Calculus of Variations *

1 Introduction

Let us first consider something very familiar to us, a simple minimization problem from differential calculus: What is the value of x that will minimize $y = 3x^2$, with $x \in [x_1,\ x_2]$?

Next, consider the integration $I = \int_{x_1}^{x_2} f(x)\,\mathrm{d}x$, which has a definite value for a particular choice of $f(x)$. However, suppose we wish to know the minimum value of I for various choices of $f(x)$. So, what we are asking is this: What is the $f(x)$ that gives the minimum value of I? Question is, what method do we use to find such a minimum? And, the answer is: *the calculus of variations.*

One may wonder in what kind of scenario would be faced with such a question. Let's take a couple of concrete examples:

- Find the curve between two points which minimizes the distance between the two points. We have

  $$L = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\ \mathrm{d}x$$

- Find the shape of a frictionless path between two points such that a particle takes the least time to slide down the path under the effect of gravity. We have

  $$T = \int_{x_1}^{x_2} \frac{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}{V}, \text{ with } V = \sqrt{2gy}. \text{ Thus, } T = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'}{2gy}}\ \mathrm{d}x$$

In the first example, the integrand is of the form $F(y')$, and in the second, it is of the form $F(y, y')$. More generally, beyond these examples, we want to minimize $I = \int_{x_1}^{x_2} F(x, y, y', y'', \dots y^{(n)})\mathrm{d}x$. Even more generally, we want to *extremize* I.

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Let us see how we can actually minimize (or, equivalently, maximize) the integration, $I(x, y, y', \dots y^{(n)})$. To keep things simple we will consider $I(x, y, y')$. Note that I is a function of functions; such an entity is referred to as a functional.

The first temptation when trying to minimize (or, maximize) a functional, I, is to try to do something like $\dfrac{\mathrm{d}I}{\mathrm{d}y}$. But this is problematic!

To see the problem, consider as an example the problem of finding the path on a plane that covers the shortest distance between two points. When we write the differential $\mathrm{d}y$, it is associated with a change in x, i.e. $\mathrm{d}x$. However, when we are considering various possibilities of the path what we are actually doing is considering different values of y for the same x. Such changes in y are completely different in meaning from $\mathrm{d}y$. The notation that we use to represent this special kind of change in y is δy.

So we certainly do *not* want $\dfrac{\mathrm{d}I}{\mathrm{d}y}$. What do we do then?

We go back to the basics and look once again into how extremization is done for functions. We will then export that idea to functionals.

2 Extremization of functions

Consider a simple function $y(x)$. Suppose this function takes on extreme values (either minimum or maximum) at $x = a$. To concretize our ideas consider the case of the minimum. We expand $y(x)$ about $x = a$ as:

$$y(x) = y(x - a + a) = y(a) + \frac{x - a}{1!} y'(a) + \frac{(x - a)^2}{2!} y'' + \cdots \tag{1}$$

Since $y(a)$ is a minimum, we must have $y(x) - y(a) > 0$. Now, both the conditions $x - a < 0$ and $x - a > 0$ are possible depending on where x is located. However, because we are considering a particular function, $y(x)$, only a particular value of $y'(a)$ is possible. Therefore, to ensure that the sign of $(x - a)$ does not influence $y(x) - y(a)$, we *need* to have $f'(a) = 0$. This condition is the necessary condition for extremum value which we are all familiar with from high school calculus.

Next, to ensure that $y(x) - y(a) > 0$, it is sufficient that $y'' > 0$ because $(x-a)^2 > 0$, always.

Now, we will export this idea of finding the conditions for extremum of functions to functionals.

3 Extremization of functionals

At the outset, note that in the previous section for the extremization of functions, we started by exploring the neighbourhood of $x = a$. Similarly for functionals, we need to explore the "neighbourhood" of the desired function or curve y that will ultimately lead to the extreme value of the functional. But since derivatives and integrations are involved we need to be careful in exploring this neighbourhood. We need to select those alternative curves which satisfy certain conditions. This set of curves are referred to as admissible curves. We consider an admissible curve $y + \varepsilon\eta$.

The value of the functional corresponding to $y + \varepsilon\eta$ is

$$\bar{I} = \int_{x_1}^{x_2} F\left(x, y + \varepsilon\eta, y' + \varepsilon\eta'\right)\, \mathrm{d}x. \tag{2}$$

For minimum, we must have

$$\bar{I} - I = \Delta I > 0. \tag{3}$$

Now,

$$\Delta I = \int_{x_1}^{x_2} \left[F(x, y + \epsilon\eta, y' + \epsilon\eta') - F(x, y, y')\right]\, \mathrm{d}x. \tag{4}$$

Using Taylor expansion on the integrand, we have

$$\begin{aligned}\Delta I = \int_{x_1}^{x_2} &\left[F(x, y, y') + \epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta'\frac{\partial F}{\partial y'} + \frac{\epsilon^2\eta^2}{2!}\frac{\partial^2 F}{\partial y^2} + \frac{\epsilon^2\eta'^2}{2!}\frac{\partial^2 F}{\partial y'^2}\right.\\ &\left. +2\frac{\epsilon\eta\eta'}{2!}\frac{\partial^2 F}{\partial y\partial y'} + O(\epsilon^3) - F(x, y, y')\right]\, \mathrm{d}x\\ = \int_{x_1}^{x_2} &\left[\epsilon\left(\eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'}\right)F + \frac{1}{2}\epsilon^2\left(\eta\frac{\partial}{\partial y} + \eta'\frac{\partial}{\partial y'}\right)^2 F\right]\, \mathrm{d}x + O(\epsilon^3).\end{aligned} \tag{5}$$

This equation is conventionally represented in the form

$$\Delta I = \delta I + \frac{1}{2}\delta^2 I + O(\epsilon^3) \tag{6}$$

The terms δI and $\delta^2 I$ are called the first and second variations of I.

If $\varepsilon \longrightarrow 0$, the term δI (which involves ε) dominates over the term $\delta^2 I$ (which involves ε^2). Now, since the sign of δI is reversed by a reversal of the sign of ε, the necessary condition for ΔI to be positive for all small values of ε is that $\delta I = 0$. Further, when $\delta I = 0$, the sufficient condition for minimum is that $\delta^2 I > 0$.

Let us look at the necessary condition in more detail. We have

$$\begin{aligned}
& \delta I = 0, \\
\text{or,} \quad & \int_{x_1}^{x_2} \varepsilon \left(\eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} \right) \, \mathrm{d}x = 0, \\
\text{or,} \quad & \int_{x_1}^{x_2} \varepsilon \eta \frac{\partial F}{\partial y} \, \mathrm{d}x + \left[\varepsilon \frac{\partial F}{\partial y'} \eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \varepsilon \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \eta \, \mathrm{d}x = 0. \\
& \text{(Using integration by parts)}
\end{aligned} \tag{7}$$

Now, we focus on the boundary term $\left[\varepsilon \frac{\partial F}{\partial y'} \eta \right]_{x_1}^{x_2}$, and consider the following cases:

Case 1: If among the conditions which define the admissibility of curves that can serve as input to our functional, we have the conditions that the values of the curves are specified at the boundaries $x = x_1$ and $x = x_2$, then we must have $\eta = 0$ at x_1 and x_2. Then the boundary term becomes zero, and we have

$$\begin{aligned}
& \int_{x_1}^{x_2} \varepsilon \eta \frac{\partial F}{\partial y} \, \mathrm{d}x - \int_{x_1}^{x_2} \varepsilon \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \eta \, \mathrm{d}x = 0, \\
\text{or,} \quad & \int_{x_1}^{x_2} \varepsilon \eta \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right] \, \mathrm{d}x = 0.
\end{aligned}$$

Since η is arbitrary, the integral can be zero only if we have the following[†]

$$\boxed{\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0,} \tag{8}$$

which is famously known as the Euler-Lagrange equation.

Case 2: If the values of the curves are *not* specified at the boundaries $x = x_1$ and $x = x_2$, then η is *not* equal to zero at the boundaries. Then the boundary terms in Eq. (7) are not zero, and so in addition to the Euler-Lagrange equations we end up with the requirement that at x_1 and x_2, $\dfrac{\partial F}{\partial y'} = 0$.

[†] Actually, there is a proof for this statement. But we state it without proof here!