Beam Theory – Bending and Buckling *

Overview:†

- Consider only lateral loading: Two methods: $\delta\Pi = 0$ and VWE
- Consider both laterial loading and axial loading i.e. both bending and stretching: again two methods: $\delta\Pi = 0$ and VWE. Both the methods will show that bending and stretching are decoupled.
- Now, suppose we switch off the lateral loading and consider axial "negative stretching", i.e. axial compression by inverting the sign of the axial load. Then again – following from what we found in the last point – we should not expect this negative stretching to have anything to do with bending. However, this expectation does not match with our physical knowledge that axial compression can indeed result in bending in the form of buckling. The root cause of the problem is the linearity in the kinematics. Buckling is an instability and to model it we need a non-linear ingredient in our theory.

1 Only lateral loading

1.1 Using principle of stationary potential energy

$$\begin{aligned}\Pi &= V_{\text{int}} + V_{\text{ext}} \\ &= \int_0^L \frac{M^2}{2EI}\,\mathrm{d}x - \int_0^L q_0 w \mathrm{d}x\end{aligned}$$

Since $MR = EI$ with $\dfrac{1}{R} \approx \dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}$, we have

$$\Pi = \int_0^L \frac{1}{2} EI \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L q_0 w \mathrm{d}x.$$

Now, by the principle of stationary potential energy (DISCUSS THIS SEPARATELY), $\delta\Pi = 0$ for a body in equilibrium. Then

$$\begin{aligned}&\delta\Pi = 0, \\ \text{or,}\quad &\int_0^L \frac{1}{2} EI \left(2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) \frac{\mathrm{d}^2 \delta w}{\mathrm{d}x^2}\,\mathrm{d}x - \int_0^L q_0 \delta w \mathrm{d}x = 0, \\ \text{or,}\quad &\left[EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}\delta w}{\mathrm{d}x}\right]_0^L - \left[EI \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\delta w\right]_0^L + \int_0^L \frac{\mathrm{d}^4 w}{\mathrm{d}x^4}\delta w\,\mathrm{d}x - \int_0^L q_0 \delta w\,\mathrm{d}x = 0.\end{aligned}$$

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†These lecture notes closely follow the presentation in Chapter 4 of Dym and Shames.

Therefore, we must have the governing equation:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q_0,$$

together with the boundary conditions that at $x = 0$ and $x = L$:

$$\begin{aligned} &\text{either} \quad EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0, \quad \text{or} \quad \frac{\mathrm{d}w}{\mathrm{d}x} \quad \text{is specified;} \\ &\text{either} \quad EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = 0, \quad \text{or} \quad w \quad \text{is specified.} \end{aligned}$$

1.2 Using virtual work equation

We have from Energy Methods:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij} = \int_S t_i \delta u_i \ \mathrm{d}S. \quad \text{(considering no body force)} \tag{1}$$

We use the following kinematical hyphotheses:

$$\begin{aligned} u &= -z\frac{\mathrm{d}w}{\mathrm{d}x}, \\ v &= 0, \\ w &\equiv w(x) \end{aligned}$$

using which we obtain from the strain-displacement relations, the following:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{2a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \tag{2b}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0, \tag{2c}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \tag{2d}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0, \tag{2e}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0. \tag{2f}$$

Then, from Eq. (1)

$$\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} b\sigma_{xx}\delta\varepsilon_{xx} \ \mathrm{d}x \ \mathrm{d}z = \int_0^L q_0 \delta w \ \mathrm{d}x,$$

We cannot proceed further without invoking the material behaviour. We assume, as before, a Hookean material, so that $\sigma_{xx} = E\varepsilon_{xx}$. Using this stress-strain relationship, we have

$$
\begin{aligned}
&\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\varepsilon_{xx}\delta\varepsilon_{xx} \,\mathrm{d}x\,\mathrm{d}z = \int_0^L q_0\delta w \,\mathrm{d}x,\\
\text{or,}\quad &\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\frac{1}{2}\delta(\varepsilon_{xx})^2 \,\mathrm{d}x\,\mathrm{d}z = \int_0^L q_0\delta w \,\mathrm{d}x,\\
\text{or,}\quad &\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\frac{1}{2}\delta\left(-z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \,\mathrm{d}x\,\mathrm{d}z = \int_0^L q_0\delta w \,\mathrm{d}x, \quad \text{(from Eq. (2a))},\\
\text{or,}\quad &\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bEz^2\frac{1}{2}\left(2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \,\mathrm{d}x\,\mathrm{d}z = \int_0^L q_0\delta w \,\mathrm{d}x,\\
\text{or,}\quad &\int_0^L E\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \,\mathrm{d}x \int_{-\frac{h}{2}}^{\frac{h}{2}} bz^2 \,\mathrm{d}z = \int_0^L q_0\delta w \,\mathrm{d}x, \quad \text{(since } w \text{ is independent of } z\text{)}\\
\text{or,}\quad &\int_0^L EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \,\mathrm{d}x = \int_0^L q_0\delta w \,\mathrm{d}x.
\end{aligned}
$$

This last equation is exactly the same one that was obtained previously when using the principle of stationary potential energy (see previous subsection). Again, integrating twice by parts we obtain the same governing equation and boundary conditions as in the previous subsection.

2 Axial and Lateral Load – both bending and stretching

2.1 Using the principle of stationary potential energy

$$
\begin{aligned}
\Pi &= V_{\text{int}} + V_{\text{ext}}\\
&= V_{\text{int,stretch}} + V_{\text{int,bend}} + V_{\text{ext,axial}} + V_{\text{ext,lateral}} \qquad (3)\\
&= \int_0^L \frac{P^2}{2EA}\,\mathrm{d}x + \int_0^L \frac{M^2}{2EI}\,\mathrm{d}x - \int_0^L P\delta_{\mathrm{D}}(x-L)u_s\,\mathrm{d}x - \int_0^L (-P)\delta_{\mathrm{D}}(x-0)u_s\,\mathrm{d}x - \int_0^L q_0 w\,\mathrm{d}x
\end{aligned}
$$

Use $MR = EI$ as before and the following:

$$
\begin{aligned}
P &= \int_A \sigma_{xx}\,\mathrm{d}A,\\
&= \int_A E\frac{\mathrm{d}u_s}{\mathrm{d}x}\,\mathrm{d}A\\
&= EA\frac{\mathrm{d}u_s}{\mathrm{d}x}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\Pi = \int_0^L \frac{1}{2}EA\left(\frac{\mathrm{d}u_s}{\mathrm{d}x}\right)^2\mathrm{d}x + \int_0^L \frac{1}{2}EI\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2\mathrm{d}x - \int_0^L P\delta_{\mathrm{D}}(x-L)u_s\,\mathrm{d}x\\
+ \int_0^L P\delta_{\mathrm{D}}(x-0)u_s\,\mathrm{d}x - \int_0^L q_0 w\,\mathrm{d}x.
\end{aligned}
$$

Now, using the principle of stationary potential energy, we have

$$
\begin{aligned}
&\delta\Pi = 0,\\
\text{or,}\quad &\int_0^L \frac{1}{2}EA\left(2\frac{\mathrm{d}u_s}{\mathrm{d}x}\right)\frac{\mathrm{d}\delta u_s}{\mathrm{d}x}\,\mathrm{d}x + \int_0^L \frac{1}{2}EI\left(2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\,\mathrm{d}x\\
&\qquad - \int_0^L P\delta_{\mathrm{D}}(x-L)\delta u_s\,\mathrm{d}x + \int_0^L P\delta_{\mathrm{D}}(x-0)u_s\,\mathrm{d}x - \int_0^L q_0 w\,\mathrm{d}x = 0,\\
\text{or,}\quad &\left[EA\frac{\mathrm{d}u_s}{\mathrm{d}x}\delta u_s\right]_0^L - \int_0^L \frac{\mathrm{d}^2 u_s}{\mathrm{d}x^2}\delta u_s\,\mathrm{d}x + \left[EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}\delta w}{\mathrm{d}x}\right]_0^L - \left[EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\delta w\right]_0^L + \int_0^L \frac{\mathrm{d}^4 w}{\mathrm{d}x^4}\delta w\,\mathrm{d}x\\
&\qquad - P\delta u_s\big|_{x=L} + P\delta u_s\big|_{x=0}\int_0^L q_0\delta w\,\mathrm{d}x = 0.
\end{aligned}
$$

Therefore, we must have the following governing equations:

$$
EA\frac{\mathrm{d}^2 u_s}{\mathrm{d}x^2} = 0, \tag{4}
$$

$$
EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = 0, \tag{5}
$$

together with the boundary conditions: At $x = L$ *and* at $x = 0$:

$$
\begin{aligned}
&\text{either}\quad EA\frac{\mathrm{d}u_s}{\mathrm{d}x} - P = 0 \quad\text{or}\quad u_s \quad\text{is specified,}\\
&\text{either}\quad EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \quad\text{or}\quad \frac{\mathrm{d}w}{\mathrm{d}x} \quad\text{is specified,}\\
&\text{either}\quad EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = 0 \quad\text{or}\quad w \quad\text{is specified.}
\end{aligned}
$$

2.2 Using virtual work equation

We again start from the virtual work equation:

$$
\int_V \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \int_S t_i\delta u_i\,\mathrm{d}S. \quad \text{(considering no body force)} \tag{6}
$$

We use the following kinematical hyphotheses:

$$
\begin{aligned}
u &= u_s - z\frac{\mathrm{d}w}{\mathrm{d}x},\\
v &= 0,\\
w &\equiv w(x)
\end{aligned}
$$

using which we obtain from the strain-displacement relations, the following:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_s}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \tag{7a}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0, \tag{7b}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0, \tag{7c}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0, \tag{7d}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0, \tag{7e}$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0. \tag{7f}$$

Then, from Eq. (6), we have

$$\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} b\sigma_{xx}\delta\varepsilon_{xx}\,\mathrm{d}x\,\mathrm{d}z = \int_0^L P\delta_{\mathrm{D}}(x-L)u_s\,\mathrm{d}x + \int_0^L (-P)\delta_{\mathrm{D}}(x-0)u_s\,\mathrm{d}x + \int_0^L q_0\delta w\,\mathrm{d}x,$$

Again, we cannot proceed further without invoking the material behaviour. We assume, as before, a Hookean material, so that $\sigma_{xx} = E\varepsilon_{xx}$. Using this stress-strain relationship, we have

$$\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\varepsilon_{xx}\delta\varepsilon_{xx}\,\mathrm{d}x\,\mathrm{d}z = \int_0^L P\delta_{\mathrm{D}}(x-L)u_s\,\mathrm{d}x - \int_0^L P\delta_{\mathrm{D}}(x-0)u_s\,\mathrm{d}x + \int_0^L q_0\delta w\,\mathrm{d}x. \tag{8}$$

Considering the l.h.s of the preceding equation, we have the following

$$\begin{aligned}
&\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\varepsilon_{xx}\delta\varepsilon_{xx}\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}bE\delta\left(\varepsilon_{xx}\right)^2\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{2}bE\delta\left(\frac{\mathrm{d}u_s}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2\,\mathrm{d}x\,\mathrm{d}z \quad \text{(using Eq. (7a))} \\
&= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\delta\left(\frac{\mathrm{d}u_s}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)\left(\frac{\mathrm{d}\delta u_s}{\mathrm{d}x} - z\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\right)\,\mathrm{d}x\,\mathrm{d}z \\
&= \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x}\,\mathrm{d}z\,\mathrm{d}x - \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bEz\frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\,\mathrm{d}z\,\mathrm{d}x \\
&\qquad - \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bEz\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x}\,\mathrm{d}z\,\mathrm{d}x + \int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bEz^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}\,\mathrm{d}z\,\mathrm{d}x \\
&= \int_0^L \frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x}E\int_{-\frac{h}{2}}^{\frac{h}{2}} b\,\mathrm{d}z\,\mathrm{d}x - \int_0^L \frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}E\int_{-\frac{h}{2}}^{\frac{h}{2}} bz\,\mathrm{d}z\,\mathrm{d}x \\
&\qquad - \int_0^L \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x}E\int_{-\frac{h}{2}}^{\frac{h}{2}} bz\,\mathrm{d}z\,\mathrm{d}x + \int_0^L \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2}E\int_{-\frac{h}{2}}^{\frac{h}{2}} bz^2\,\mathrm{d}z\,\mathrm{d}x
\end{aligned}$$

Noting that

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} b \, \mathrm{d}z = A,$$
$$\int_{-\frac{h}{2}}^{\frac{h}{2}} bz \, \mathrm{d}z = 0,$$
$$\int_{-\frac{h}{2}}^{\frac{h}{2}} bz^2 \, \mathrm{d}z = I,$$

we have

$$\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} bE\varepsilon_{xx}\delta\varepsilon_{xx} \, \mathrm{d}x \, \mathrm{d}z = \int_0^L EA\frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x} \, \mathrm{d}x + \int_0^L EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \, \mathrm{d}x, \tag{9}$$

so that from Eq. (8), we have

$$\int_0^L EA\frac{\mathrm{d}u_s}{\mathrm{d}x}\frac{\mathrm{d}\delta u_s}{\mathrm{d}x} \, \mathrm{d}x + \int_0^L EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2\delta w}{\mathrm{d}x^2} \, \mathrm{d}x - \int_0^L P\delta_{\mathrm{D}}(x-L)u_s \, \mathrm{d}x + \int_0^L P\delta_{\mathrm{D}}(x-0)u_s \, \mathrm{d}x - \int_0^L q_0\delta w \, \mathrm{d}x = 0. \tag{10}$$

Note that this equation is exactly the same as Eq. (3) and so the exact same governing equations and boundary conditions will be obtained as in the previous section.

VERY IMPORTANT POINT: What the above derivation shows is that based on our kinematical hypothesis, the stretching (i.e. the solution involving u_s) decouples from the bending (i.e. the solution involving w).

3 Buckling

We start with the following nonlinear strain-displacement relation[‡]

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{11}$$

As before, we consider the following kinematical hypothesis:

$$u = u_s - z\frac{\mathrm{d}w}{\mathrm{d}x},$$
$$v = 0,$$
$$w \equiv w(x).$$

[‡]Refer to the lecture notes on kinematics available at http://www.facweb.iitkgp.ac.in/~jeevanjyoti/teaching/mechsolids/2019/kinem.pdf

However, to address buckling we additionally assume the following:

$$\frac{\partial w}{\partial x} \gg \frac{\partial u_s}{\partial x} \tag{12}$$

$$\begin{aligned}
E_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
E_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} = 0, \\
E_{zz} &= \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
E_{xy} &= \frac{1}{2}\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\} = 0, \\
E_{yz} &= \frac{1}{2}\left\{\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right\} = 0, \\
E_{zx} &= \frac{1}{2}\left\{\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right\} = 0,
\end{aligned}$$

The notes for this section are incomplete.

4 Elastica

The exact version of

$$\frac{1}{2}EI\int_0^L \delta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \frac{1}{2}P\int_0^L \delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 = 0$$

is the following:

$$\begin{aligned}
&\frac{1}{2}EI\int_0^L \delta\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - P\int_0^L \delta\left(\mathrm{d}s - \cos\theta\mathrm{d}s\right) = 0, \\
\text{or,}\quad &\frac{1}{2}EI\int_0^L \delta\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - P\int_0^L \sin\delta\theta\mathrm{d}s = 0, \\
\text{or,}\quad &EI\int_0^L \frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\delta\theta}{\mathrm{d}s}\,\mathrm{d}s - P\int_0^L \sin\delta\theta\,\mathrm{d}s = 0, \\
\text{or,}\quad &\left[EI\frac{\mathrm{d}\theta}{\mathrm{d}s}\delta\theta\right]_0^L - EI\int_0^L \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}\delta\theta\,\mathrm{d}s - P\int_0^L \sin\theta\delta\theta\,\mathrm{d}s = 0
\end{aligned} \tag{13}$$

From the above, we can extract the following governing differential equation and boundary conditions as follows:

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\sin\theta = 0, \tag{14}$$

$$\text{either} \quad EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0 \quad \text{or} \quad \theta \quad \text{is specified} \tag{15}$$

Considering the governing differential equation, we can proceed towards a solution through a standard trick of multiplying throughout by $\frac{\mathrm{d}\theta}{\mathrm{d}s}$:

$$\begin{aligned} EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}\frac{\mathrm{d}\theta}{\mathrm{d}s} + P\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}s} &= 0, \\ \text{or,} \quad \frac{1}{2}EI\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 - P\frac{\mathrm{d}}{\mathrm{d}s}(\cos\theta) &= 0, \end{aligned}$$

$$\text{or,} \quad \frac{1}{2}EI\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = P\cos\theta + c \tag{16}$$

At $x = 0$ and $x = L$, θ is not specified; so $EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0$. However, we denote θ at $x = 0$ as α and θ at $x = L$ as $-\alpha$. Thus, we obtain

$$0 = P\cos\alpha + c \tag{17}$$

Therefore, we have

$$\begin{aligned} &\frac{1}{2}EI\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = P\left(\cos\theta - \cos\alpha\right), \\ \text{or,} \quad &\frac{\mathrm{d}\theta}{\mathrm{d}s} = -\left[\frac{2P}{EI}\left(\cos\theta - \cos\alpha\right)\right]^{\frac{1}{2}} \qquad (\text{−ve because } \theta \text{ decreases from } \alpha \text{ to } -\alpha \text{ through } 0) \\ \text{or,} \quad &\int_0^L \mathrm{d}s = -\int_\alpha^{-\alpha}\sqrt{\frac{EI}{2P}}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\alpha}}, \\ \text{or,} \quad &L = -\int_\alpha^{-\alpha}\sqrt{\frac{EI}{4P}}\frac{\mathrm{d}\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}, \\ \text{or,} \quad &L = -\int_\alpha^{0}\sqrt{\frac{EI}{4P}}\frac{\mathrm{d}2\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}, \\ \text{or,} \quad &L = \int_0^{\alpha}\sqrt{\frac{EI}{4P}}\frac{\mathrm{d}2\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}. \end{aligned}$$

Set $\sin\frac{\theta}{2} = K\sin\varphi$, where $K = \sin\frac{\alpha}{2}$. Then, we have

$$\frac{1}{2}\cos\frac{\theta}{2}\mathrm{d}\theta = K\cos\varphi\mathrm{d}\varphi \tag{18}$$

Furthermore, for the limits of the integration, we have

$$\begin{aligned}
&\text{When} \quad \theta = \alpha, \qquad K \sin\varphi = \sin\frac{\alpha}{2} \implies \sin\varphi = 1 \implies \varphi = \frac{\pi}{2},\\
&\text{When} \quad \theta = 0, \qquad K \sin\varphi = 0 \implies \varphi = 0.
\end{aligned}$$

Utilizing the above in the integration, we obtain

$$\begin{aligned}
L &= \sqrt{\frac{EI}{P}} \int_0^{\alpha} \frac{\mathrm{d}\theta}{\sqrt{K^2 - K^2 \sin^2\varphi}},\\
&= \sqrt{\frac{EI}{P}} \int_0^{\pi/2} \frac{2K\cos\varphi \mathrm{d}\varphi}{\cos\frac{\theta}{2} K \cos\varphi},\\
&= \sqrt{\frac{EI}{P}} \int_0^{\pi/2} \frac{2\mathrm{d}\varphi}{\cos\frac{\theta}{2}},\\
&= \sqrt{\frac{EI}{P}} \int_0^{\pi/2} \frac{2\mathrm{d}\varphi}{\sqrt{1 - K^2\sin^2\varphi}},\\
&= \sqrt{P_{\mathrm{Euler}}P} \left(\frac{L}{\pi}\right) \int_0^{\pi/2} \frac{2\mathrm{d}\varphi}{\sqrt{1 - K^2\sin^2\varphi}},
\end{aligned}$$

where $P_{\mathrm{Euler}} = \sqrt{\frac{\pi^2 EI}{L^2}}$, and the integration $\int_0^{\pi/2} \frac{2\mathrm{d}\varphi}{\sqrt{1 - K^2\sin^2\varphi}}$ is known as the complete elliptic integral of the first kind. We finally obtain

$$\frac{P}{P_{\mathrm{Euler}}} = \frac{2}{\pi} \int_0^{\pi/2} \frac{2\mathrm{d}\varphi}{\sqrt{1 - K^2\sin^2\varphi}}. \tag{19}$$

In a given physical situation, we know the load P, so we know the LHS of the above equation. Thus we have to find K such that the RHS = LHS.

Note that $\sin\theta = \frac{\mathrm{d}w}{\mathrm{d}s}$. Therefore,

$$EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\sin\theta = 0, \tag{20}$$

$$\implies EI\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + P\frac{\mathrm{d}w}{\mathrm{d}s} = 0. \tag{21}$$

Integrating once with respect to s, we obtain

$$EI\frac{\mathrm{d}\theta}{\mathrm{d}s} + Pw = c. \tag{22}$$

At $x = 0$ and $x = L$, $EI\frac{\mathrm{d}\theta}{\mathrm{d}s} = 0$ and $w = 0$. Therefore, $c = 0$, and we have

$$w = -\frac{EI}{P}\frac{\mathrm{d}\theta}{\mathrm{d}s}. \tag{23}$$

However, we had found earlier that

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}s} &= -\left[\frac{2P}{EI}(\cos\theta - \cos\alpha)\right]^{\frac{1}{2}},\\ \frac{\mathrm{d}\theta}{\mathrm{d}s} &= -\sqrt{\frac{4P}{EI}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)},\\ w &= \sqrt{\frac{4P}{EI}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)},\end{aligned}$$

Since we know K, we can find w as a function of θ where $\theta \in [-\alpha, \quad \alpha]$. This means that know w along the entire length length of the beam, i.e. we know the deflection of the beam.

It is useful to also know the maximum value of the deflection. Clearly the maximum value of the deflection is where the slope vanishes, i.e. $\theta = 0$. Thus, we have

$$\begin{aligned} w_{\text{max}} &= \sqrt{\frac{4EI}{P}\sin^2\frac{\alpha}{2}},\\ &= \sqrt{\frac{4EI}{P}}K,\\ &= \sqrt{\frac{P_{\text{Euler}}}{P}}\frac{2L}{\pi}K. \end{aligned} \tag{24}$$

Thus, the maximum deflection normalized by the length of the beam can be written as

$$\frac{w_{\text{max}}}{L} = \frac{2}{\pi}\frac{K}{\sqrt{P/P_{\text{Euler}}}}. \tag{25}$$