

LECTURE NOTES ON FÖPPL-VON KARMAN PLATE THEORY

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We start from the following kinematical hypothesis:

$$u = u_s - z \frac{\partial w}{\partial x}, \quad (1)$$

$$v = v_s - z \frac{\partial w}{\partial y}, \quad (2)$$

$$w \equiv w(x, y). \quad (3)$$

Just like in the case of buckling of beams, we use the nonlinear strain-displacement relations. We also assume the following

$$\frac{\partial w}{\partial x} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial w}{\partial y} \gg \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}. \quad (4)$$

$$\begin{aligned}
E_{xx} &\equiv \frac{\partial u}{\partial x} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right\} \\
&\approx \frac{\partial u_s}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2,
\end{aligned} \tag{5}$$

$$\begin{aligned}
E_{yy} &\equiv \frac{\partial v}{\partial y} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right\} \\
&\approx \frac{\partial v_s}{\partial y} - z \frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2,
\end{aligned} \tag{6}$$

$$\begin{aligned}
E_{zz} &\equiv \frac{\partial w}{\partial z} + \frac{1}{2} \left\{ \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right\} \\
&= \frac{1}{2} \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right\},
\end{aligned} \tag{7}$$

$$\begin{aligned}
E_{xy} &\equiv \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \\
&= \frac{1}{2} \left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right),
\end{aligned} \tag{8}$$

$$\begin{aligned}
E_{yz} &\equiv \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \right) \\
&\approx \frac{1}{2} \left(-\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} + \left(\frac{\partial u_s}{\partial y} - z \frac{\partial^2 w}{\partial x \partial y} \right) \left(-\frac{\partial w}{\partial x} \right) + \left(\frac{\partial v_s}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \right) \left(-\frac{\partial w}{\partial y} \right) + 0 \right) \\
&\approx 0,
\end{aligned} \tag{9}$$

$$\begin{aligned}
E_{zx} &\equiv \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z} \frac{\partial w}{\partial x} \right) \\
&\approx \frac{1}{2} \left(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} + \left(-\frac{\partial w}{\partial x} \right) \left(\frac{\partial u_s}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \right) + \left(-\frac{\partial w}{\partial y} \right) \left(\frac{\partial v_s}{\partial x} - z \frac{\partial^2 w}{\partial x \partial y} \right) + 0 \right) \\
&\approx 0.
\end{aligned} \tag{10}$$

We represent the terms E_{xx} , E_{yy} , and E_{xy} as

$$E_{xx} = E_{xx}^0 - z \frac{\partial^2 w}{\partial x^2}, \tag{11a}$$

$$E_{yy} = E_{yy}^0 - z \frac{\partial^2 w}{\partial y^2}, \tag{11b}$$

$$E_{xy} = E_{xy}^0 - z \frac{\partial^2 w}{\partial x \partial y}, \tag{11c}$$

where

$$E_{xx}^0 = \frac{\partial u_s}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \tag{12a}$$

$$E_{yy}^0 = \frac{\partial v_s}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \tag{12b}$$

$$E_{xy}^0 = \frac{1}{2} \left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right), \tag{12c}$$

where, notably, each of E_{xx}^0 , E_{yy}^0 , and E_{xy}^0 is independent of z .

Considering the virtual work equation:

$$\int_V \sigma_{ij} \delta E_{ij} \, dV = \int_A t_i \delta u_i \, dA,$$

and again forcibly assuming $\sigma_{zz} = 0$ (just as in the classical plate theory), we have from the left hand side

$$\begin{aligned} \text{LHS} &= \int_V \sigma_{ij} \delta E_{ij} \, dV \\ &= \int_V [\sigma_{xx} \delta E_{xx} + \sigma_{yy} \delta E_{yy} + 2\sigma_{xy} \delta E_{xy}] \, dV \\ &= \int_A \int_{-h/2}^{h/2} \sigma_{xx} \left(E_{xx}^0 - z \frac{\partial^2 w}{\partial x^2} \right) + \sigma_{yy} \left(E_{yy}^0 - z \frac{\partial^2 w}{\partial y^2} \right) + 2\sigma_{xy} \left(E_{xy}^0 - z \frac{\partial^2 w}{\partial x \partial y} \right) \, dz \, dA \\ &= \underbrace{\int_A [(N_x \delta E_{xx}^0 + N_y \delta E_{yy}^0 + 2N_{xy} \delta E_{xy}^0)] \, dA}_{\text{LHS}_1} + \underbrace{\int_A \left[- \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \right] \, dA}_{\text{LHS}_2}. \end{aligned}$$

To write the last step, we have used the following definitions:

$$\begin{aligned} \int_{-h/2}^{h/2} \sigma_{xx} \, dz &= N_x; & \int_{-h/2}^{h/2} z \sigma_{xx} \, dz &= M_x; \\ \int_{-h/2}^{h/2} \sigma_{yy} \, dz &= N_y; & \int_{-h/2}^{h/2} z \sigma_{yy} \, dz &= M_y; \\ \int_{-h/2}^{h/2} \sigma_{xy} \, dz &= N_{xy}; & \int_{-h/2}^{h/2} z \sigma_{xy} \, dz &= M_{xy}. \end{aligned}$$

We now consider the two integrals LHS_1 and LHS_2 separately.

$$\begin{aligned}
\text{LHS}_1 &= \int_A [N_x \delta E_{xx}^0 + N_y \delta E_{yy}^0 + 2N_{xy} \delta E_{xy}^0] \, dA \\
&= \int_A \left[N_x \left(\frac{\partial \delta u_s}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right) + N_y \left(\frac{\partial \delta v_s}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \right) \right. \\
&\quad \left. + 2N_{xy} \frac{1}{2} \left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial x} \right) \right] \, dA \\
&= \int_A \left[N_x \frac{\partial \delta u_s}{\partial x} + N_y \frac{\partial \delta v_s}{\partial y} + N_{xy} \frac{\partial \delta u_s}{\partial y} + N_{xy} \frac{\partial \delta v_s}{\partial x} \right. \\
&\quad \left. + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \frac{\partial \delta w}{\partial y} \right] \, dA \\
&= \int_A \left[\underbrace{\frac{\partial}{\partial x} (N_x \delta u_s)}_{\textcircled{1}} - \frac{\partial N_x}{\partial x} \delta u_s + \underbrace{\frac{\partial}{\partial y} (N_y \delta v_s)}_{\textcircled{4}} - \frac{\partial N_y}{\partial y} \delta v_s \right. \\
&\quad \left. + \underbrace{\frac{\partial}{\partial y} (N_{xy} \delta u_s)}_{\textcircled{5}} - \frac{\partial N_{xy}}{\partial y} \delta u_s + \underbrace{\frac{\partial}{\partial x} (N_{xy} \delta v_s)}_{\textcircled{2}} - \frac{\partial N_{xy}}{\partial x} \delta v_s \right. \\
&\quad \left. + \underbrace{\frac{\partial}{\partial x} \left\{ \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right\}}_{\textcircled{3}} - \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right. \\
&\quad \left. + \underbrace{\frac{\partial}{\partial y} \left\{ \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\}}_{\textcircled{6}} - \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right] \, dA
\end{aligned}$$

We consider the terms $\textcircled{1}$, $\textcircled{2}$, $\textcircled{3}$ together and the terms $\textcircled{4}$, $\textcircled{5}$, $\textcircled{6}$ together, use the Green's theorem alongwith the relations $n_x = \frac{dy}{ds}$ and $n_y = -\frac{dx}{ds}$. We also rearrange the remaining terms. Thus, we obtain

$$\begin{aligned}
\text{LHS}_1 &= \oint \left[\left\{ N_x \delta u_s + N_{xy} \delta v_s + \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) \delta w \right\} \, dy \right. \\
&\quad \left. - \left\{ N_y \delta v_s + N_{xy} \delta u_s + \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \delta w \right\} \, dx \right] \\
&\quad - \int_A \left[\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u_s + \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v_s \right. \\
&\quad \left. + \left\{ \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) \right\} \delta w \right] \, dA
\end{aligned}$$

$$\begin{aligned} \text{LHS}_2 = & - \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} + M_{xy} \frac{\partial \delta w}{\partial y} - \frac{\partial M_x}{\partial x} \delta w - \frac{\partial M_{xy}}{\partial y} \delta w \right) dy \right. \\ & \left. - \left(M_y \frac{\partial \delta w}{\partial y} + M_{xy} \frac{\partial \delta w}{\partial x} - \frac{\partial M_y}{\partial y} \delta w - \frac{\partial M_{xy}}{\partial x} \delta w \right) dx \right] \\ & - \int_A \left[\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right] \delta w \, dA \end{aligned}$$

For the right hand side of the virtual work equation we have

$$\text{RHS} \equiv \int_A t_i \delta u_i \, dA = \int_A q \delta w \, dA.$$

Substituting the expressions of LHS₁, LHS₂, and the RHS into the virtual work equation, collecting the coefficients of δu_s , δv_s , and δw_s in the area integrals, and setting them individually to zero we have the following governing equations:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \quad (13a)$$

$$\frac{\partial N_{xy}}{\partial y} + \frac{\partial N_y}{\partial x} = 0, \quad (13b)$$

$$- \frac{\partial}{\partial x} \left(N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) - \frac{\partial}{\partial y} \left(N_y \frac{\partial w}{\partial y} + N_{xy} \frac{\partial w}{\partial x} \right) - \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = q \quad (13c)$$

Now, we expand (13c) and use (13a) and (13b) in it. We also use the the following

$$- \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) = D \nabla^4 w,$$

to obtain the following:

$$- \left(N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} \right) + D \nabla^4 w - q = 0. \quad (14)$$

We set N_x , N_y , and N_{xy} in terms of a scalar function F as

$$N_x = \frac{\partial^2 F}{\partial x^2}, \quad (15a)$$

$$N_y = \frac{\partial^2 F}{\partial y^2}, \quad (15b)$$

$$N_{xy} = - \frac{\partial^2 F}{\partial x \partial y} \quad (15c)$$

and substitute in (14) to obtain

$$\boxed{D \nabla^4 w = q + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2}.} \quad (16)$$

This is the first of the Föppl-von Karman equations. In this equation, both w and F are unknown. So we need another equation to solve for w and F .

We know that the first of the compatibility equations is

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}, \quad (17)$$

where ε_{xx} , ε_{yy} , and ε_{xy} are infinitesimal strain tensor components. These are related to E_{xx} , E_{yy} , and E_{xy} as

$$\begin{aligned} E_{xx} &= \varepsilon_{xx} - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2, \\ E_{yy} &= \varepsilon_{yy} - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2, \\ E_{xy} &= \varepsilon_{xy} - \frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}. \end{aligned}$$

Therefore, from (17), we have

$$\begin{aligned} &\frac{\partial^2 E_{xx}}{\partial y^2} - \frac{\partial^2}{\partial y^2} \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \frac{\partial^2 E_{yy}}{\partial x^2} - \frac{\partial^2}{\partial x^2} \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} = 2 \frac{\partial^2 E_{xy}}{\partial x \partial y} - 2 \frac{\partial^2}{\partial x \partial y} \left(\frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right), \\ \text{or, } &\frac{\partial^2 E_{xx}}{\partial y^2} + \frac{\partial^2 E_{yy}}{\partial x^2} - 2 \frac{\partial^2 E_{xy}}{\partial x \partial y} = \frac{\partial^2}{\partial y^2} \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} + \frac{\partial^2}{\partial x^2} \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} - \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right). \end{aligned}$$

Using (11), expanding the terms and simplifying, we obtain the following

$$\frac{\partial^2 E_{xx}^0}{\partial y^2} + \frac{\partial^2 E_{yy}^0}{\partial x^2} - 2 \frac{\partial^2 E_{xy}^0}{\partial x \partial y} = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}. \quad (18)$$

Now,

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_{xx} \, dz, \\ &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (E_{xx} + \nu E_{yy}) \, dz, \\ &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(E_{xx}^0 - z \frac{\partial^2 w}{\partial x^2} + \nu E_{yy}^0 - \nu z \frac{\partial^2 w}{\partial y^2} \right) dz, \\ &= \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} (E_{xx}^0 + \nu E_{yy}^0) \, dz, \\ &= \frac{Eh}{1-\nu^2} (E_{xx}^0 + \nu E_{yy}^0), \\ &= C (E_{xx}^0 + \nu E_{yy}^0). \end{aligned}$$

Setting $C = \frac{Eh}{1-\nu^2}$, and expanding similarly for N_y and N_{xy} , we obtain

$$N_x = C (E_{xx}^0 + \nu E_{yy}^0), \quad (19a)$$

$$N_y = C (E_{yy}^0 + \nu E_{xx}^0), \quad (19b)$$

$$N_{xy} = C(1-\nu)E_{xy}^0. \quad (19c)$$

Inverting the relations (19a), (19b), and (19c), we have

$$E_{xx}^0 = \frac{1}{Eh} (N_x - \nu N_y), \quad (20a)$$

$$E_{yy}^0 = \frac{1}{Eh} (N_y - \nu N_x), \quad (20b)$$

$$E_{xy}^0 = \frac{(1 + \nu)}{Eh} N_{xy}. \quad (20c)$$

Substituting these expressions of E_{xx}^0 , E_{yy}^0 , and E_{xy}^0 in (14), we have

$$\frac{1}{Eh} \left[\frac{\partial^2 N_x}{\partial y^2} - \nu \frac{\partial^2 N_y}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} - \nu \frac{\partial^2 N_x}{\partial x^2} - 2(1 + \nu) \frac{\partial^2 N_{xy}}{\partial x \partial y} \right] = \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}.$$

Finally, using the relations from Eqs (15a), (15b), and (15c), we have

$$\boxed{\nabla^4 F = Eh \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right]}. \quad (21)$$

This equation is the second of the Föppl-von Karman equations.