Problem Sheet 1: Energy Methods

For first Asssignment submission: Problem numbers 4, 6, and 8.
See submission instructions at the end.

1. During lecture, it was shown that if a body occupying a volume V and subjected to body forces (per unit volume) ρb_i (where ρ is the mass density) and tractions $t_i = \sigma_{ji} n_j$ specified on a part of the surface S_1 together with displacements u_i specified on the remaining part of the surface S_2, is in static equilibrium, then the virtual work associated with the external forces is given by $\delta W_e = \int_V \sigma_{ij} \delta\varepsilon_{ij} \, \mathrm{d}V$. This statement is the necessity condition for equilibrium. Show that the converse also holds; thus: If we have $\delta W_e = \int_V \sigma_{ij} \delta\varepsilon_{ij} \, \mathrm{d}V$, then the body will be in equilibrium, i.e. $\rho b_i + \dfrac{\partial \sigma_{ij}}{\partial x_j} = 0$ *and* $t_i = \sigma_{ji} n_j$.

   Hint: See §3.2 in Dym and Shames.

2. A cantilever beam is loaded as shown in Figure 1 by a point load P. Determine the vertical deflection at the point of application of the load. Also, determine the vertical deflection at the cantilever tip.

$$\left[d_P = \frac{P(L-b)^3}{3EI},\ d_{\mathrm{tip}} = \frac{P(L^3-b^3)}{3EI} - \frac{Pb(L^2-b^2)}{2EI} \right]$$

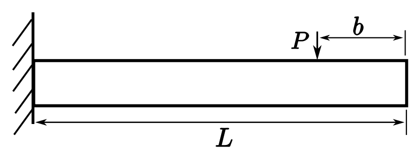

Figure 1

3. A uniformly loaded beam is fixed at both ends as shown in Figure 2. Without taking advantage of symmetry and using appropriate energy methods, determine the reaction components on the left side of the beam due to the uniformly distributed load q_0 [N/m].

$$\left[R = \frac{q_0 L}{2},\ M = \frac{q_0 L^2}{12} \right]$$

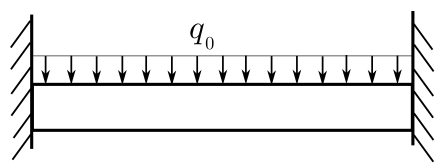

Figure 2

4. For the tapered cantilever beam made of Hookean material shown in Figure 3, the second moment of area is given by $I = (ax+b)^{-1}$, where a and b are constants. Using Castigliano's theorem, find the deflection of the free end. $\left[\dfrac{PaL^4}{12E} + \dfrac{PbL^3}{3E} \right]$

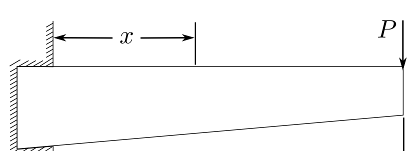

Figure 3

5. The uniform semi-circular beam made of Hookean material is loaded by a vertical force P as shown in Figure 4. Using Castigliano's theorem, determine the horizontal displacement at the free end.

$$\left[\frac{Pa^3(\pi-1)}{2EI}\right]$$

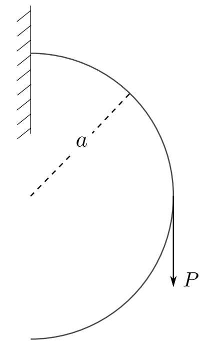

Figure 4

6. Consider the beam shown in Figure 5 where q_0 [N/m] represents a uniformly distributed load. The beam material behaves according to the stress-strain law $\sigma_{xx} = k\varepsilon_{xx}^{1/3}$. The beam has a rectangular cross-section of width b and height h. Find the algebraic equation that needs to be solved in order to find the reaction force at the left end. How does this equation change with the values of k, b, and h?

[The algebraic equation resulting from $\int_0^L \left(R - \frac{q_0 x}{2}\right)^3 x^4 \, \mathrm{d}x = 0$. Equation does not change.]

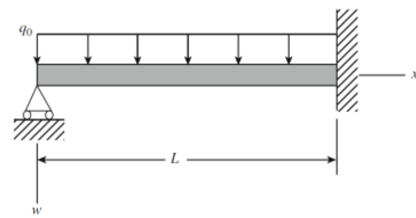

Figure 5

7. For the structure shown in Figure 6, the material has Young's modulus E. The second moment of area I is the same throughout. The structure is loaded by a horizontal force Q at C and by a uniform load w [N/m] distributed over AB. Determine the horizontal and vertical displacement components of point C in terms of Q, w, L_1, L_2, E, and I. Note that the cross-sectional dimensionals are much smaller than L_1 and L_2.

$$\left[d_{\text{hor}} = \frac{QL_2^3}{3EI} + \frac{QL_1L_2^2}{EI} + \frac{wL_1^3L_2}{6EI},\ d_{\text{vert}} = \frac{QL_1^2L_2}{2EI} + \frac{wL_1^4}{8EI}\ \text{(down)}\right]$$

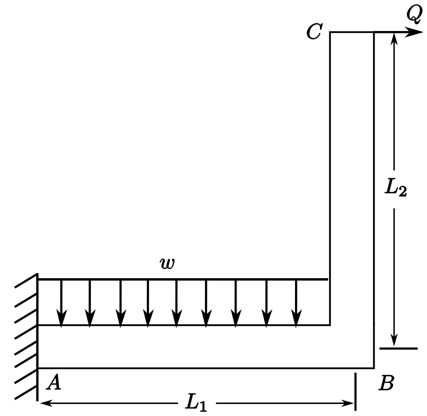

Figure 6

8. Consider a deformable body in two different equilibrium states (1) and (2). State (1) is subjected to body forces $b_i^{(1)}$ per unit volume and surface tractions $t_i^{(1)}$ leading to the displacement field $u_i^{(1)}$. State (2) is subjected to subjected to body forces $b_i^{(2)}$ per unit volume and surface tractions $t_i^{(2)}$ leading to the displacement field $u_i^{(2)}$.

   (a) Show that the work that would be done if $b_i^{(1)}$ and $t_i^{(1)}$ acted over the displacement $u_i^{(2)}$ is

   $$\int_V b_i^{(1)} u_i^{(2)} \, \mathrm{d}V + \int_S t_i^{(1)} u_i^{(2)} \, \mathrm{d}S = \int_V \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_j} \, \mathrm{d}V, \tag{1}$$

   where $\sigma_{ij}^{(1)}$ is the stress field in the first equilibrium state. *Hint:* Use divergence theorem and the equation of equilibrium.

   (b) Use the constitutive equation: $\sigma_{ij}^{(1)} = \lambda \varepsilon_{kk}^{(1)} + 2G\varepsilon_{ij}^{(1)}$ (where $\varepsilon_{ij}^{(1)}$ is the strain field in the first equilibrium state; λ and G are the Lamé parameters) in the right hand side of Eq. (1) to express it in terms of strains and displacement variables. Next, proceeding in a similar way for the work done if $b_i^{(2)}$ and $t_i^{(2)}$ acted over the displacement $u_i^{(1)}$, show that

   $$\int_V b_i^{(1)} u_i^{(2)} \, \mathrm{d}V + \int_S t_i^{(1)} u_i^{(2)} \, \mathrm{d}S = \int_V b_i^{(2)} u_i^{(1)} \, \mathrm{d}V + \int_S t_i^{(2)} u_i^{(1)} \, \mathrm{d}S.$$

Very Important Instructions:

- Assignment submissions must be done *strictly* by **the student himself/herself**. Assignments sent through someone else will be given zero marks. However, if, unfortunately, a student suffers due to some medical reason or family emergency, then (s)he should submit through a friend. In such a case, the student should inform me via email beforehand.
- The deadline for Assignment submission with full credit is February 10 (Monday) at 11:20 AM in class. Two other deadlines with corresponding penalty are as follows:
  - Assignments submitted by end of February 10 class (i.e. 12 noon) will carry a 20% penalty.
  - Assignments submitted by end of February 11 class (i.e. 10 AM next day) will carry a 70% penalty. No submission will be considered after this deadline.
- Discussions and brainstorming is always allowed while solving the problems. However, the students must write down the solutions individually and without discussion. Copying can be easily detected and will be severely punished. **Cases of copying will be summarily given zero marks.**
- The assignment submission must be made in a neat and clean fashion. The solutions must follow the sequence of the question numbers.
- The solution steps must be clearly explained. Just writing some formulae and presenting an answer after a messy derivation/working out will be severely penalised.
- *It is* **not** *the responsibility of the grader to extract meaning out of the student's work. It is the responsibility of the student to present everything clearly.* Clear communication of one's ideas is an essential part of training to be an engineer.