + \left[\varepsilon \eta \frac{\partial F}{\partial y'}\right]_{x_1}^{x_2} = 0$$

This will be 0 if

$$\underbrace{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}_{\text{Euler-Lagrange eqn}} \quad \text{AND} \quad \frac{\partial F}{\partial y'} = 0 \text{ at } x = x_1 \text{ and } x = x_2$$