Classical Plate Theory

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We use the following kinematical hypothesis:

$$
\begin{aligned}
u &= u_s - z\frac{\partial w}{\partial x}, \\
v &= v_s - z\frac{\partial w}{\partial y}, \\
w &\equiv w(x, y),
\end{aligned}
$$

using which we obtain from the strain-displacement relationships, the following:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_s}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \\
\varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v_s}{\partial y} - z\frac{\partial^2 w}{\partial y^2}, \\
\varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0, \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x}\right) - z\frac{\partial^2 w}{\partial x \partial y}, \\
\varepsilon_{yz} &= \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\left(-\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\right) = 0, \\
\varepsilon_{zx} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \frac{1}{2}\left(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x}\right) = 0.
\end{aligned}
$$

Since ε_{yz} and ε_{zx} are zero, from Hooke's law, it follows that σ_{yz} and σ_{zx} are also zero. Additionally, since $\varepsilon_{zz} = 0$, so $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. Now, σ_{xx} and σ_{yy} are not zero, therefore σ_{zz} is not zero. However, we forcibly assume plane stress conditions and take $\sigma_{zz} = 0$.

Using $\sigma_{zz} = 0$ in the following relations from Hooke's law:

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right],
\end{aligned}
$$

we have

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right), \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left(\varepsilon_{yy} + \nu\varepsilon_{xx}\right).
\end{aligned}
$$

Considering the virtual work equation:

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}\ \mathrm{d}V = \int_A t_i\delta u_i\ \mathrm{d}A,$$

we have from the left hand side

$$\begin{aligned}
\text{LHS} &= \int_V \sigma_{ij}\delta\varepsilon_{ij}\ \mathrm{d}V \\
&= \int_V \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy}\right)\ \mathrm{d}v \\
&= \int_A\int_{-h/2}^{h/2}\left[\sigma_{xx}\left(\frac{\partial\delta u_s}{\partial x} - z\frac{\partial^2\delta w}{\partial x^2}\right) + \sigma_{yy}\left(\frac{\partial\delta v_s}{\partial y} - z\frac{\partial^2\delta w}{\partial y^2}\right) + 2\sigma_{xy}\frac{1}{2}\left(\frac{\partial\delta u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z\frac{\partial^2\delta w}{\partial x\partial y}\right)\right]\ \mathrm{d}z\mathrm{d}A \\
&= \int_A\int_{-h/2}^{h/2}\left[\sigma_{xx}\frac{\partial\delta u_s}{\partial x} + \sigma_{yy}\frac{\partial\delta v_s}{\partial y} + \sigma_{xy}\left(\frac{\partial\delta u_s}{\partial y} + \frac{\partial\delta v_s}{\partial x}\right)\right]\ \mathrm{d}z\mathrm{d}A \\
&\quad - \int_A\int_{-h/2}^{h/2}\sigma_{xx}z\frac{\partial^2\delta w}{\partial x^2}\ \mathrm{d}z\mathrm{d}A - \int_A\int_{-h/2}^{h/2}\sigma_{yy}z\frac{\partial^2\delta w}{\partial y^2}\ \mathrm{d}z\mathrm{d}A - 2\int_A\int_{-h/2}^{h/2}\sigma_{xy}z\frac{\partial^2\delta w}{\partial x\partial y}\ \mathrm{d}z\mathrm{d}A
\end{aligned}$$

We note in the last step that stretching and bending are completely decoupled. Considering only bending and using the following definitions

$$M_x = \int_{-h/2}^{h/2}\sigma_{xx}z\ \mathrm{d}z, \quad M_y = \int_{-h/2}^{h/2}\sigma_{yy}z\ \mathrm{d}z, \quad M_{xy} = \int_{-h/2}^{h/2}\sigma_{xy}z\ \mathrm{d}z,$$

we obtain the following:

$$\begin{aligned}
\text{LHS}_{\text{bending}} &= -\int_A M_x\frac{\partial^2\delta w}{\partial x^2}\ \mathrm{d}A - \int_A M_y\frac{\partial^2\delta w}{\partial y^2}\ \mathrm{d}A - 2\int_A M_{xy}\frac{\partial^2\delta w}{\partial x\partial y}\ \mathrm{d}A \\
\text{or,}\quad -\text{LHS}_{\text{bending}} &= \int_A\left[\frac{\partial}{\partial x}\left(M_x\frac{\partial\delta w}{\partial x}\right) + \frac{\partial}{\partial y}\left(M_y\frac{\partial\delta w}{\partial y}\right)\right]\mathrm{d}A + \int_A\left[\frac{\partial}{\partial x}\left(M_{xy}\frac{\partial\delta w}{\partial y}\right) + \frac{\partial}{\partial y}\left(M_{xy}\frac{\partial\delta w}{\partial x}\right)\right]\mathrm{d}A \\
&\quad - \int_A\frac{\partial M_x}{\partial x}\frac{\partial\delta w}{\partial x}\mathrm{d}A - \int_A\frac{\partial M_y}{\partial y}\frac{\partial\delta w}{\partial y}\mathrm{d}A - \int_A\frac{\partial M_{xy}}{\partial x}\frac{\partial\delta w}{\partial y}\mathrm{d}A - \int_A\frac{\partial M_{xy}}{\partial y}\frac{\partial\delta w}{\partial x}\mathrm{d}A \\
&= \oint\left[\left(M_x\frac{\partial\delta w}{\partial x}\right)n_x + \left(M_y\frac{\partial\delta w}{\partial y}\right)n_y\right]\mathrm{d}s + \oint\left[\left(M_{xy}\frac{\partial\delta w}{\partial y}\right)n_x + \left(M_{xy}\frac{\partial\delta w}{\partial x}\right)n_y\right]\mathrm{d}s \\
&\quad - \int_A\left[\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial M_x}{\partial x}\delta w\right)}_{①} - \frac{\partial^2 M_x}{\partial x^2}\delta w\right]\mathrm{d}A - \int_A\left[\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial M_y}{\partial y}\delta w\right)}_{②} - \frac{\partial^2 M_y}{\partial y^2}\delta w\right]\mathrm{d}A \\
&\quad - \int_A\left[\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial M_{xy}}{\partial x}\delta w\right)}_{③} - \frac{\partial^2 M_{xy}}{\partial y\partial x}\delta w\right]\mathrm{d}A - \int_A\left[\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial M_{xy}}{\partial y}\delta w\right)}_{④} - \frac{\partial^2 M_{xy}}{\partial x\partial y}\delta w\right]\mathrm{d}A \qquad (1)
\end{aligned}$$

Considering the terms ① with ② and ③ with ④, and using Green's theorem, we obtain (after transposing the negative sign from the left hand side)

$$
\begin{aligned}
\text{LHS}_{\text{bending}} = &- \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} \right) n_x + \left(M_y \frac{\partial \delta w}{\partial y} \right) n_y \right] \mathrm{d}s - \oint \left[\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) n_x + \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) n_y \right] \mathrm{d}s \\
&+ \oint \left[\left(\frac{\partial M_x}{\partial x} \delta w \right) n_x + \left(\frac{\partial M_y}{\partial y} \delta w \right) n_y \right] \mathrm{d}s + \oint \left[\left(\frac{\partial M_{xy}}{\partial x} \delta w \right) n_y + \left(\frac{\partial M_{xy}}{\partial y} \delta w \right) n_x \right] \mathrm{d}s \\
&- \int_A \left[\left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) \delta w \right] \mathrm{d}A
\end{aligned}
\tag{2}
$$

We note that the terms on the right hand side can be classified as boundary integral terms (those within $\oint$) and domain integral terms (those within $\int_A$). The domain integral terms expressed in terms of M_x, M_y, and M_{xy} can be rewritten in terms of the displacement component w; thus

$$
\begin{aligned}
M_x &= \int_{-h/2}^{h/2} \sigma_{xx} z \, \mathrm{d}z \\
&= \int_{-h/2}^{h/2} \frac{E}{1 - \nu^2} \left(\varepsilon_{xx} + \nu \varepsilon_{yy} \right) z \, \mathrm{d}z \\
&= \frac{E}{1 - \nu^2} \int_{-h/2}^{h/2} \left(\frac{\partial u_s}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial v_s}{\partial y} - \nu z \frac{\partial^2 w}{\partial y^2} \right) z \, \mathrm{d}z \\
&= - \frac{E}{1 - \nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \int_{-h/2}^{h/2} z^2 \, \mathrm{d}z \\
&= - \frac{Eh^3}{12(1 - \nu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)
\end{aligned}
$$

Setting $D = \dfrac{Eh^3}{12(1 - \nu^2)}$ (it is referred to as the bending rigidity), and proceeding similarly as above for M_y and M_{xy}, we have

$$
M_x = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \tag{3a}
$$

$$
M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \tag{3b}
$$

$$
M_{xy} = -D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \tag{3c}
$$

Using these expressions of M_x, M_y, and M_{xy}, we have

$$
\begin{aligned}
&\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \\
&= - D \left(\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial y^2 \partial x^2} + 2(1 - \nu) \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\
&= -D \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\
&= -D \nabla^4 w
\end{aligned}
$$

Therefore, we have the following

$$
\begin{aligned}
\text{LHS}_{\text{bending}} = &-\oint\left[\left(M_x\frac{\partial\delta w}{\partial x}\right)n_x+\left(M_y\frac{\partial\delta w}{\partial y}\right)n_y\right]\mathrm{d}s-\oint\left[\left(M_{xy}\frac{\partial\delta w}{\partial y}\right)n_x+\left(M_{xy}\frac{\partial\delta w}{\partial x}\right)n_y\right]\mathrm{d}s\\
&+\oint\left[\left(\frac{\partial M_x}{\partial x}\delta w\right)n_x+\left(\frac{\partial M_y}{\partial y}\delta w\right)n_y\right]\mathrm{d}s+\oint\left[\left(\frac{\partial M_{xy}}{\partial y}\delta w\right)n_x+\left(\frac{\partial M_{xy}}{\partial x}\delta w\right)n_y\right]\mathrm{d}s\\
&+\int_A D\nabla^4 w\,\mathrm{d}A
\end{aligned}
\tag{4}
$$

Now, going back to the right hand side of the virtual work equation and considering the contribution due only to bending we have

$$
\text{RHS}_{\text{bending}} = \int_A t_i\delta u_i\ \mathrm{d}A = \int_A q\delta w\ \mathrm{d}A.
$$

Bringing together the left hand and right hand sides, we thus have

$$
\text{LHS}_{\text{bending}} = \text{RHS}_{\text{bending}},
$$

from which we obtain the following:

$$
\begin{aligned}
&\int_A\left(D\nabla^4 w-q\right)\delta w\mathrm{d}A-\oint\left[M_x\frac{\partial\delta w}{\partial x}n_x+M_y\frac{\partial\delta w}{\partial y}n_y\right]\mathrm{d}s-\oint\left[M_{xy}\frac{\partial\delta w}{\partial y}n_x+M_{xy}\frac{\partial\delta w}{\partial x}n_y\right]\mathrm{d}s\\
&\quad+\oint\left[\underbrace{\frac{\partial M_x}{\partial x}\delta w n_x}_{①}+\underbrace{\frac{\partial M_y}{\partial y}\delta w n_y}_{③}\right]\mathrm{d}s+\oint\left[\underbrace{\frac{\partial M_{xy}}{\partial y}\delta w n_x}_{②}+\underbrace{\frac{\partial M_{xy}}{\partial x}\delta w n_y}_{④}\right]\mathrm{d}s=0
\end{aligned}
\tag{5}
$$

Now, consider the x-component of the mechanical equilibrium equations, and integrate as follows:

$$
\begin{aligned}
&\int_{-h/2}^{h/2}\left[\frac{\partial\sigma_{xx}}{\partial x}z+\frac{\partial\sigma_{xy}}{\partial y}z+\frac{\partial\sigma_{xz}}{\partial z}z\right]\ \mathrm{d}z=0\\
\text{or,}\quad&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}+\left[z\sigma_{xz}\right]_{-h/2}^{h/2}-\int_{-h/2}^{h/2}\sigma_{xz}\ \mathrm{d}z=0\\
\text{or,}\quad&\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}=Q_x\qquad\left(\text{using }\sigma_{xz}=0\text{ and }Q_x=\int_{-h/2}^{h/2}\sigma_{xz}\ \mathrm{d}z\right)
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}=Q_y,
$$

where $Q_y=\int_{-h/2}^{h/2}\sigma_{yz}\ \mathrm{d}z$.

Going back to Eq. (5) and considering the terms ① together with ②, and ③ together with ④, we obtain

$$
\begin{aligned}
&\int_A\left(D\nabla^4 w-q\right)\delta w\mathrm{d}A-\oint\left[M_x\frac{\partial\delta w}{\partial x}n_x+M_y\frac{\partial\delta w}{\partial y}n_y\right]\mathrm{d}s-\oint\left[M_{xy}\frac{\partial\delta w}{\partial y}n_x+M_{xy}\frac{\partial\delta w}{\partial x}n_y\right]\mathrm{d}s\\
&\quad+\oint Q_x\delta w n_x\mathrm{d}s+\oint Q_y\delta w n_y\mathrm{d}s=0.
\end{aligned}
\tag{6}
$$

Now we want to convert the preceding equation from the (x, y) coordinate system to the (s, n) coordinate system where s is the coordinate along the periphery of the plate and n is the coordinate perpendicular to it. Towards that end, we first establish the relationship between $\frac{\partial}{\partial s}, \frac{\partial}{\partial n}$ and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

For any point on the periphery given by $\boldsymbol{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, the unit normal is $\hat{\mathbf{e}}_n = n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}} \equiv \frac{\mathrm{d}y}{\mathrm{d}s}\hat{\mathbf{i}} - \frac{\mathrm{d}x}{\mathrm{d}s}\hat{\mathbf{j}}$. Then for an elemental line segment at $\boldsymbol{r}$ we have,

$$\begin{aligned} \mathrm{d}s\,\hat{\mathbf{e}}_s &= \mathrm{d}x\hat{\mathbf{i}} + \mathrm{d}y\hat{\mathbf{j}}, \\ \text{or,}\quad \hat{\mathbf{e}}_s &= \frac{\mathrm{d}x}{\mathrm{d}s}\hat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}s}\hat{\mathbf{j}} \equiv -n_y\hat{\mathbf{i}} + n_x\hat{\mathbf{j}}. \end{aligned}$$

Consider the gradient of any arbitrary scalar ϕ first in the (x, y) coordinate system and next in the (s, n) coordinate system. Thus we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}},$$

and

$$\begin{aligned} \nabla\phi &= \frac{\partial\phi}{\partial s}\hat{\mathbf{e}}_s + \frac{\partial\phi}{\partial n}\hat{\mathbf{e}}_n, \\ &= \frac{\partial\phi}{\partial s}\left(-n_y\hat{\mathbf{i}} + n_x\hat{\mathbf{j}}\right) + \frac{\partial\phi}{\partial n}\left(n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}\right), \\ &= \left(-\frac{\partial\phi}{\partial s}n_y + \frac{\partial\phi}{\partial n}n_x\right)\hat{\mathbf{i}} + \left(\frac{\partial\phi}{\partial s}n_x + \frac{\partial\phi}{\partial n}n_y\right)\hat{\mathbf{j}} \end{aligned}$$

Comparing the two expressions of $\nabla\phi$, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} &= -n_y\frac{\partial}{\partial s} + n_x\frac{\partial}{\partial n}, \\ \frac{\partial}{\partial y} &= n_x\frac{\partial}{\partial s} + n_y\frac{\partial}{\partial n}. \end{aligned}$$

Then from Eq. (6), we obtain

$$\begin{aligned} &\int_A \left(D\nabla^4 w - q\right)\delta w\mathrm{d}A - \oint\left[M_x\left(-n_y\frac{\partial\delta w}{\partial s} + n_x\frac{\partial\delta w}{\partial n}\right)\mathrm{d}y - M_y\left(n_x\frac{\partial\delta w}{\partial s} + n_y\frac{\partial\delta w}{\partial n}\right)\mathrm{d}x\right] \\ &\qquad - \oint\left[M_{xy}\left(n_x\frac{\partial\delta w}{\partial s} + n_y\frac{\partial\delta w}{\partial n}\right)\mathrm{d}y - M_{xy}\left(-n_y\frac{\partial\delta w}{\partial s} + n_x\frac{\partial\delta w}{\partial n}\right)\mathrm{d}x\right] \\ &\qquad + \oint Q_x\delta w\mathrm{d}y - \oint Q_y\delta w\mathrm{d}x = 0, \\ \text{or,}\quad &\int_A \left(D\nabla^4 w - q\right)\delta w\mathrm{d}A - \oint M_x\left(-n_y\frac{\partial\delta w}{\partial s} + n_x\frac{\partial\delta w}{\partial n}\right)n_x\mathrm{d}s + M_y\left(n_x\frac{\partial\delta w}{\partial s} + n_y\frac{\partial\delta w}{\partial n}\right)(-n_y)\mathrm{d}s \\ &\qquad - \oint M_{xy}\left(n_x\frac{\partial\delta w}{\partial s} + n_y\frac{\partial\delta w}{\partial n}\right)n_x\mathrm{d}s + M_{xy}\left(-n_y\frac{\partial\delta w}{\partial s} + n_x\frac{\partial\delta w}{\partial n}\right)(-n_y)\mathrm{d}s \\ &\qquad + \oint Q_x\delta w n_x\mathrm{d}s - \oint Q_y\delta w(-n_y)\mathrm{d}s = 0, \\ \text{or,}\quad &\int_A \left(D\nabla^4 w - q\right)\delta w\mathrm{d}A + \oint\left(M_x n_y n_x - M_y n_x n_y - M_{xy}n_x^2 + M_{xy}n_y^2\right)\frac{\partial\delta w}{\partial s}\mathrm{d}s \\ &\qquad + \oint\left(-M_x n_x^2 - M_y n_y^2 - 2M_{xy}n_x n_y\right)\frac{\partial\delta w}{\partial n}\mathrm{d}s \\ &\qquad + \oint\left(Q_x n_x + Q_y n_y\right)\delta w\mathrm{d}s = 0. \end{aligned} \tag{7}$$

From stress-transformation we have

$$\sigma_{nn} = n_x^2\sigma_{xx} + 2n_x n_y\sigma_{xy} + n_y^2\sigma_{yy}, \tag{8a}$$

$$\sigma_{ns} = n_x n_y\left(\sigma_{yy} - \sigma_{xx}\right) + \left(n_x^2 - n_y^2\right)\sigma_{xy}, \tag{8b}$$

from which we obtain

$$M_n = n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y, \tag{9a}$$

$$M_{ns} = n_x n_y\left(M_y - M_x\right) + \left(n_x^2 - n_y^2\right) M_{xy}. \tag{9b}$$

Using Eqns (9a) and (9b) in Eq. (7), we obtain

$$\int_A \left(D\nabla^4 w - q\right)\delta w \mathrm{d}A - \oint M_{ns}\frac{\partial \delta w}{\partial s}\mathrm{d}s - \oint M_n \frac{\partial \delta w}{\partial n}\mathrm{d}s + \oint Q_n \delta w \mathrm{d}s = 0. \tag{10}$$

Now, $\oint M_{ns}\frac{\partial \delta w}{\partial s}\mathrm{d}s = [M_{ns}\delta w]_1^2 - \oint \frac{\partial M_{ns}}{\partial s}\delta w \mathrm{d}s$. For a closed contour, $[M_{ns}\delta w]_1^2 = 0$; therefore

$$\int_A \left(D\nabla^4 w - q\right)\delta w \mathrm{d}A + \oint \frac{\partial M_{ns}}{\partial s}\delta w \mathrm{d}s - \oint M_n \frac{\partial \delta w}{\partial n}\mathrm{d}s + \oint Q_n \delta w \mathrm{d}s = 0,$$

$$\text{or,}\quad \int_A \left(D\nabla^4 w - q\right)\delta w \mathrm{d}A + \oint \left(\frac{\partial M_{ns}}{\partial s} + Q_n\right)\delta w \mathrm{d}s - \oint M_n \frac{\partial \delta w}{\partial n}\mathrm{d}s = 0. \tag{11}$$

So, the governing equation is

$$D\nabla^4 w = q, \tag{12}$$

and the boundary conditions are given by

$$\text{Either}\quad \frac{\partial M_{ns}}{\partial s} + Q_n = 0 \quad \text{or,} \quad w \quad \text{is specified,} \tag{13a}$$

$$\text{Either}\quad M_n = 0 \quad \text{or,} \quad \frac{\partial w}{\partial n} \quad \text{is specified.} \tag{13b}$$