Lecture Notes on
Föppl-von Karman Plate Theory

Jeevanjyoti Chakraborty
jeevan@mech.iitkgp.ac.in

We start from the following kinematical hypothesis:

$$u = u_s - z\frac{\partial w}{\partial x}, \tag{1}$$

$$v = v_s - z\frac{\partial w}{\partial y}, \tag{2}$$

$$w \equiv w(x, y). \tag{3}$$

Just like in the case of buckling of beams, we use the nonlinear strain-displacement relations. We also assume the following

$$\frac{\partial w}{\partial x} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y} \gg \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}. \tag{4}$$

$$
\begin{aligned}
E_{xx} &\equiv \frac{\partial u}{\partial x} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right\} \\
&\approx \frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad (5) \\
E_{yy} &\equiv \frac{\partial v}{\partial y} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\
&\approx \frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad (6) \\
E_{zz} &\equiv \frac{\partial w}{\partial z} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} \\
&= \frac{1}{2}\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}, \qquad (7) \\
E_{xy} &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \qquad (8) \\
E_{yz} &\equiv \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right) \\
&\approx \frac{1}{2}\left(-\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} + \left(\frac{\partial u_s}{\partial y} - z\frac{\partial^2 w}{\partial x \partial y}\right)\left(-\frac{\partial w}{\partial x}\right) + \left(\frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right)\left(-\frac{\partial w}{\partial y}\right) + 0\right) \\
&\approx 0, \qquad (9) \\
E_{zx} &\equiv \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial x}\right) \\
&\approx \frac{1}{2}\left(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} + \left(-\frac{\partial w}{\partial x}\right)\left(\frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \left(-\frac{\partial w}{\partial y}\right)\left(\frac{\partial v_s}{\partial x} - z\frac{\partial^2 w}{\partial x \partial y}\right) + 0\right) \\
&\approx 0. \qquad (10)
\end{aligned}
$$

We represent the terms E_{xx}, E_{yy}, and E_{xy} as

$$E_{xx} = E^0_{xx} - z\frac{\partial^2 w}{\partial x^2}, \qquad (11a)$$

$$E_{yy} = E^0_{yy} - z\frac{\partial^2 w}{\partial y^2}, \qquad (11b)$$

$$E_{xy} = E^0_{xy} - z\frac{\partial^2 w}{\partial x \partial y}, \qquad (11c)$$

where

$$E^0_{xx} = \frac{\partial u_s}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad (12a)$$

$$E^0_{yy} = \frac{\partial v_s}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \qquad (12b)$$

$$E^0_{xy} = \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \qquad (12c)$$

where, notably, each of E^0_{xx}, E^0_{yy}, and E^0_{xy} is independent of z.

Considering the virtual work equation:

$$\int_V \sigma_{ij}\delta E_{ij}\ \mathrm{d}V = \int_A t_i \delta u_i\ \mathrm{d}A,$$

and again forcibly assuming $\sigma_{zz} = 0$ (just as in the classical plate theory), we have from the left hand side

$$\begin{aligned}
\mathrm{LHS} &= \int_V \sigma_{ij}\delta E_{ij}\ \mathrm{d}V \\
&= \int_V \left[\sigma_{xx}\delta E_{xx} + \sigma_{yy}\delta E_{yy} + 2\sigma_{xy}\delta E_{xy}\right]\ \mathrm{d}V \\
&= \int_A \int_{-h/2}^{h/2} \sigma_{xx}\delta\left(E^0_{xx} - z\frac{\partial^2 w}{\partial x^2}\right) + \sigma_{yy}\delta\left(E^0_{yy} - z\frac{\partial^2 w}{\partial y^2}\right) + 2\sigma_{xy}\delta\left(E^0_{xy} - z\frac{\partial^2 w}{\partial x \partial y}\right)\ \mathrm{d}z\ \mathrm{d}A \\
&= \underbrace{\int_A \left[\left(N_x\delta E^0_{xx} + N_y\delta E^0_{yy} + 2N_{xy}\delta E^0_{xy}\right)\right]\ \mathrm{d}A}_{\mathrm{LHS}_1} + \underbrace{\int_A \left[-\left(M_x\frac{\partial^2 w}{\partial x^2} + M_y\frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)\right]\ \mathrm{d}A}_{\mathrm{LHS}_2}.
\end{aligned}$$

To write the last step, we have used the following definitions:

$$\begin{aligned}
\int_{-h/2}^{h/2} \sigma_{xx}\ \mathrm{d}z = N_x; &\quad \int_{-h/2}^{h/2} z\sigma_{xx}\ \mathrm{d}z = M_x; \\
\int_{-h/2}^{h/2} \sigma_{yy}\ \mathrm{d}z = N_y; &\quad \int_{-h/2}^{h/2} z\sigma_{yy}\ \mathrm{d}z = M_y; \\
\int_{-h/2}^{h/2} \sigma_{xy}\ \mathrm{d}z = N_{xy}; &\quad \int_{-h/2}^{h/2} z\sigma_{xy}\ \mathrm{d}z = M_{xy}.
\end{aligned}$$

We now consider the two integrals LHS_1 and LHS_2 separately.

$$
\begin{aligned}
\mathrm{LHS}_1 &= \int_A \left[N_x \delta E^0_{xx} + N_y E^0_{yy} + 2N_{xy} E^0_{xy} \right] \,\mathrm{d}A \\
&= \int_A \left[N_x \left(\frac{\partial \delta u_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} \right) + N_y \left(\frac{\partial \delta v_s}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial y} \right) \right. \\
&\qquad \left. + 2N_{xy} \frac{1}{2} \left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \delta w}{\partial x} \right) \right] \,\mathrm{d}A \\
&= \int_A \left[N_x \frac{\partial \delta u_s}{\partial x} + N_y \frac{\partial \delta v_s}{\partial y} + N_{xy} \frac{\partial \delta u_s}{\partial y} + N_{xy} \frac{\partial \delta v_s}{\partial x} \right. \\
&\qquad \left. + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \frac{\partial \delta w}{\partial y} \right] \,\mathrm{d}A \\
&= \int_A \left[\underbrace{\frac{\partial}{\partial x}(N_x \delta u_s)}_{①} - \frac{\partial N_x}{\partial x} \delta u_s + \underbrace{\frac{\partial}{\partial y}(N_y \delta v_s)}_{④} - \frac{\partial N_y}{\partial y} \delta v_s \right. \\
&\qquad + \underbrace{\frac{\partial}{\partial y}(N_{xy} \delta u_s)}_{⑤} - \frac{\partial N_{xy}}{\partial y} \delta u_s + \underbrace{\frac{\partial}{\partial x}(N_{xy} \delta v_s)}_{②} - \frac{\partial N_{xy}}{\partial x} \delta v_s \\
&\qquad + \underbrace{\frac{\partial}{\partial x} \left\{ \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right\}}_{③} - \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \\
&\qquad \left. + \underbrace{\frac{\partial}{\partial y} \left\{ \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\}}_{⑥} - \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right] \,\mathrm{d}A
\end{aligned}
$$

We consider the terms ①, ②, ③ together and the terms ④, ⑤, ⑥ together, use the Green's theorem alongwith the relations $n_x = \frac{\mathrm{d}y}{\mathrm{d}s}$ and $n_y = -\frac{\mathrm{d}x}{\mathrm{d}s}$. We also rearrange the remaining terms. Thus, we obtain

$$
\begin{aligned}
\mathrm{LHS}_1 = \oint & \left[\left\{ N_x \delta u_s + N_{xy} \delta v_s + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right\} \,\mathrm{d}y \right. \\
& \left. - \left\{ N_y \delta v_s + N_{xy} \delta u_s + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\} \,\mathrm{d}x \right] \\
& - \int_A \left[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_s + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v_s \right. \\
& \left. + \left\{ \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \right\} \delta w \right] \,\mathrm{d}A
\end{aligned}
$$

$$\begin{aligned}\text{LHS}_2 = -\oint \Bigg[& \left(M_x \frac{\partial \delta w}{\partial x} + M_{xy} \frac{\partial \delta w}{\partial y} - \frac{\partial M_x}{\partial x} \delta w - \frac{\partial M_{xy}}{\partial y} \delta w \right) \mathrm{d}y \\ & - \left(M_y \frac{\partial \delta w}{\partial y} + M_{xy} \frac{\partial \delta w}{\partial x} - \frac{\partial M_y}{\partial y} \delta w - \frac{\partial M_{xy}}{\partial x} \delta w \right) \mathrm{d}x \Bigg] \\ & - \int_A \left[\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right] \delta w \, \mathrm{d}A\end{aligned}$$

For the right hand side of the virtual work equation we have

$$\text{RHS} \equiv \int_A t_i \delta u_i \, \mathrm{d}A = \int_A q \delta w \, \mathrm{d}A.$$

Substituting the expressions of LHS_1, LHS_2, and the RHS into the virtual work equation, collecting the coefficients of δu_s, δv_s, and δw_s in the area integrals, and setting them individually to zero we have the following governing equations:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{13a}$$

$$\frac{\partial N_{xy}}{\partial y} + \frac{\partial N_y}{\partial y} = 0, \tag{13b}$$

$$-\frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) - \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = q \tag{13c}$$

Now, we expand (13c) and use (13a) and (13b) in it. We also use the the following

$$-\left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = D \nabla^4 w,$$

to obtain the following:

$$-\left(N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) + D \nabla^4 w - q = 0. \tag{14}$$

We set N_x, N_y, and N_{xy} in terms of a scalar function F as

$$N_x = \frac{\partial^2 F}{\partial x^2}, \tag{15a}$$

$$N_y = \frac{\partial^2 F}{\partial y^2}, \tag{15b}$$

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{15c}$$

and substitute in (14) to obtain

$$\boxed{D \nabla^4 w = q + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2}.} \tag{16}$$

This is the first of the Föppl-von Karman equations. In this equation, both w and F are unknown. So we need another equation to solve for w and F.

We know that the first of the compatibility equations is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \tag{17}$$

where ε_{xx}, ε_{yy}, and ε_{xy} are infinitesimal strain tensor components. These are related to E_{xx}, E_{yy}, and E_{xy} as

$$\begin{aligned}
E_{xx} &= \varepsilon_{xx} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\
E_{yy} &= \varepsilon_{yy} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\
E_{xy} &= \varepsilon_{xy} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.
\end{aligned}$$

Therefore, from (17), we have

$$\frac{\partial^2 E_{xx}}{\partial y^2} - \frac{\partial^2}{\partial y^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{\partial^2 E_{yy}}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} = 2\frac{\partial^2 E_{xy}}{\partial x \partial y} - 2\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right),$$

or, $$\frac{\partial^2 E_{xx}}{\partial y^2} + \frac{\partial^2 E_{yy}}{\partial x^2} - 2\frac{\partial^2 E_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + \frac{\partial^2}{\partial x^2}\left\{\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\} - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right).$$

Using (11), expanding the terms and simplifying, we obtain the following

$$\frac{\partial^2 E^0_{xx}}{\partial y^2} + \frac{\partial^2 E^0_{yy}}{\partial x^2} - 2\frac{\partial^2 E^0_{xy}}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}. \tag{18}$$

Now,

$$\begin{aligned}
N_x &= \int_{-h/2}^{h/2} \sigma_{xx}\,\mathrm{d}z, \\
&= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left(E_{xx} + \nu E_{yy}\right)\,\mathrm{d}z, \\
&= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left(E^0_{xx} - z\frac{\partial^2 w}{\partial x^2} + \nu E^0_{yy} - \nu z\frac{\partial^2 w}{\partial y^2}\right)\,\mathrm{d}z, \\
&= \frac{E}{1-\nu^2}\int_{-h/2}^{h/2}\left(E^0_{xx} + \nu E^0_{yy}\right)\,\mathrm{d}z, \\
&= \frac{Eh}{1-\nu^2}\left(E^0_{xx} + \nu E^0_{yy}\right), \\
&= C\left(E^0_{xx} + \nu E^0_{yy}\right).
\end{aligned}$$

Setting $C = \dfrac{Eh}{1-\nu^2}$, and expanding similarly for N_y and N_{xy}, we obtain

$$N_x = C\left(E^0_{xx} + \nu E^0_{yy}\right), \tag{19a}$$

$$N_y = C\left(E^0_{yy} + \nu E^0_{xx}\right), \tag{19b}$$

$$N_{xy} = C(1-\nu)E^0_{xy}. \tag{19c}$$

Inverting the relations (19a), (19b), and (19c), we have

$$E^0_{xx} = \frac{1}{Eh}\left(N_x - \nu N_y\right), \tag{20a}$$

$$E^0_{yy} = \frac{1}{Eh}\left(N_y - \nu N_x\right), \tag{20b}$$

$$E^0_{xy} = \frac{(1+\nu)}{Eh} N_{xy}. \tag{20c}$$

Substituting these expressions of E^0_{xx}, E^0_{yy}, and E^0_{xy} in (14), we have

$$\frac{1}{Eh}\left[\frac{\partial^2 N_x}{\partial y^2} - \nu\frac{\partial^2 Ny}{\partial y^2} + \frac{\partial^2 Ny}{\partial x^2} - \nu\frac{\partial^2 N_x}{\partial x^2} - 2(1+\nu)\frac{\partial^2 N_{xy}}{\partial x \partial y}\right] = \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}.$$

Finally, using the relations from Eqs (15a), (15b), and (15c), we have

$$\boxed{\nabla^4 F = Eh\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right].} \tag{21}$$

This equation is the second of the Föppl-von Karman equations.