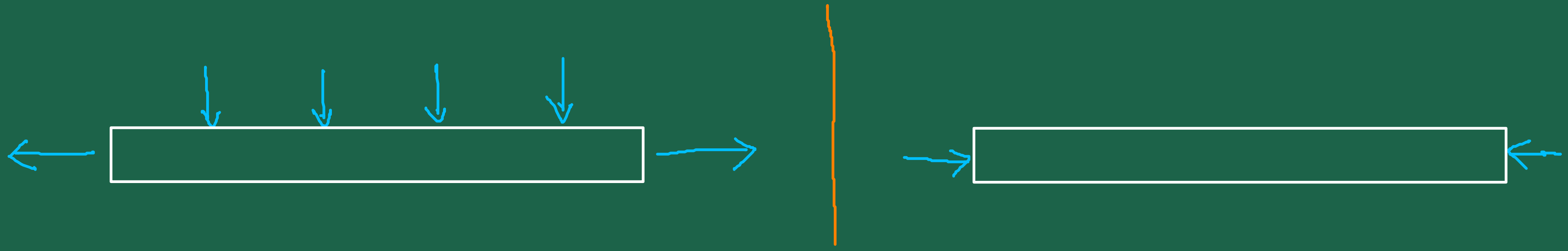
Beam Theory – Bending and Buckling



Lateral loading (Bending) → Principle of stationary P.E.

Principle of virtual work

Both lateral loading (Bending) + Axial loading (Stretching) (Tensile first) → P. St. P.E.

P. VW

↳ Bending & Stretching will be decoupled

Lateral loading + Axial loading (Compressive) → Problematic !

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Principle of VW : $\delta W = 0$

$$\Rightarrow \delta W_{int} + \delta W_{ext} = 0$$

Now, consider the forces are conservative

$$W_{int} = -V_{int} \quad , \quad W_{ext} = -V_{ext}$$

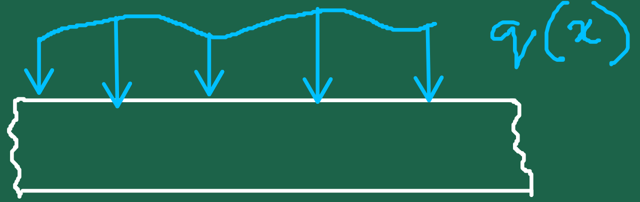
$$\delta\left(-V_{int} - V_{ext}\right) = 0$$

$$\Rightarrow \delta\left(V_{int} + V_{ext}\right) = 0$$

$$\Rightarrow \delta\Pi = 0$$ → Principle of stationary P.E.

Lateral Loading (only)

* Principle of Stationary P.E.



w : deflection

Euler – Bernoulli Beam Hypothesis

Linear, Elastic Material Behaviour

$$\Pi = V_{int} + V_{ext}$$

$$= \int_0^L \frac{M^2}{2EI}\,dx - \int_0^L q\,w\,dx$$

$$\delta\Pi = 0 \Rightarrow \delta\left[\int_0^L \frac{M^2}{2EI}\,dx - \int_0^L q\,w\,dx\right] = 0$$

$$\Rightarrow \delta\left[\int_0^L \frac{1}{2}EI\frac{1}{R^2}\,dx - \int_0^L q\,w\,dx\right]$$

$$MR = EI$$

$$\Rightarrow M = \frac{EI}{R}$$

$$\frac{1}{R} = \frac{\frac{d^2w}{dx^2}}{\left[1+\left(\frac{dw}{dx}\right)^2\right]^{3/2}}$$

$$\approx \frac{d^2w}{dx^2} \quad \left(\frac{dw}{dx} \ll 1\right)$$

$$\delta\left[\int_0^L \frac{1}{2} EI \frac{1}{R^2}\, dx - \int_0^L q\, w\, dx\right] = 0$$

$$\Rightarrow \delta \int_0^L \frac{1}{2} EI \left(\frac{d^2 w}{dx^2}\right)^2 dx - \delta \int_0^L q\, w\, dx = 0$$

$$\Rightarrow \int_0^L \frac{1}{2} EI \times 2 \frac{d^2 w}{dx^2} \times \frac{d^2 \delta w}{dx^2}\, dx - \int_0^L q\, \delta w\, dx = 0$$

$$\Rightarrow \int_0^L EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2}\, dx - \int_0^L q\, \delta w\, dx = 0$$

(* → compare with PVW)

$$\Rightarrow \left[EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}\right]_0^L - \int_0^L \frac{d}{dx}\left(EI \frac{d^2 w}{dx^2}\right) \frac{d\delta w}{dx}\, dx - \int_0^L q\, \delta w\, dx = 0$$

$$\Rightarrow \left[EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx}\right]_0^L - \left[EI \frac{d^3 w}{dx^3}\, \delta w\right]_0^L + \int_0^L \frac{d}{dx}\left(EI \frac{d^3 w}{dx^3}\right) \delta w\, dx - \int_0^L q\, \delta w\, dx = 0$$

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$$\left[EI\frac{d^2w}{dx^2}\frac{d\delta w}{dx}\right]_0^L - \left[EI\frac{d^3w}{dx^3}\delta w\right]_0^L + \int_0^L\left[EI\frac{d^4w}{dx^4} - q\right]\delta w\,dx = 0$$

$\therefore$ Governing eqn: $EI\dfrac{d^4w}{dx^4} - q = 0$

Boundary conditions:

At $x=0$ and $x=L$,

Either $EI\dfrac{d^2w}{dx^2} = 0$ or $\dfrac{dw}{dx}$ is specified

At $x=0$ and $x=L$,

Either $EI\dfrac{d^3w}{dx^3} = 0$ or w is specified

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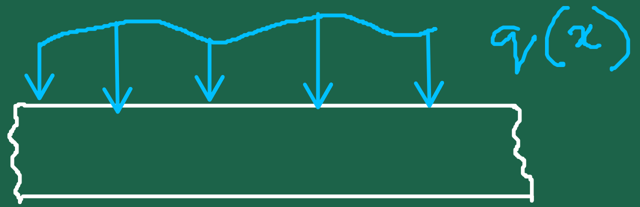
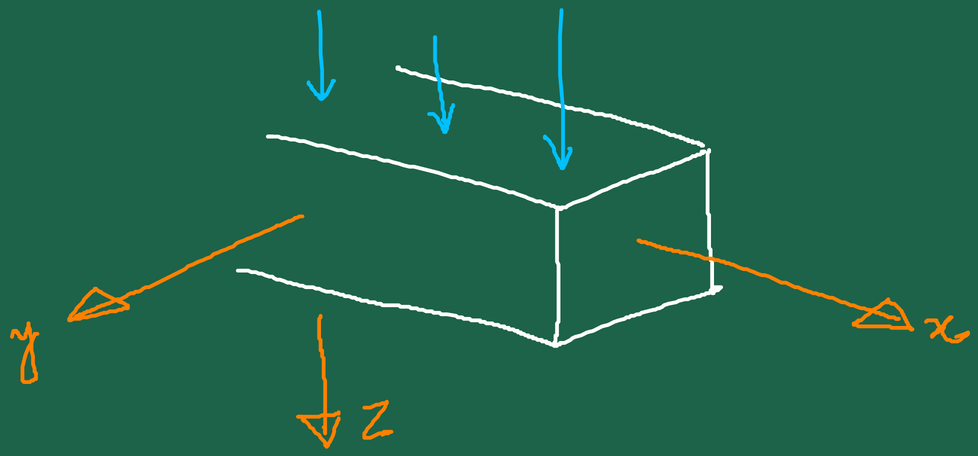
Lateral Loading (only)

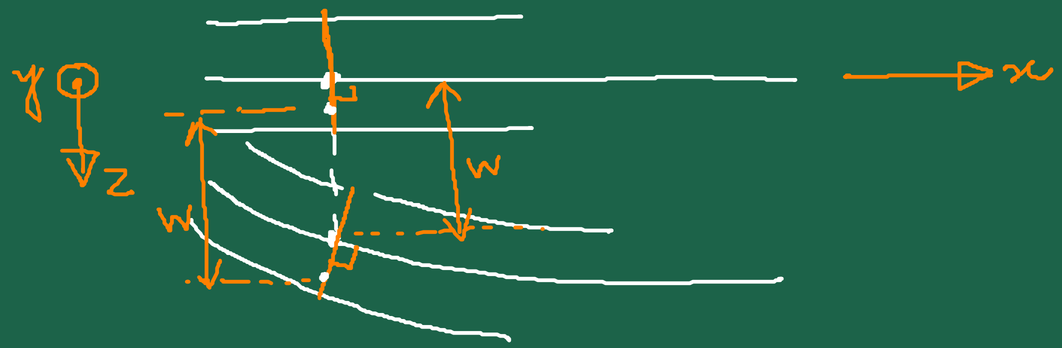
Using Principle of Virtual Work



$$-\delta W_i = \delta W_e = \underbrace{\int_V \rho b_i \, \delta u_i \, dV}_{\text{Neglect body forces}} + \int_S t_i \, \delta u_i \, dS = \int_V \sigma_{ij} \, \delta \varepsilon_{ij} \, dV$$

$$\int_S t_i \, \delta u_i \, dS = \int_V \sigma_{ij} \, \delta \varepsilon_{ij} \, dV$$



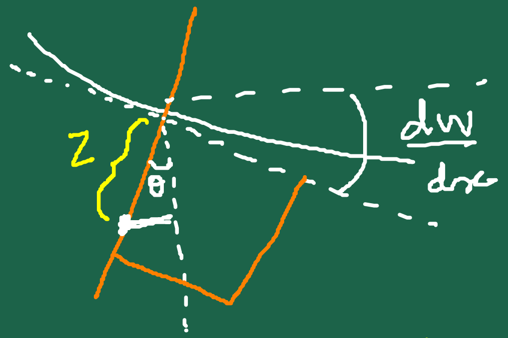


Kinematical hypothesis

$$u = -z\theta = -z\frac{dw}{dx}$$

$$v = 0$$

$$w = w(x)$$



$$\theta \approx \tan\theta = \frac{dw}{dx}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2}, \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0, \qquad \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(\frac{dw}{dx} - \frac{dw}{dx}\right) = 0$$

We use within the scope of the classical beam theory

$$\sigma_{xx} = E\varepsilon_{xx}$$

We sub. $\sigma_{xx} = E\varepsilon_{xx}$ in $\int_S t_i \delta u_i \, dS = \int_V \sigma_{ij} \delta\varepsilon_{ij} \, dV$

$$\int_0^L q\,\delta w \, dx = \int \sigma_{xx} \delta\varepsilon_{xx} \, dV$$

$$= \int E\,\varepsilon_{xx} \delta\varepsilon_{xx} \, dV$$

$$= \int_0^L \int_{-h/2}^{h/2} b\,E\,\varepsilon_{xx} \delta\varepsilon_{xx} \, dz \, dx$$

[b: width of the beam, along the y-dir^n]

$$= \int_0^L \int_{-h/2}^{h/2} b\,E \left(-z \frac{d^2 w}{dx^2}\right)\left(-z \frac{d^2 \delta w}{dx^2}\right) dz \, dx$$

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$$\int q\,\delta w\,dx = \int_0^L \int_{-h/2}^{h/2} b\,E\left(-z\frac{d^2w}{dx^2}\right)\left(-z\frac{d^2\delta w}{dx^2}\right)dz\,dx$$

$$= \int_0^L \int_{-h/2}^{h/2} E\,b\,z^2\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx^2}\,dz\,dx$$

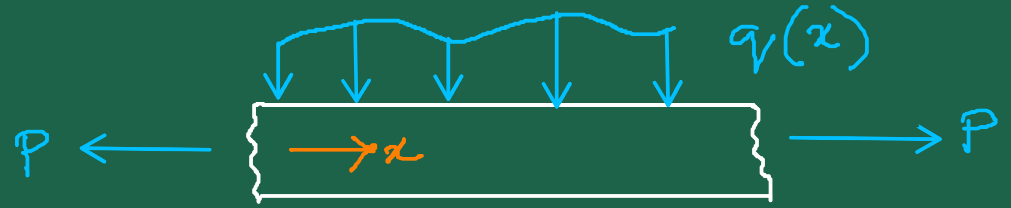
$$= \int_0^L E\,b\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx^2}\left(\int_{-h/2}^{h/2} z^2\,dz\right)dx$$

$$= \int_0^L EI\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx^2}\,dx \qquad \left[\,I = b\int_{-h/2}^{h/2} z^2\,dz\,\right]$$

$$\therefore \int_0^L EI\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx}\,dx - \int_0^L q\,\delta w\,dx = 0$$

Compare this with an intermediate step in the application of P. of st. P.E. → (*)
(prev. video)

Lateral and Axial Load – Bending and Stretching Using the Principle of Stationary Potential Energy



$$\Pi = V_{int} + V_{ext}$$

$$= V_{int,stretch} + V_{int,\,bending} + V_{ext,\,axial} + V_{ext,\,lateral}$$

$$= \int_0^L \frac{P^2}{2EA}\,dx + \int_0^L \frac{M^2}{2EI}\,dx - \int_0^L P\,\delta_D(x-L)\,u_s\,dx - \int_0^L (-P)\,\delta_D(x-0)\,u_s\,dx - \int_0^L q\,w\,dx$$

$$P = \int_A \sigma_{xx}\,dA = \int_A E\,\varepsilon_{xx}\,dA = \int_A E\,\frac{du_s}{dx}\,dA$$

$$= EA\,\frac{du_s}{dx}$$

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$$\delta\pi = 0$$

$$\Rightarrow \delta\left[\int_0^L \frac{1}{2}EA\left(\frac{du_s}{dx}\right)^2 dx + \int_0^L \frac{1}{2}EI\left(\frac{d^2w}{dx^2}\right)^2 dx - \int_0^L P\,\delta_D(x-L)\,u_s\,dx + \int_0^L P\,\delta_D(x-0)\,u_s\,dx - \int_0^L q\,w\,dx\right] = 0$$

(***)

$$\Rightarrow \int_0^L EA\,\frac{du_s}{dx}\,\frac{d\,\delta u_s}{dx}\,dx + \left(\text{same as before}\right) - \int_0^L P\,\delta_D(x-L)\,\delta u_s\,dx + \int_0^L P\,\delta_D(x-0)\,\delta u_s\,dx - \int_0^L q\,\delta w\,dx = 0$$

$$\Rightarrow \left[EA\,\frac{du_s}{dx}\,\delta u_s\right]_0^L - \int_0^L \frac{d}{dx}\left(EA\,\frac{du_s}{dx}\right)\delta u_s\,dx - \int_0^L P\,\delta_D(x-L)\,\delta u_s\,dx + \int_0^L P\,\delta_D(x-0)\,\delta u_s + \left(\text{same as before}\right) - \int_0^L q\,\delta w\,dx = 0$$

$$-\int_0^L P\,\delta_D(x-L)\,\delta u_s\,dx + \int_0^L P\,\delta_D(x-0)\,\delta u_s\,dx = -P\,\delta u_s\big|_{x=L} + P\,\delta u_s\big|_{x=0} = -\left[P\,\delta u_s\right]_0^L$$

$$\left[EA\frac{du_s}{dx}\,\delta u_s\right]_0^L - \int_0^L EA\frac{d^2u_s}{dx^2}\,\delta u_s\,dx - \left[P\,\delta u_s\right]_0^L + \left(\begin{array}{c}\text{all terms}\\ \text{involving } w\end{array}\right) = 0$$

$\therefore$ The governing eqns are:

$$EA\frac{d^2u_s}{dx^2} = 0$$

$$EI\frac{d^4w}{dx^4} = q$$

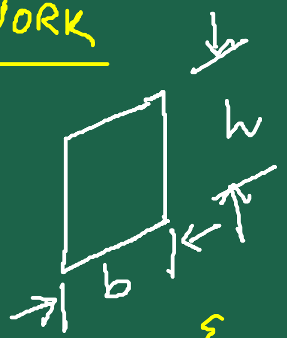
Boundary conditions:

At $x = 0$ and $x = L$:

$$\left\{\begin{array}{l} \text{Either } EA\dfrac{du_s}{dx} - P = 0 \text{ or } u_s \text{ is specified} \\[2ex] \text{Either } EI\dfrac{d^2w}{dx^2} = 0 \text{ or } \dfrac{dw}{dx} \text{ is specified} \\[2ex] \text{Either } EI\dfrac{d^3w}{dx^3} = 0 \text{ or } w \text{ is specified} \end{array}\right.$$

Lateral and Axial Load — Bending and Stretching

Using the Principle of Virtual Work



$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV = \int_S t_i\,\delta u_i\,dS$$

Kinematical hypothesis

$$u = -z\frac{dw}{dx} + u_s$$

$$v = 0$$

$$w = w(x)$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2w}{dx^2} + \frac{du_s}{dx}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(\frac{dw}{dx} - \frac{dw}{dx}\right) = 0$$

For the material behaviour: $\sigma_{xx} = E\,\varepsilon_{xx}$

Now, $\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dv = \int_S t_i\,\delta u_i\,dS$

$$\Rightarrow \int_V \underset{E\varepsilon_{xx}}{\cancel{\sigma_{xx}}}\,\delta\varepsilon_{xx}\,dV = \int_0^L P\,\delta_D(x-L)\,\delta u_s\,dx + \int_0^L (-P)\,\delta_D(x-0)\,\delta u_s\,dx + \int_0^L q\,\delta w\,ds$$

$$\text{LHS} = \int_0^L \int_{-h/2}^{h/2} E\left(-z\frac{d^2 w}{dx^2} + \frac{du_s}{dx}\right)\left(-z\frac{d^2\delta w}{dx^2} + \frac{d\delta u_s}{dx}\right) b\,dz\,dx$$

$$= \int_0^L \int_{-h/2}^{h/2} E\left[z^2 \frac{d^2 w}{dx^2}\frac{d^2\delta w}{dx^2} - z\frac{d^2 w}{dx^2}\frac{d\delta u_s}{dx} - z\frac{du_s}{dx}\frac{d^2\delta w}{dx^2} + \frac{du_s}{dx}\frac{d\delta u_s}{dx}\right] b\,dz\,dx$$

$$= \int_0^L Eb\,\frac{d^2 w}{dx^2}\frac{d^2\delta w}{dx^2}\int_{-h/2}^{h/2} z^2\,dz\,dx - \int_0^L Eb\left(\frac{d^2 w}{dx^2}\frac{d\delta u_s}{dx} + \frac{du_s}{dx}\frac{d^2\delta w}{dx^2}\right)\int_{-h/2}^{h/2} z\,dz\,dx + \int_0^L Eb\,\frac{du_s}{dx}\frac{d\delta u_s}{dx}\int_{-h/2}^{h/2} dz\,dx$$

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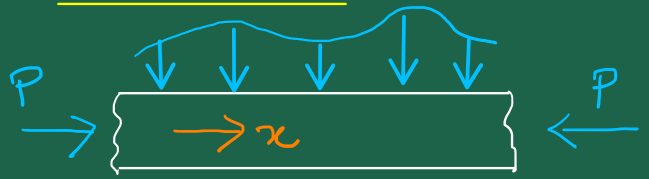
$$LHS = \int_0^L EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2} dx + \int_0^L EA \frac{du_s}{dx} \frac{d\delta u_s}{dx} dx$$

Now, if we consider the LHS and the RHS together and compare with the intermediate step (***) (from pg. 11) from the use of the principle of stationary P.E., we realize that they are exactly identical.

Finally, we are going to end up with the same governing eqns and BCs.

BUCKLING



Kinematical hypothesis

$$u = u_s - z\frac{dw}{dx}$$

$$v = 0$$

$$w = w(x)$$

Strain-displacement relations → non-linearity

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\}$$

$$E_{yy} = \ldots$$

$$E_{zz} = \ldots$$

$$\vdots$$

Important assumption:

$$\frac{\partial w}{\partial x} \gg \frac{\partial u}{\partial x}$$

$$\therefore E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$= \frac{du_s}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2$$

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Use the principle of virtual work :

$$\int_V \sigma_{ij}\, \delta \varepsilon_{ij}\, dV = \int_S t_i\, \delta u_i\, dS$$

$$LHS = \int_0^L \int_{-h/2}^{h/2} b\, \sigma_{xx}\, \delta \varepsilon_{xx}\, dz\, dx$$

$$= \int_0^L \int_{-h/2}^{h/2} E b\, \varepsilon_{xx}\, \delta \varepsilon_{xx}\, dz\, dx$$

$$= \int_0^L \int_{-h/2}^{h/2} E b \left(\frac{du_s}{dx} - z \frac{d^2 w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right) \left(\frac{d\delta u_s}{dx} - z \frac{d^2 \delta w}{dx^2} + \frac{dw}{dx}\frac{d\delta w}{dx} \right) dz\, dx$$

$$= \int_0^L \int_{-h/2}^{h/2} E b \left\{ \frac{du_s}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \right\} \left\{ \frac{d\delta u_s}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx} \right\} dz\, dx + 0 + \int_0^L \int_{-h/2}^{h/2} E b z^2 \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2}\, dz\, dx$$

$$\text{LHS} = \int_0^L EA\left\{\frac{du_s}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\}\left\{\frac{d\delta u_s}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right\}dx + \int_0^L EI\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx^2}\,dx$$

$$= \int_0^L EA\{\_\}\,\frac{d\delta u_s}{dx}\,dx + \int_0^L EA\{\_\}\,\frac{dw}{dx}\,\frac{d\delta w}{dx}\,dx + \int_0^L EI\,\frac{d^2w}{dx^2}\,\frac{d^2\delta w}{dx^2}\,dx$$

$$= \Big[EA\{\ \}\,\delta u_s\Big]_0^L - \int_0^L \frac{d}{dx}\Big[EA\{\ \}\Big]\,\delta u_s\,dx + \Big[EA\{\ \}\,\frac{dw}{dx}\,\delta w\Big]_0^L$$

$$- \int_0^L \frac{d}{dx}\Big[EA\{\ \}\,\frac{dw}{dx}\Big]\,\delta w\,dx + \Big[EI\,\frac{d^2w}{dx^2}\,\frac{d\delta w}{dx}\Big]_0^L - \Big[EI\,\frac{d^3w}{dx^3}\,\delta w\Big]_0^L$$

$$+ \int_0^L EI\,\frac{d^4w}{dx^4}\,\delta w\,dx$$

$$\text{RHS} = -P\,\delta u_s\Big|_{x=L} + P\,\delta u_s\Big|_{x=0} + \int_0^L q\,\delta w\,dx \;\Rightarrow\; = -\Big[P\,\delta u_s\Big]_0^L + \int_0^L q\,\delta w\,dx$$

Governing eqns: $\frac{d}{dx}\left[EA\{\ \}\right] = 0$

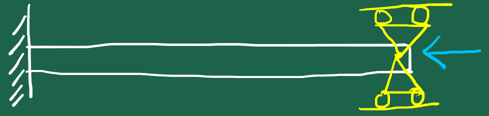
$$-\frac{d}{dx}\left[EA\{\ \}\frac{dw}{dx}\right] + EI\frac{d^4 w}{dx^4} = q$$

Boundary conditions: At $x=0$ and $x=L$

Either $EA\{\ \} = -P$ or u_s is specified

Either $EA\{\ \}\frac{dw}{dx} - EI\frac{d^3 w}{dx^3} = 0$ or w is specified

Either $EI\frac{d^2 w}{dx^2} = 0$ or $\frac{dw}{dx}$ is specified



At $x=L$, u_s is not specified

$\therefore EA\{\ \} = -P$ (at $x=L$)

Note $\frac{d}{dx}\left[EA\{\ \}\right] = 0$, $0 \le x \le L$

$\Rightarrow EA\{\ \} = \text{const.}$ $0 \le x \le L$

$\Rightarrow EA\{\ \} = -P$

$0 \le x \le L$

$$-\frac{d}{dx}\left[EA\{\ \}\frac{dw}{dx}\right] + EI\frac{d^4w}{dx^4} = q$$

$$\Rightarrow -\frac{d}{dx}\left[(-P)\frac{dw}{dx}\right] + EI\frac{d^4w}{dx^4} = q$$

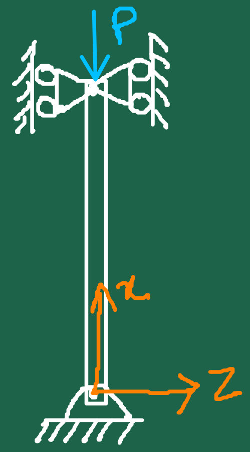
$$\Rightarrow \boxed{EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = q}$$

Boundary Conditions: At $x=0$ and $x=L$:

Either $-P\dfrac{dw}{dx} - EI\dfrac{d^3w}{dx^3} = 0$ or w is specified

Either $EI\dfrac{d^2w}{dx^2} = 0$ or $\dfrac{dw}{dx}$ is specified

Example



$$\boxed{EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = 0}$$

Boundary Conditions: At $x=0$ and $x=L$:

Either $-P\frac{dw}{dx} - EI\frac{d^3w}{dx^3} = 0$ or $\underline{w \text{ is specified}}$ $= 0$

Either $\underline{EI\frac{d^2w}{dx^2} = 0}$ or $\frac{dw}{dx}$ is specified

$$\frac{d^4w}{dx^4} + \frac{P}{EI}\frac{d^2w}{dx^2} = 0$$

$$\Rightarrow \frac{d^4w}{dx^4} + k^2\frac{d^2w}{dx^2} = 0$$

$\rightarrow$ To solve: Assume a sol^n form : $w = e^{mx}$

$$m^4 + k^2 m^2 = 0$$

$$\Rightarrow m^2 (m^2 + k^2) = 0$$

$$\Rightarrow m = 0, 0, \pm iK, \quad i = \sqrt{-1}$$

$$\therefore W = A' e^{iKx} + B' e^{-iKx} + Cx + D$$

$$\Rightarrow W = A \cos Kx + B \sin Kx + Cx + D \Rightarrow \frac{d^2w}{dx^2} = -K^2 A \cos Kx - K^2 B \sin Kx$$

At $x = 0$, $w = 0 \Rightarrow A + D = 0 \Rightarrow D = -A = 0$

At $x = L$, $w = 0 \Rightarrow A \cos KL + B \sin KL + CL + D = 0 \Rightarrow CL = -B \sin KL \Rightarrow C = 0$

At $x = 0$, $EI \frac{d^2w}{dx^2} = 0 \Rightarrow -K^2 A = 0 \Rightarrow A = 0$

At $x = L$, $EI \frac{d^2w}{dx^2} = 0 \Rightarrow -K^2 \underset{0}{\cancel{A}} \cos KL - K^2 B \sin KL = 0 \Rightarrow B \sin KL = 0$

$\therefore \sin KL = 0$ [For non-trivial sol^n]

$\Rightarrow KL = n\pi, \quad n \in \mathbb{Z}$

$$KL = n\pi$$

$$\Rightarrow K^2L^2 = n^2\pi^2$$

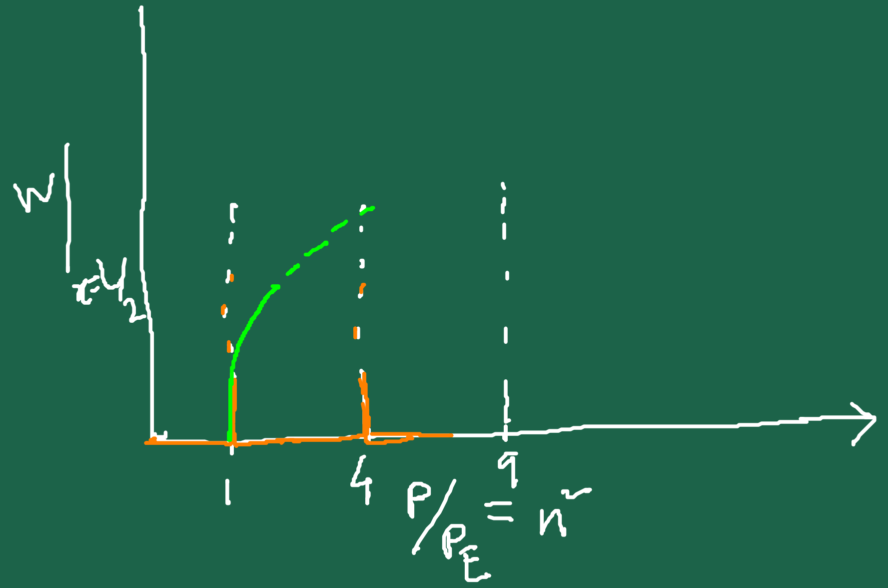
$$\Rightarrow \frac{P}{EI} = \frac{n^2\pi^2}{L^2}$$

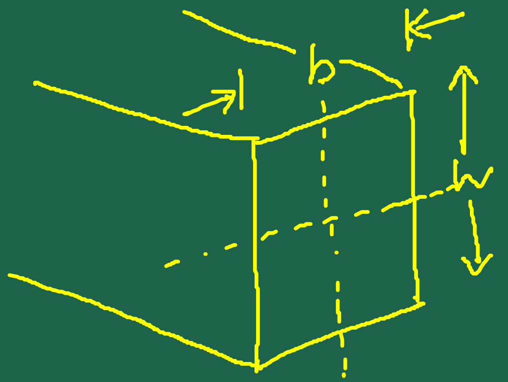
$$\Rightarrow \boxed{P = \frac{n^2\pi^2 EI}{L^2}}$$

For these values of P, we are going to have non-zero values of w → Buckling takes place!

$$P_E = \frac{\pi^2 EI}{L^2}$$

$$P/P_E = n^2$$





$$I = \frac{1}{12} b h^3$$

$$I = \frac{1}{12} h b^3$$

Buckling – Alternative Formulation

Set $\dfrac{du_s}{dx} = -\dfrac{1}{2}\left(\dfrac{dw}{dx}\right)^2 \rightarrow$ To be justified later

$$\int_0^L \int_{-h/2}^{h/2} E\, b\, \varepsilon_{xx}\, \delta\varepsilon_{xx}\, dz\, dx = -\Big[P\,\delta u_s\Big]_0^L + \int_0^L q\,\delta w\, dx$$

$$\varepsilon_{xx} = \frac{du_s}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 = -z\frac{d^2w}{dx^2}$$

$$\int_0^L \int_{-h/2}^{h/2} E\, b\, z^2 \frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}\, dz\, dx = -\int_0^L P\frac{d\delta u_s}{dx}\, dx + \int_0^L q\,\delta w\, dx$$

$$\Rightarrow \int_0^L EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}\, dx = \int_0^L P\frac{dw}{dx}\frac{d\delta w}{dx}\, dx + \int_0^L q\,\delta w\, dx$$

$$\Rightarrow \left[EI \frac{d^2 w}{dx^2} \frac{d\delta w}{dx} \right]_0^L - \left[\frac{d}{dx}\left(EI \frac{d^2 w}{dx^2} \right) \delta w \right]_0^L + \int_0^L \frac{d^2}{dx^2}\left(EI \frac{d^2 w}{dx^2} \right) \delta w \, dx$$

$$= \left[P \frac{dw}{dx} \delta w \right]_0^L - \int_0^L \frac{d}{dx}\left(P \frac{dw}{dx} \right) \delta w \, dx + \int_0^L q \, \delta w \, dx$$

G.D.E. : $EI \dfrac{d^4 w}{dx^4} + P \dfrac{d^2 w}{dx^2} = q$

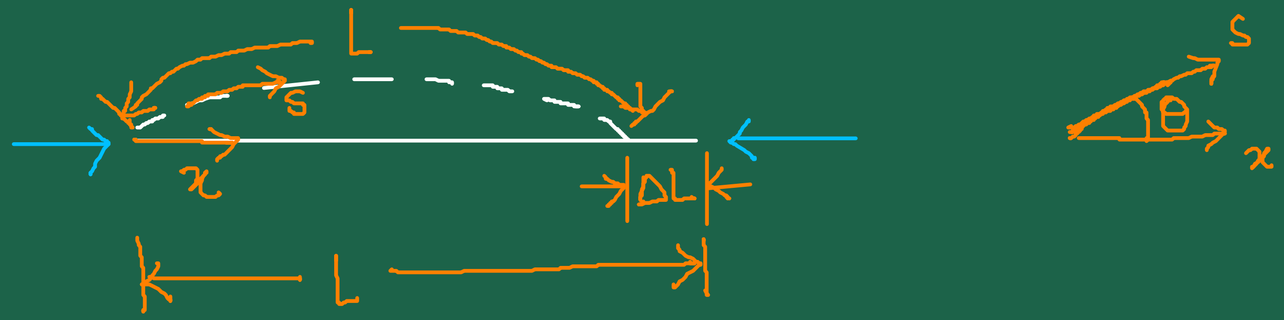
BCs : At $x = 0$ and $x = L$,

Either $EI \dfrac{d^2 w}{dx^2} = 0$ or $\dfrac{dw}{dx}$ is specified

Either $EI \dfrac{d^3 w}{dx^3} + P \dfrac{dw}{dx} = 0$ or w is specified

Same set of eqns obtained earlier

Justification for $\frac{du}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \rightarrow$ It represents the shortening of the beam



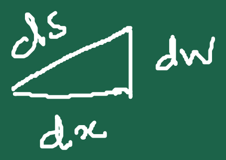
$$\Delta L = L - (L - \Delta L)$$

$$= \int_0^L dx - \int_0^{L-\Delta L} dx$$

Use a body-fitted coordinate variable, s

$$\Delta L = \int_0^L ds - \int_0^L ds \cos\theta$$

$$\Delta L = \int_0^{L} ds - \int_0^{L-\Delta L} dx$$



$$ds = \sqrt{dx^2 + dw^2} = \sqrt{1 + \left(\frac{dw}{dx}\right)^2}\, dx$$

$$= \int_0^{L-\Delta L} \sqrt{1 + \left(\frac{dw}{dx}\right)^2}\, dx - \int_0^{L-\Delta L} dx$$

$$\approx \int_0^{L-\Delta L} \left\{1 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right\} dx - \int_0^{L-\Delta L} dx$$

$$= \int_0^{L-\Delta L} \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \quad \text{— } (\#_1)$$

$$\Delta L = -\int_0^{-\Delta L} du_s = -\int_0^{L-\Delta L} \frac{du_s}{dx}\, dx \quad \text{— } (\#_2)$$

Comparing $(\#_1)$ with $(\#_2)$, we obtain $\displaystyle \frac{du_s}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2$

ELASTICA

$$\int_0^L EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2} dx = \int_0^L P \frac{dw}{dx} \frac{d\delta w}{dx} dx + \cancel{\int_0^L q\, \delta w\, dx} \to 0$$

$$\Rightarrow \int_0^L EI \frac{d^2 w}{dx^2} \frac{d^2 \delta w}{dx^2} dx = \frac{1}{2} \int_0^L P\, \delta \left(\frac{dw}{dx}\right)^2 dx$$

LHS: $\frac{d^2 w}{dx^2}$ is an approximation of the curvature of the beam: $\frac{d^2w/dx^2}{\left\{1 + \left(dw/dx\right)^2\right\}^{3/2}}$

In the s-θ coordinate system, the curvature is given by $\frac{d\theta}{ds}$

RHS: $\int_0^L \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx = -\int_0^L \frac{du}{dx} dx = \Delta L$ (a bit of mismatch with what we had shown earlier!)

$$= \int_0^L ds - \int_0^L ds \cos\theta$$

$$\int_0^L EI \frac{d\theta}{ds}\frac{d\delta\theta}{ds}\,ds = P\,\delta\left[\int_0^L ds - \int_0^L ds\cos\theta\right]$$

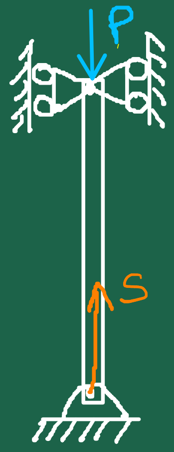
$$\Rightarrow \left[EI\frac{d\theta}{ds}\delta\theta\right]_0^L - \int_0^L \frac{d}{ds}\left(EI\frac{d\theta}{ds}\right)\delta\theta\,ds = +P\int_0^L ds\,\sin\theta\,\delta\theta$$

G.D.E. : $\quad EI\dfrac{d^2\theta}{ds^2} + P\sin\theta = 0$

B.C.s : At $\cancel{x}^{\,s}=0$ and $\cancel{x}^{\,s}=L$,

Either $\quad EI\dfrac{d\theta}{ds} = 0 \quad$ or $\quad \theta$ is specified

ELASTICA — EXAMPLE



G.D.E. : $EI\dfrac{d^2\theta}{ds^2} + P\sin\theta = 0$

B.C.S. : At $s=0$ and $s=L$: Either $EI\dfrac{d\theta}{ds}=0$ or $\underline{\theta \text{ is specified}}$ ✗

$$EI\frac{d^2\theta}{ds^2} + P\sin\theta = 0$$

$$\Rightarrow EI\frac{d^2\theta}{ds^2}\frac{d\theta}{ds} + P\sin\theta\frac{d\theta}{ds} = 0$$

$$\Rightarrow \frac{1}{2}EI\frac{d}{ds}\left\{\left(\frac{d\theta}{ds}\right)^2\right\} + P\frac{d}{ds}(-\cos\theta) = 0$$

$$\Rightarrow \frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2 - P\cos\theta = C$$

Since the ends at $s=0$ and $s=L$ are pinned, θ is not specified

$\therefore$ At $s=0$ and $s=L$, $EI\dfrac{d\theta}{ds}=0$

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At $s=0$, let $\theta = \alpha$ (unknown).

At $s=L$, $\theta = -\alpha$

$$\left.\begin{aligned} &\text{At } s=0, \quad \tfrac{1}{2}EI\cancel{\left(\frac{d\theta}{ds}\right)^2}^{\,0} - P\cos\alpha = C \\ &\text{At } s=L, \quad \tfrac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2 - P\cos(-\alpha) = C \end{aligned}\right\} \rightarrow C = -P\cos\alpha$$

$$\therefore \quad \frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2 - P\cos\theta = -P\cos\alpha$$

$$\Rightarrow \frac{1}{2}EI\left(\frac{d\theta}{ds}\right)^2 = P\cos\theta - P\cos\alpha$$

$$\Rightarrow \frac{d\theta}{ds} = -\sqrt{\frac{2P}{EI}(\cos\theta - \cos\alpha)}$$

[–ve because θ is decreasing from α through 0 to $-\alpha$]

$$\frac{d\theta}{ds} = -\sqrt{\frac{2P}{EI}\left(\cos\theta - \cos\alpha\right)}$$

$$\Rightarrow \frac{d\theta}{ds} = -\sqrt{\frac{2P}{EI}}\sqrt{2\sin^2\frac{\alpha}{2} - 2\sin^2\frac{\theta}{2}}$$

$$\Rightarrow \int_0^L ds = -\sqrt{\frac{EI}{4P}}\int_\alpha^{-\alpha}\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

$$\Rightarrow \int_0^L ds = -\sqrt{\frac{EI}{4P}}\; 2\int_\alpha^{0}\frac{d\theta}{\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}}$$

$$\Rightarrow L = -\sqrt{\frac{EI}{P}}\int_\alpha^{0}\frac{d\theta}{\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}}$$

Make the substitution $\sin\frac{\theta}{2} = K\sin\phi$ where $K = \sin\frac{\alpha}{2}$

$\boxed{33}$

$$\sin\frac{\theta}{2} = K\sin\phi \Rightarrow \frac{1}{2}\cos\frac{\theta}{2}\,d\theta = K\cos\phi\,d\phi$$

$$\Rightarrow \quad d\theta = \frac{2K\cos\phi\,d\phi}{\cos\theta/2}$$

$$\therefore \quad L = -\sqrt{\frac{EI}{P}}\int_{\pi/2}^{0}\frac{2K\cos\phi/\cos\theta/2\;d\phi}{\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\sin^2\phi\right)^{1/2}}$$

$$= -\sqrt{\frac{EI}{P}}\int_{\pi/2}^{0}\frac{2\cancel{K}\cos\phi/\cos\theta/2}{\cancel{\sin\frac{\alpha}{2}}\left(1-\sin^2\phi\right)^{1/2}}\,d\phi$$

$$= -\sqrt{\frac{EI}{P}}\int_{\pi/2}^{0}\frac{2\cos\phi}{\cos\phi\,\cos\theta/2}\,d\phi$$

$\theta = \alpha$

$\sin\frac{\alpha}{2} = K\sin\phi$

$\Rightarrow \sin\phi = 1$

$\Rightarrow \phi = \pi/2$

$\theta = 0$

$0 = K\sin\phi$

$\Rightarrow \phi = 0$

$$\therefore \quad L = -\sqrt{\frac{EI}{P}} \int_{\pi/2}^{0} \frac{2\, d\phi}{\sqrt{1 - \sin^2 \theta/2}}$$

$$= -2\sqrt{\frac{EI}{P}} \int_{\pi/2}^{0} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

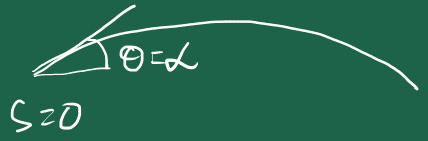
$$L = 2\sqrt{\frac{EI}{P}} \underbrace{\int_{0}^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}}_{\text{Complete elliptic integral of the 1st kind}} \quad \text{—} \quad (\#\#)$$

Given a value of P, we can find a value k such that (##) is satisfied $\rightarrow$ we can find value of α because $k = \sin\frac{\alpha}{2}$

$$L = 2\sqrt{\frac{EI}{P}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$\Rightarrow \sqrt{\frac{PL^2}{EI}} = 2\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$\Rightarrow \sqrt{\frac{P}{\pi^2 EI/L^2}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$



$$\Rightarrow \sqrt{\frac{P}{P_{Euler}}} = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}$$

$$\left(k = \sin\frac{\alpha}{2}\right)$$

$$\alpha > 0$$

$$\Rightarrow \sqrt{\frac{P}{P_{Euler}}} - \frac{2}{\pi}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = 0$$

$$EI\frac{d^2\theta}{ds^2} + P\sin\theta = 0$$

Note $\sin\theta = \dfrac{dw}{ds}$

$$EI\frac{d^2\theta}{ds^2} + P\frac{dw}{ds} = 0$$

$$\Rightarrow EI\frac{d\theta}{ds} + Pw = C_2$$

At $s=0$, we had already found that $EI\dfrac{d\theta}{ds} = 0$

" $w = 0$

$\therefore C_2 = 0$

$$\therefore \quad EI\frac{d\theta}{ds} + Pw = 0$$

$$\Rightarrow \quad w = -\frac{EI}{P}\frac{d\theta}{ds}$$

$$= +\frac{EI}{P}\sqrt{\frac{2P}{EI}}\sqrt{2\sin^2\frac{\alpha}{2} - 2\sin^2\frac{\theta}{2}}$$

$$= 2\sqrt{\frac{EI}{P}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}$$

$$= 2\sqrt{\frac{EI}{P}}\sqrt{k^2 - \sin^2\frac{\theta}{2}}$$

As θ goes from α to $-\alpha$ (through 0), we can find the corresponding values of w.

$$W = 2\sqrt{\frac{EI}{P}}\sqrt{k^2 - \sin^2\frac{\theta}{2}}$$

$$= 2\sqrt{\frac{\pi^2 EI/L^2}{\pi^2 P/L^2}}\sqrt{k^2 - k^2\sin^2\phi}$$

$$= \frac{2L}{\pi}\sqrt{\frac{P_{Euler}}{P}}\sqrt{k^2 - k^2\sin^2\phi}$$

$$\bar{W} = \frac{W}{L} = \frac{2}{\pi}\sqrt{\frac{1}{P/P_{Euler}}}\sqrt{k^2 - k^2\sin^2\phi}$$

$$\sin\frac{\theta}{2} = k\sin\phi$$

$$\theta = -\alpha$$

$$-\sin\frac{\alpha}{2} = \sin\frac{\alpha}{2}\sin\phi$$

$$\Rightarrow \sin\phi = -1 \Rightarrow \phi = -\pi/2$$

$$\theta = \alpha$$

$$\sin\alpha/2 = \sin\alpha/2 \, \sin\phi$$

$$\Rightarrow \sin\phi = 1 \Rightarrow \phi = \pi/2$$

As θ goes from $-\alpha$ to α, ϕ goes from $-\pi/2$ to $\pi/2$