Analysis of Stress

Jeevanjyoti Chakraborty
jeevan@mech.iitkgp.ac.in

1 Traction across a plane at a point

Consider an area A in a given plane and containing a point P within a body as shown in Fig. 1. Suppose the plane divides the body into two regions, Region I and Region II. Consider Region I. Draw a normal $\hat{\mathbf{n}}$ to the plane at P and pointing from Region I towards Region II. Over the area A, Region II exerts forces on Region I. Suppose this system of forces is statically equivalent to a force $\mathbf{F}$ acting at P in a definite direction and a couple $\mathbf{C}$ about a definite axis. Let us make the area A small, ensuring that the point P is always inside it. Then the force $\mathbf{F}$ and the couple $\mathbf{C}$ tend to zero limits and the direction of $\mathbf{F}$ tends to a limiting direction. We assume that as A tends to zero, the number $|\mathbf{F}|/A$ tends to a non-zero limit while $|\mathbf{C}|/A$ tends to 0 (which is sensible because smaller the area, the smaller will be the distance from the definite axis referred to earlier leading to a couple that vanishes).

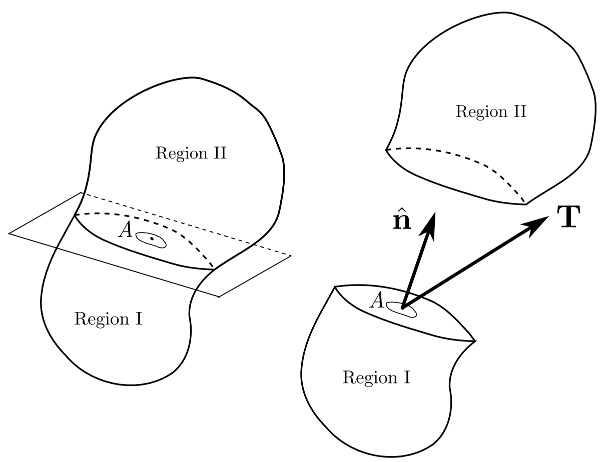

Figure 1: Traction

We define a vector

$$\mathbf{T} = \lim_{A \to 0} \frac{\mathbf{F}}{A} \tag{1}$$

called the *stress vector* or the *traction vector*.[1]

[1] The way in which most modern mechanicians present the concepts of traction and stress can be traced back to the way it was presented in the classic, "A Treatise on the Mathematical Theory of Elasticity" by A. E. H. Love. My way is no exception.

Note that the traction vector depends on the location of P as well as the choice of the plane on which A is located. Since the orientation of this plane is given by $\hat{\mathbf{n}}$, so $\mathbf{T}$ depends the position vector of P *and* $\hat{\mathbf{n}}$. If $\hat{\mathbf{n}}$ is different, $\mathbf{T}$ will be different.

Note also that just as Region II exerts forces on Region I, so also Region I exerts forces on Region II over the area; these forces must necessarily be equal in magnitude and opposite in direction from Newton's third law (important: it is not necessary for the whole body to be in equilibrium for these forces to be equal and opposite). Thus, in a fashion identical to what was discussed previously, a stress vector or traction vector can be defined on the plane and considering the unit vector, $-\hat{\mathbf{n}}$. We then have

$$\mathbf{T}\left(\mathbf{x}, \hat{\mathbf{n}}\right) = -\mathbf{T}\left(\mathbf{x}, -\hat{\mathbf{n}}\right), \tag{2}$$

where it is important to note that the position vector $\mathbf{x}$ is the same for both traction vectors because we are considering the same point P just from two different sides.

2 Surface tractions

The nature of the action between two bodies in contact is assumed to be of the same nature as the action between two portions of the same body separated by an imaginary surface. If the point P in the previous discussion is moved to a point P' on the bounding surface of the body with the position vector $\mathbf{x}$ changing to $\mathbf{x}'$ and $\hat{\mathbf{n}}$ changing to $\hat{\mathbf{n}}'$ that coincides with the unit outward normal at P', the resulting traction vector $\mathbf{T}'\left(\mathbf{x}', \hat{\mathbf{n}}'\right)$ is referred to as the *surface traction.*

Very important: Whether it is the traction across an imaginary plane inside a body or the surface traction which acts at the actual bounding surface of a body, the direction of the traction vector does not, in general, coincide with that of $\hat{\mathbf{n}}$.

The traction vector can be decomposed into a component normal to the plane (defined by $\hat{\mathbf{n}}$) and a component parallel to the plane.

3 Law of equilibrium of tractions on small volumes

Consider the linear momentum equation applied to a material volume element in integral form[2]

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_{\mathrm{m}}(t)} \rho\mathbf{v}\,\mathrm{d}V = \int_{V_{\mathrm{m}}(t)} \rho\mathbf{b}\,\mathrm{d}V + \int_{S_{\mathrm{m}}(t)} \mathbf{T}\,\mathrm{d}S, \tag{3}$$

where $V_{\mathrm{m}}(t)$ is the domain contained in the material volume element, S_{m} is the bounding surface, ρ is the density of the material, $\mathbf{v}$ is the velocity, $\mathbf{b}$ is the body force per unit mass acting at a generic point within the volume element, and $\mathbf{T}$ is the traction acting at a generic point on the bounding surface.

Using the Reynolds' transport theorem[3] on the l.h.s. we can take the material time derivative inside the

[2]Refer to your notes in Fluid Mechanics - it's the same thing.
[3]Refer again to your Fluid Mechanics notes

integration to obtain

$$
\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\int_{V_{\mathrm{m}}(t)} \rho\mathbf{v}\,\mathrm{d}V &= \int_{V_{\mathrm{m}}(t)} \frac{\partial(\rho\mathbf{v})}{\partial t}\,\mathrm{d}V + \int_{S_{\mathrm{m}}(t)} (\rho\mathbf{v})\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S, \\
&= \int_{V_{\mathrm{m}}(t)} \left[\frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot\{(\rho\mathbf{v})\otimes\mathbf{v}\}\right]\mathrm{d}V \qquad \text{(Using Gauss' divergence theorem)} \\
&= \int_{V_{\mathrm{m}}(t)} \left[\mathbf{v}\left\{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\right\} + \rho\left\{\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right\}\right]\mathrm{d}V \\
&= \int_{V_{\mathrm{m}}(t)} \left[\mathbf{v}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\right]\mathrm{d}V \qquad \left(\text{Using}\frac{\mathrm{D}()}{\mathrm{D}t} \equiv \frac{\partial()}{\partial t} + \mathbf{v}\cdot\nabla()\right) \\
&= \int_{V_{\mathrm{m}}(t)} \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}V \qquad \left(\text{Mass conservation implies}\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0\right)
\end{aligned}
$$

So, from Eq. (3), we have

$$\int_{V_{\mathrm{m}}(t)} \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\,\mathrm{d}V = \int_{V_{\mathrm{m}}(t)} \rho\mathbf{b}\,\mathrm{d}V + \int_{S_{\mathrm{m}}(t)} \mathbf{T}\,\mathrm{d}S. \tag{4}$$

If the characteristic dimension of the volume element is l, the volume integral terms can be viewed as the average value of the integrands multiplied by l^3; thus from the above equation we have

$$\langle\text{integrand 1}\rangle l^3 = \langle\text{integrand 2}\rangle l^3 + \int_{S_{\mathrm{m}}(t)} \mathbf{T}\,\mathrm{d}S. \tag{5}$$

Now, let the volume be shrunk to a very small size by letting l tend to 0. Then, dividing throughbout by l^2 and taking the limit $l \to 0$, we have

$$\lim_{l\to 0}\frac{1}{l^2}\int_{S_{\mathrm{m}}(t)} \mathbf{T}\mathrm{d}S = 0. \tag{6}$$

The above equation says that the tractions on the area elements of the surface of an infinitesimal body are, approximately, a system in equilibrium.

The most important thing to note here is that this equilibrium of tractions holds even if the infinitesimal body is itself not in static equilibrium, i.e. even if it is accelerating.

4 The state of stress at a point

An infinite number of planes can be drawn through the point P, and corresponding to each such plane, we have a $\hat{\mathbf{n}}$ and a $\mathbf{T}$. The complete specification of the state of stress at P involves the knowledge of the traction at P across all these planes.

We use the law of equilibrium of surface tractions to express the traction at P across any plane in terms of the components of the tractions across planes that are parallel to the coordinate planes.

Consider the equilibrium of a tetrahedral portion of the body having one vertex at P and the three edges that meet at this vertex to be parallel to the coordinate axes.

Referring to Fig. 2, for the force equilibrium along direction-1, we have:

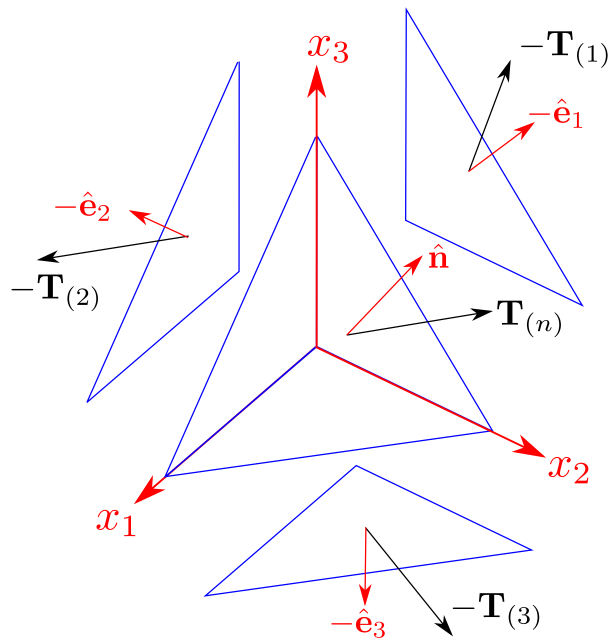

Figure 2: Stress equilibrium

$$T_{(n)1}\Delta A - T_{(1)1}\Delta A_1 - T_{(2)1}\Delta A_2 - T_{(3)1}\Delta A_3 = 0. \tag{7}$$

Now,

$$\Delta A_1 = n_1\Delta A, \quad \Delta A_2 = n_2\Delta A, \quad \Delta A_3 = n_3\Delta A. \tag{8}$$

Therefore,

$$T_{(n)1} - T_{(1)1}n_1 - T_{(2)1}n_2 - T_{(3)1}n_3 = 0 \tag{9}$$

So, writing generally for any direction-i, we have

$$\begin{aligned} T_{(n)i} &= T_{(1)i}n_1 + T_{(2)i}n_2 + T_{(3)i}n_3, \\ \text{or,} \quad T_{(n)i} &= T_{(j)i}n_j \qquad \text{(using indical notation)} \end{aligned} \tag{10}$$

Here, $T_{(j)i}$ represents the component of the traction vector T_j along the i-th direction, and is denoted, alternatively, as σ_{ji}. Thus,

$$T_{(n)i} = \sigma_{ji}n_j, \tag{11}$$

which we identify as a dot product (because one of the indices is repeated) and so rewrite in vector (or, compact notation) as

$$\mathbf{T}_{(n)} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \tag{12}$$

or, equivalently in matrix representation as

$$[\mathbf{T}_{(n)}] = [\boldsymbol{\sigma}]^{\mathsf{T}}[\hat{\mathbf{n}}], \tag{13}$$

where the 'T' in the superscript refers to transpose. To understand why the transpose comes, refer to the "Mathematical Preliminaries" document.

VERY IMPORTANT: We state (without proving) that conservation of angular momentum in the absence of body couples leads to the conclusion that the stress tensor is symmetric, i.e. $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$.

In expanded form, we have from Eq. (13)

$$T_{(n)1} = \sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3, \tag{14a}$$

$$T_{(n)2} = \sigma_{12}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3, \tag{14b}$$

$$T_{(n)3} = \sigma_{13}n_1 + \sigma_{23}n_2 + \sigma_{33}n_3. \tag{14c}$$

Eqs (11), (12), (13), and (14) are different forms of what are referred to as the Cauchy's formula (sometimes Cauchy's stress theorem or Cauchy's law).

5 Cauchy's equation of motion and mechanical equilibrium equations

Going back to Eq. (4), we have

$$\begin{aligned} \int_{V_{\mathrm{m}}} \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}\, \mathrm{d}V &= \int_{V_{\mathrm{m}}} \rho\mathbf{b}\, \mathrm{d}V + \int_{S_{\mathrm{m}}} \mathbf{T}\, \mathrm{d}S \\ &= \int_{V_{\mathrm{m}}} \rho\mathbf{b}\, \mathrm{d}V + \int_{S_{\mathrm{m}}} \boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\, \mathrm{d}S \\ &= \int_{V_{\mathrm{m}}} \rho\mathbf{b}\, \mathrm{d}V + \int_{V_{\mathrm{m}}} \nabla\cdot\boldsymbol{\sigma}\, \mathrm{d}V \quad \text{(Using divergence theorem)} \end{aligned} \tag{15}$$

Thus we can write

$$\int_{V_{\mathrm{m}}} \left(\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho\mathbf{b} - \nabla\cdot\boldsymbol{\sigma}\right) \mathrm{d}V = 0, \tag{16}$$

or, using the arbitrariness of the material volume we have

$$\boxed{\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} - \rho\mathbf{b} - \nabla\cdot\boldsymbol{\sigma} = \mathbf{0}.} \tag{17}$$

If the body is in equilibrium, we have

$$\boxed{\rho\mathbf{b} + \nabla\cdot\boldsymbol{\sigma} = 0.} \tag{18}$$

Referring to a rectangular Cartesian coordinate system, Eq. (18) can be expressed in component form as

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} + \rho b_1 = 0, \tag{19a}$$

$$\frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \frac{\partial\sigma_{23}}{\partial x_3} + \rho b_2 = 0, \tag{19b}$$

$$\frac{\partial\sigma_{13}}{\partial x_1} + \frac{\partial\sigma_{23}}{\partial x_2} + \frac{\partial\sigma_{33}}{\partial x_3} + \rho b_3 = 0. \tag{19c}$$

VERY IMPORTANT: Eq. (18) or, equivalently, the three equations collectively in in Eqs (19) is known as the mechanical equilibrium equations.

6 Normal and shear components of traction

It has already been pointed out that, in general, the traction vector $\mathbf{T}_{(n)}$ acting at a point in a plane with unit normal $\hat{\mathbf{n}}$ is not parallel to $\hat{\mathbf{n}}$. So, it is possible to resolve $\mathbf{T}_{(n)}$ into components parallel and perpendicular to $\hat{\mathbf{n}}$.

We denote the component parallel to $\hat{\mathbf{n}}$ as T^N and call it the normal component. We have

$$T^N = \mathbf{T}_{(n)} \cdot \hat{\mathbf{n}} = (\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}) \cdot \hat{\mathbf{n}}, \tag{20}$$

$$\text{or, in indical notation} \quad T^N = T_{(n)i} n_i = \sigma_{ji} n_j n_i, \tag{21}$$

$$\text{or, in matrix representation} \quad T^N = \left([\sigma]^{\mathsf{T}}[\hat{\mathbf{n}}]\right)^{\mathsf{T}} [\hat{\mathbf{n}}] \equiv [\hat{\mathbf{n}}]^{\mathsf{T}}[\boldsymbol{\sigma}][\hat{\mathbf{n}}] \tag{22}$$

$$\text{or, in expanded form} \quad T^N = \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 + 2\sigma_{12} n_1 n_2 + 2\sigma_{23} n_2 n_3 + 2\sigma_{13} n_1 n_3. \tag{23}$$

Note that since T^N is the component of the traction $\mathbf{T}_{(n)}$ along $\hat{\mathbf{n}}$, T^N may be equivalently denoted by σ_{nn}, i.e. $T^N \equiv \sigma_{nn}$.

Note also that the expressions for T^N (or, σ_{nn}) are exactly like the ones we had found, in the previous chapter, for the engineering strain along a particular direction.

Likewise, we denote the component of $\mathbf{T}_{(n)}$ perpendicular to $\hat{\mathbf{n}}$ and lying in the same plane as $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$ as T^S and call it the shear component. We have

$$\left(T^S\right)^2 = \left|\mathbf{T}_{(n)}\right|^2 - \left(T^N\right)^2 \tag{24}$$

$$\text{or,} \quad \left(T^S\right)^2 = \left\{(T_{(n)1})^2 + (T_{(n)2})^2 + (T_{(n)3})^2\right\} - \left(T^N\right)^2. \tag{25}$$

Substituting the expressions for $T_{(n)1}$, $T_{(n)2}$, and $T_{(n)3}$ from Eqs (14) and the expression for T^N from Eq. (23), we can obtain T^S.

When we use Eq. (24), we are implicitly saying that T^N and T^S are in the same plane contained by $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$. So another way of finding T^S would be by taking the dot product $\mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_s$, where $\hat{\mathbf{e}}_s$ is perpendicular to $\hat{\mathbf{n}}$ and is contained in the plane formed by $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$.

Let us now try to find an expression for $\hat{\mathbf{e}}_s$. We first note that the unit vector along $\mathbf{T}_{(n)}$ is $\dfrac{\mathbf{T}_{(n)}}{|\mathbf{T}_{(n)}|}$. The unit vector that is perpendicular to both $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$ is $\dfrac{\mathbf{T}_{(n)}}{|\mathbf{T}_{(n)}|} \times \hat{\mathbf{n}}$. Now, $\hat{\mathbf{e}}_s$ is the unit vector that should be perpendicular to both this newly found unit vector and $\hat{\mathbf{n}}$, so that $\hat{\mathbf{e}}_s = \hat{\mathbf{n}} \times \left(\dfrac{\mathbf{T}_{(n)}}{|\mathbf{T}_{(n)}|} \times \hat{\mathbf{n}}\right)$.

In terms of unit vectors, therefore, we have the following:

$$\mathbf{T}_{(n)} = T^N \hat{\mathbf{n}} + T^S \hat{\mathbf{e}}_s. \tag{26}$$

Just as we had used σ_{nn} to denote the component of $\mathbf{T}_{(n)}$ along $\hat{\mathbf{n}}$, we can use σ_{ns} to denote the component of $\mathbf{T}_{(n)}$ along $\hat{\mathbf{e}}_s$.

Now, consider another unit vector, say, $\hat{\mathbf{e}}_t$ that is lying in the plane perpendicular to $\hat{\mathbf{n}}$, has a common origin as $\hat{\mathbf{n}}$ and $\mathbf{T}_{(n)}$ but, unlike $\hat{\mathbf{e}}_s$, is not coplanar with $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$. If we take the dot product $\mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_t$, then the resulting component would also lie on the plane perpendicular to $\hat{\mathbf{n}}$. It is important to note that while $\hat{\mathbf{e}}_s$ is unique, we can have infinite such $\hat{\mathbf{e}}_t$. In fact, $\hat{\mathbf{e}}_s$ is a special case of $\hat{\mathbf{e}}_t$ distinguished by its requirement to be coplanar with $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$.

The component of $\mathbf{T}_{(n)}$ given by the dot product $\mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_s$ is also a shearing component of $\mathbf{T}_{(n)}$. However, $T^S = \mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_s$ is a special shearing component distinguished by its requirement to be coplanar with $\mathbf{T}_{(n)}$ and $\hat{\mathbf{n}}$.

Just as we had used σ_{nn} and σ_{ns} to denote the components of $\mathbf{T}_{(n)}$ along $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_s$, respectively, we use σ_{nt} to denote the component of $\mathbf{T}_{(n)}$ along $\hat{\mathbf{e}}_t$.

VERY IMPORTANT: In the previous chapter on Kinematics, the shear component of strain tensor was physically interpreted by referring to elemental line segments along two perpendicular directions. Similarly, here, the shearing components of $\mathbf{T}_{(n)}$ can be related to two perpendicular directions. We can say that $\sigma_{nt} = \mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_t$ is related $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_t$ while $\sigma_{ns} \equiv T^S = \mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_s$ is related to $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_s$. We have the following:

$$\begin{aligned}
\sigma_{nt} &= \left[\mathbf{T}_{(n)} \cdot \hat{\mathbf{e}}_t\right], \\
&= \left[\mathbf{T}_{(n)}\right]^{\mathsf{T}} \left[\hat{\mathbf{e}}_t\right], \\
&= \left[\boldsymbol{\sigma} \cdot \hat{\mathbf{n}}\right]^{\mathsf{T}} \left[\hat{\mathbf{e}}_t\right], \\
&= \left(\left[\boldsymbol{\sigma}\right]^{\mathsf{T}} \left[\hat{\mathbf{n}}\right]\right)^{\mathsf{T}} \left[\hat{\mathbf{e}}_t\right], \\
&= \left[\hat{\mathbf{n}}\right]^{\mathsf{T}} \left[\boldsymbol{\sigma}\right] \left[\hat{\mathbf{e}}_t\right].
\end{aligned}$$

Similarly, we have $\sigma_{ns} \equiv T^S = [\hat{\mathbf{n}}]^{\mathsf{T}} [\boldsymbol{\sigma}] [\hat{\mathbf{e}}_s]$.

7 Principal stress

In the previous chapter on Kinematics, after we had found an expression for the normal or engineering strain along a given direction (or, unit vector) in terms of a given strain tensor, we had set about the problem of finding the directions along which the normal strain was maximum - these strains being referred to as the principal strains. We are at a corresponding point in this chapter. We have in our hands the expression for the normal stress $\sigma_{nn} \equiv T^N$, and we set about the following problem:

Given a state of stress $\boldsymbol{\sigma}$ *referred to coordinate axes along the directions* $\hat{\mathbf{e}}_1$*,* $\hat{\mathbf{e}}_2$*, and* $\hat{\mathbf{e}}_3$, which $\hat{\mathbf{n}}$ maximizes $\sigma_{nn} \equiv T^N$?

We could proceed exactly as in the previous chapter by using the method of Lagrange multiplier but we take up another method as follows:

We first note that for T^N to be maximum, $\mathbf{T}_{(n)}$ must be parallel to $\hat{\mathbf{n}}$.

Now, referred to $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$, we have:

$$\hat{\mathbf{n}} = \hat{\mathbf{e}}_1 n_1 + \hat{\mathbf{e}}_2 n_2 + \hat{\mathbf{e}}_3 n_3.$$

Therefore, $\mathbf{T}_{(n)} \parallel \hat{\mathbf{n}}$ with its magnitude entirely contributed by T^N must be

$$\begin{aligned}
&\mathbf{T}_{(n)} = T^N \hat{\mathbf{n}}, \\
\text{or,} \qquad &\mathbf{T}_{(n)} = \hat{\mathbf{e}}_1 (T^N n_1) + \hat{\mathbf{e}}_2 (T^N n_2) + \hat{\mathbf{e}}_3 (T^N n_3).
\end{aligned} \tag{27}$$

The plane defined by $\hat{\mathbf{n}}$ is the principal plane and T^N is the principal stress. Henceforth, we will denote T^N by σ. Using this new notation, Eq. (27) can be written in component form as

$$T_{(n)1} = \sigma n_1, \tag{28a}$$
$$T_{(n)2} = \sigma n_2, \tag{28b}$$
$$T_{(n)3} = \sigma n_3. \tag{28c}$$

Furthermore, using from Cauchy's formula (see Eq. (14)),

$$T_{(\mathbf{n})\mathbf{1}} = \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3, \tag{29a}$$
$$T_{(\mathbf{n})\mathbf{2}} = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3, \tag{29b}$$
$$T_{(\mathbf{n})\mathbf{3}} = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3. \tag{29c}$$

Now, (29a) - (28a), (29b) - (28b), and (29c) - (28c) gives us

$$(\sigma_{11} - \sigma)\, n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 = 0, \tag{30a}$$
$$\sigma_{12} n_1 + (\sigma_{22} - \sigma)\, n_2 + \sigma_{32} n_3 = 0, \tag{30b}$$
$$\sigma_{13} n_1 + \sigma_{23} n_2 + (\sigma_{33} - \sigma)\, n_3 = 0. \tag{30c}$$

For non-trivial solutions of n_1, n_2, and n_3, we must have

$$\begin{vmatrix} \sigma_{11} - \sigma & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{vmatrix} = 0. \tag{31}$$

On expanding,

$$\begin{aligned} \sigma^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\, \sigma^2 + \left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2\right) \sigma \\ - \left(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2\right) = 0, \\ \text{or,} \qquad \sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0, \end{aligned} \tag{32}$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \tag{33a}$$
$$\begin{aligned} I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \\ &= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{33} & \sigma_{31} \\ \sigma_{31} & \sigma_{11} \end{vmatrix}, \end{aligned} \tag{33b}$$
$$\begin{aligned} I_3 &= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2, \\ &= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}. \end{aligned} \tag{33c}$$

Here, I_1, I_2, and I_3 are stress invariants. These are not the only stress invariants. Other invariants can be formed from them. For instance, $2I_1^2 - 6I_2$ is another stress invariant.

There are a couple of important facts associated with principal stresses and principal directions that follow from general theory of eigenvalues (covered in Mathematics II in First Year):

(i) Eigenvalues of a real, symmetric matrix are real. The stress matrix is real and symmetric. So the principal stresses are always real (as one would, of course, expect!)

(ii) The eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. So if the principal stress values are all different, then the principal directions are mutually perpendicular to each other.

8 State of stress referred to principal directions

We can choose to orient the coordinate axes along three mutually perpendicular principal directions. In that case, the state of stress shapes up (in matrix representation) as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma^{(1)} & 0 & 0 \\ 0 & \sigma^{(2)} & 0 \\ 0 & 0 & \sigma^{(3)} \end{bmatrix}, \tag{34}$$

where $\sigma^{(1)}$, $\sigma^{(2)}$, $\sigma^{(3)}$ are the three principal stresses. The fact that the stress matrix referred to axes that are directed along the principal directions must be diagonal is embedded in the definition of the principal stress itself. The principal stresses are defined to be the normal components of the traction vectors on those planes where the traction vector is parallel to the unit normal to the plane itself. In other words, a component along a direction perpendicular to this unit normal (i.e. along the plane itself) must be necessarily zero. So, if the principal directions themselves are chosen as the coordinate axes, then the traction vector corresponding to each principal plane will be entirely along the axis perpendicular to the plane and along the plane there will be no component - meaning that along the other two coordinate axes which necessarily must lie on the plane, there can be no component of the traction. Thus, stress components along these two directions (the shear directions) must be zero.

9 Octahedral stress

Consider the coordinate axes aligned along the principal directions. A plane that is equally aligned to these axes is called an octahedral plane. For such a plane, $|n_1| = |n_2| = |n_3|$. Now, since we must have $n_1^2 + n_2^2 + n_3^2 = 1$, therefore

$$|n_1| = |n_2| = |n_3| = \frac{1}{\sqrt{3}}. \tag{35}$$

Note that there can be eight such planes and together they form an octahedron.

Normal and shear stress on each of these planes are referred to as octahedral normal stress and octahedral shear stress

$$\begin{aligned} \sigma_{\text{oct}} &= \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + \sigma_{33} n_3^2 \\ &= \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right) = \frac{1}{3} I_1. \qquad (36) \\ \tau_{\text{oct}}^2 &= \left(\sigma_{11} - \sigma_{22}\right)^2 n_1^2 n_2^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 n_2^2 n_3^2 + \left(\sigma_{33} - \sigma_{11}\right)^2 n_3^2 n_1^2 \\ &= \frac{1}{9}\left[\left(\sigma_{11} - \sigma_{22}\right)^2 + \left(\sigma_{22} - \sigma_{33}\right)^2 + \left(\sigma_{33} - \sigma_{11}\right)^2\right] \\ &= \frac{1}{9}\left[2\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)^2 - 6\left(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11}\right)\right] \\ &= \frac{1}{9}\left(2I_1^2 - 6I_2\right) \qquad (37) \end{aligned}$$

From the last equation we have

$$|\tau_{\text{oct}}| = \frac{\sqrt{2}}{3}\left(I_1^2 - 3I_2\right)^{1/2} \tag{38}$$

10 Decomposition into mean and deviatoric parts

A general state of stress matrix can be decomposed as follows:

$$[\boldsymbol{\sigma}] \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{\mathrm{m}} & 0 & 0 \\ 0 & \sigma_{\mathrm{m}} & 0 \\ 0 & 0 & \sigma_{\mathrm{m}} \end{bmatrix}}_{[\boldsymbol{\sigma}^{\mathrm{M}}]} + \underbrace{\begin{bmatrix} \sigma_{11} - \sigma_{\mathrm{m}} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma_{\mathrm{m}} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma_{\mathrm{m}} \end{bmatrix}}_{[\boldsymbol{\sigma}^{\mathrm{D}}]}, \tag{39}$$

where $\boldsymbol{\sigma}_{\mathrm{m}}$ is taken as $\sigma_{\mathrm{m}} = \dfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \equiv \dfrac{1}{3} I_1$ with I_1 being the first stress invariant of $\boldsymbol{\sigma}$. Equivalently, $\sigma_{\mathrm{m}} = \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma})$ where tr is the trace. In the above, $\boldsymbol{\sigma}^{\mathrm{M}}$ is the mean stress tensor and $\boldsymbol{\sigma}^{\mathrm{D}}$ is the deviatoric stress tensor. Note also that the first stress invariant of the deviatoric stress tensor, $\boldsymbol{\sigma}^{\mathrm{D}}$, is 0 while the first invariant of the mean stress tensor, $\boldsymbol{\sigma}^{\mathrm{M}}$ is I_1 itself.