

CLASSICAL PLATE THEORY

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We use the following kinematical hypothesis:

$$\begin{aligned}u &= u_s - z \frac{\partial w}{\partial x}, \\v &= v_s - z \frac{\partial w}{\partial y}, \\w &\equiv w(x, y),\end{aligned}$$

using which we obtain from the strain-displacement relationships, the following:

$$\begin{aligned}\varepsilon_{xx} &= \frac{\partial u}{\partial x} = \frac{\partial u_s}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} = \frac{\partial v_s}{\partial y} - z \frac{\partial^2 w}{\partial y^2}, \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} = 0, \\ \varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} \left(\frac{\partial u_s}{\partial y} + \frac{\partial v_s}{\partial x} \right) - z \frac{\partial^2 w}{\partial x \partial y}, \\ \varepsilon_{yz} &= \frac{1}{2} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \left(-\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} \right) = 0, \\ \varepsilon_{zx} &= \frac{1}{2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = \frac{1}{2} \left(-\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} \right) = 0.\end{aligned}$$

Since ε_{yz} and ε_{zx} are zero, from Hooke's law, it follows that σ_{yz} and σ_{zx} are also zero. Additionally, since $\varepsilon_{zz} = 0$, so $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. Now, σ_{xx} and σ_{yy} are not zero, therefore σ_{zz} is not zero. However, we forcibly assume plane stress conditions and take $\sigma_{zz} = 0$.

Using $\sigma_{zz} = 0$ in the following relations from Hooke's law:

$$\begin{aligned}\varepsilon_{xx} &= \frac{1}{E} \left[\sigma_{xx} - \nu (\sigma_{yy} + \sigma_{zz}) \right], \\ \varepsilon_{yy} &= \frac{1}{E} \left[\sigma_{yy} - \nu (\sigma_{xx} + \sigma_{zz}) \right],\end{aligned}$$

we have

$$\begin{aligned}\sigma_{xx} &= \frac{E}{1 - \nu^2} (\varepsilon_{xx} + \nu \varepsilon_{yy}), \\ \sigma_{yy} &= \frac{E}{1 - \nu^2} (\varepsilon_{yy} + \nu \varepsilon_{xx}).\end{aligned}$$

Considering the virtual work equation:

$$\int_V \sigma_{ij} \delta \varepsilon_{ij} dV = \int_A t_i \delta u_i dA,$$

we have from the left hand side

$$\begin{aligned} \text{LHS} &= \int_V \sigma_{ij} \delta \varepsilon_{ij} dV \\ &= \int_V (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + 2\sigma_{xy} \delta \varepsilon_{xy}) dv \\ &= \int_A \int_{-h/2}^{h/2} \left[\sigma_{xx} \left(\frac{\partial \delta u_s}{\partial x} - z \frac{\partial^2 \delta w}{\partial x^2} \right) + \sigma_{yy} \left(\frac{\partial \delta v_s}{\partial y} - z \frac{\partial^2 \delta w}{\partial y^2} \right) + 2\sigma_{xy} \frac{1}{2} \left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} - 2z \frac{\partial^2 \delta w}{\partial x \partial y} \right) \right] dz dA \\ &= \int_A \int_{-h/2}^{h/2} \left[\sigma_{xx} \frac{\partial \delta u_s}{\partial x} + \sigma_{yy} \frac{\partial \delta v_s}{\partial y} + \sigma_{xy} \left(\frac{\partial \delta u_s}{\partial y} + \frac{\partial \delta v_s}{\partial x} \right) \right] dz dA \\ &\quad - \int_A \int_{-h/2}^{h/2} \sigma_{xx} z \frac{\partial^2 \delta w}{\partial x^2} dz dA - \int_A \int_{-h/2}^{h/2} \sigma_{yy} z \frac{\partial^2 \delta w}{\partial y^2} dz dA - 2 \int_A \int_{-h/2}^{h/2} \sigma_{xy} z \frac{\partial^2 \delta w}{\partial x \partial y} dz dA \end{aligned}$$

We note in the last step that stretching and bending are completely decoupled. Considering only bending and using the following definitions

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} z dz, \quad M_y = \int_{-h/2}^{h/2} \sigma_{yy} z dz, \quad M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz,$$

we obtain the following:

$$\begin{aligned} \text{LHS}_{\text{bending}} &= - \int_A M_x \frac{\partial^2 \delta w}{\partial x^2} dA - \int_A M_y \frac{\partial^2 \delta w}{\partial y^2} dA - 2 \int_A M_{xy} \frac{\partial^2 \delta w}{\partial x \partial y} dA \\ \text{or, } -\text{LHS}_{\text{bending}} &= \int_A \left[\frac{\partial}{\partial x} \left(M_x \frac{\partial \delta w}{\partial x} \right) + \frac{\partial}{\partial y} \left(M_y \frac{\partial \delta w}{\partial y} \right) \right] dA + \int_A \left[\frac{\partial}{\partial x} \left(M_{xy} \frac{\partial \delta w}{\partial y} \right) + \frac{\partial}{\partial y} \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) \right] dA \\ &\quad - \int_A \frac{\partial M_x}{\partial x} \frac{\partial \delta w}{\partial x} dA - \int_A \frac{\partial M_y}{\partial y} \frac{\partial \delta w}{\partial y} dA - \int_A \frac{\partial M_{xy}}{\partial x} \frac{\partial \delta w}{\partial y} dA - \int_A \frac{\partial M_{xy}}{\partial y} \frac{\partial \delta w}{\partial x} dA \\ &= \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} \right) n_x + \left(M_y \frac{\partial \delta w}{\partial y} \right) n_y \right] ds + \oint \left[\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) n_x + \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) n_y \right] ds \\ &\quad - \int_A \left[\underbrace{\frac{\partial}{\partial x} \left(\frac{\partial M_x}{\partial x} \delta w \right)}_{\textcircled{1}} - \frac{\partial^2 M_x}{\partial x^2} \delta w \right] dA - \int_A \left[\underbrace{\frac{\partial}{\partial y} \left(\frac{\partial M_y}{\partial y} \delta w \right)}_{\textcircled{2}} - \frac{\partial^2 M_y}{\partial y^2} \delta w \right] dA \\ &\quad - \int_A \left[\underbrace{\frac{\partial}{\partial y} \left(\frac{\partial M_{xy}}{\partial x} \delta w \right)}_{\textcircled{3}} - \frac{\partial^2 M_{xy}}{\partial y \partial x} \delta w \right] dA - \int_A \left[\underbrace{\frac{\partial}{\partial x} \left(\frac{\partial M_{xy}}{\partial y} \delta w \right)}_{\textcircled{4}} - \frac{\partial^2 M_{xy}}{\partial x \partial y} \delta w \right] dA \end{aligned} \quad (1)$$

Considering the terms ① with ② and ③ with ④, and using Green's theorem, we obtain

$$\begin{aligned}
-\text{LHS}_{\text{bending}} = & \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} \right) n_x + \left(M_y \frac{\partial \delta w}{\partial y} \right) n_y \right] ds + \oint \left[\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) n_x + \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) n_y \right] ds \\
& - \oint \left[\left(\frac{\partial M_x}{\partial x} \delta w \right) n_x + \left(\frac{\partial M_y}{\partial y} \delta w \right) n_y \right] ds - \oint \left[\left(\frac{\partial M_{xy}}{\partial x} \delta w \right) n_y + \left(\frac{\partial M_{xy}}{\partial y} \delta w \right) n_x \right] ds \\
& + \int_A \left[\left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) \delta w \right] dA
\end{aligned} \tag{2}$$

Now, using $n_x = \frac{dy}{ds}$ and $n_y = -\frac{dx}{ds}$, and transposing the negative sign from the left hand side, we have

$$\begin{aligned}
\text{LHS}_{\text{bending}} = & - \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} \right) dy - \left(M_y \frac{\partial \delta w}{\partial y} \right) dx \right] - \oint \left[\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) dy - \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) dx \right] \\
& + \oint \left[\left(\frac{\partial M_x}{\partial x} \delta w \right) dy - \left(\frac{\partial M_y}{\partial y} \delta w \right) dx \right] - \oint \left[\left(\frac{\partial M_{xy}}{\partial x} \delta w \right) dx - \left(\frac{\partial M_{xy}}{\partial y} \delta w \right) dy \right] \\
& - \int_A \left[\left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) \delta w \right] dA
\end{aligned} \tag{3}$$

We note that the terms on the right hand side can be classified as boundary integral terms (those within $\oint$) and domain integral terms (those within $\int_A$). The domain integral terms expressed in terms of M_x , M_y , and M_{xy} can be rewritten in terms of the displacement component w ; thus

$$\begin{aligned}
M_x = & \int_{-h/2}^{h/2} \sigma_{xx} z \, dz \\
= & \int_{-h/2}^{h/2} \frac{E}{1-\nu^2} (\epsilon_{xx} + \nu \epsilon_{yy}) z \, dz \\
= & \frac{E}{1-\nu^2} \int_{-h/2}^{h/2} \left(\frac{\partial u_s}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial v_s}{\partial y} - \nu z \frac{\partial^2 w}{\partial y^2} \right) z \, dz \\
= & -\frac{E}{1-\nu^2} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \int_{-h/2}^{h/2} z^2 \, dz \\
= & -\frac{Eh^3}{12(1-\nu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)
\end{aligned}$$

Setting $D = \frac{Eh^3}{12(1-\nu^2)}$ (it is referred to as the bending rigidity), and proceeding similarly as above for M_y and M_{xy} , we have

$$M_x = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \tag{4a}$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right), \tag{4b}$$

$$M_{xy} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y} \tag{4c}$$

Using these expressions of M_x , M_y , and M_{xy} , we have

$$\begin{aligned}
& \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} \\
&= -D \left(\frac{\partial^4 w}{\partial x^4} + \nu \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \nu \frac{\partial^4 w}{\partial y^2 \partial x^2} + 2(1-\nu) \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\
&= -D \left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} \right) \\
&= -D \nabla^4 w
\end{aligned}$$

Therefore, we have the following

$$\begin{aligned}
\text{LHS}_{\text{bending}} &= - \oint \left[\left(M_x \frac{\partial \delta w}{\partial x} \right) dy - \left(M_y \frac{\partial \delta w}{\partial y} \right) dx \right] - \oint \left[\left(M_{xy} \frac{\partial \delta w}{\partial y} \right) dy - \left(M_{xy} \frac{\partial \delta w}{\partial x} \right) dx \right] \\
&+ \oint \left[\left(\frac{\partial M_x}{\partial x} \delta w \right) dy - \left(\frac{\partial M_y}{\partial y} \delta w \right) dx \right] + \oint \left[\left(\frac{\partial M_{xy}}{\partial x} \delta w \right) dy - \left(\frac{\partial M_{xy}}{\partial y} \delta w \right) dx \right] \\
&+ \int_A D \nabla^4 w \delta w dA
\end{aligned} \tag{5}$$

Now, going back to the right hand side of the virtual work equation and considering the contribution due only to bending we have

$$\text{RHS}_{\text{bending}} = \int_A t_i \delta u_i dA = \int_A q \delta w dA.$$

Bringing together the left hand and right hand sides, we thus have

$$\text{LHS}_{\text{bending}} = \text{RHS}_{\text{bending}},$$

from which we obtain the following:

$$\begin{aligned}
& \int_A (D \nabla^4 w - q) \delta w dA - \oint \left[M_x \frac{\partial \delta w}{\partial x} dy - M_y \frac{\partial \delta w}{\partial y} dx \right] - \oint \left[M_{xy} \frac{\partial \delta w}{\partial y} dy - M_{xy} \frac{\partial \delta w}{\partial x} dx \right] \\
&+ \oint \left[\underbrace{\frac{\partial M_x}{\partial x} \delta w dy}_{\textcircled{1}} - \underbrace{\frac{\partial M_y}{\partial y} \delta w dx}_{\textcircled{3}} \right] + \oint \left[\underbrace{\frac{\partial M_{xy}}{\partial y} \delta w dy}_{\textcircled{2}} - \underbrace{\frac{\partial M_{xy}}{\partial x} \delta w dx}_{\textcircled{4}} \right] = 0
\end{aligned} \tag{6}$$

Now, consider the x -component of the mechanical equilibrium equations, and integrate as follows:

$$\begin{aligned}
& \int_{-h/2}^{h/2} \left[\frac{\partial \sigma_{xx}}{\partial x} z + \frac{\partial \sigma_{xy}}{\partial y} z + \frac{\partial \sigma_{xz}}{\partial z} z \right] dz = 0 \\
\text{or, } & \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + [z \sigma_{xz}]_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \sigma_{xz} dz = 0 \\
\text{or, } & \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = Q_x \quad (\text{using } \sigma_{xz} = 0 \text{ and } Q_x = \int_{-h/2}^{h/2} \sigma_{xz} dz)
\end{aligned}$$

Similarly, we have

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = Q_y,$$

where $Q_y = \int_{-h/2}^{h/2} \sigma_{yz} dz$.

Going back to Eq. (6) and considering the terms ① together with ②, and ③ together with ④, we obtain

$$\begin{aligned} \int_A (D\nabla^4 w - q) \delta w dA - \oint \left[M_x \frac{\partial \delta w}{\partial x} dy - M_y \frac{\partial \delta w}{\partial y} dx \right] - \oint \left[M_{xy} \frac{\partial \delta w}{\partial y} dy - M_{xy} \frac{\partial \delta w}{\partial x} dx \right] \\ + \oint Q_x \delta w dy - \oint Q_y \delta w dx = 0. \end{aligned} \quad (7)$$

Now we want to convert the preceding equation from the (x, y) coordinate system to the (s, n) coordinate system where s is the coordinate along the periphery of the plate and n is the coordinate perpendicular to it. Towards that end, we first establish the relationship between $\frac{\partial}{\partial s}, \frac{\partial}{\partial n}$ and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$.

For any point on the periphery given by $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, the unit normal is $\hat{\mathbf{e}}_n = n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}} \equiv \frac{dy}{ds}\hat{\mathbf{i}} - \frac{dx}{ds}\hat{\mathbf{j}}$. Then for an elemental line segment at $\mathbf{r}$ we have,

$$\begin{aligned} ds \hat{\mathbf{e}}_s &= dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}, \\ \text{or, } \hat{\mathbf{e}}_s &= \frac{dx}{ds}\hat{\mathbf{i}} + \frac{dy}{ds}\hat{\mathbf{j}} \equiv -n_y\hat{\mathbf{i}} + n_x\hat{\mathbf{j}}. \end{aligned}$$

Consider the gradient of any arbitrary scalar ϕ first in the (x, y) coordinate system and next in the (s, n) coordinate system. Thus we have

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{\mathbf{i}} + \frac{\partial \phi}{\partial y} \hat{\mathbf{j}},$$

and

$$\begin{aligned} \nabla \phi &= \frac{\partial \phi}{\partial s} \hat{\mathbf{e}}_s + \frac{\partial \phi}{\partial n} \hat{\mathbf{e}}_n, \\ &= \frac{\partial \phi}{\partial s} (-n_y\hat{\mathbf{i}} + n_x\hat{\mathbf{j}}) + \frac{\partial \phi}{\partial n} (n_x\hat{\mathbf{i}} + n_y\hat{\mathbf{j}}), \\ &= \left(-\frac{\partial \phi}{\partial s} n_y + \frac{\partial \phi}{\partial n} n_x \right) \hat{\mathbf{i}} + \left(\frac{\partial \phi}{\partial s} n_x + \frac{\partial \phi}{\partial n} n_y \right) \hat{\mathbf{j}} \end{aligned}$$

Comparing the two expressions of $\nabla \phi$, we obtain

$$\begin{aligned} \frac{\partial}{\partial x} &= -n_y \frac{\partial}{\partial s} + n_x \frac{\partial}{\partial n}, \\ \frac{\partial}{\partial y} &= n_x \frac{\partial}{\partial s} + n_y \frac{\partial}{\partial n}. \end{aligned}$$

Then from Eq. (7), we obtain

$$\begin{aligned}
& \int_A (D\nabla^4 w - q) \delta w dA - \oint \left[M_x \left(-n_y \frac{\partial \delta w}{\partial s} + n_x \frac{\partial \delta w}{\partial n} \right) dy - M_y \left(n_x \frac{\partial \delta w}{\partial s} + n_y \frac{\partial \delta w}{\partial n} \right) dx \right] \\
& - \oint \left[M_{xy} \left(n_x \frac{\partial \delta w}{\partial s} + n_y \frac{\partial \delta w}{\partial n} \right) dy - M_{xy} \left(-n_y \frac{\partial \delta w}{\partial s} + n_x \frac{\partial \delta w}{\partial n} \right) dx \right] \\
& + \oint Q_x \delta w dy - \oint Q_y \delta w dx = 0, \\
\text{or, } & \int_A (D\nabla^4 w - q) \delta w dA - \oint M_x \left(-n_y \frac{\partial \delta w}{\partial s} + n_x \frac{\partial \delta w}{\partial n} \right) n_x ds + M_y \left(n_x \frac{\partial \delta w}{\partial s} + n_y \frac{\partial \delta w}{\partial n} \right) (-n_y) ds \\
& - \oint M_{xy} \left(n_x \frac{\partial \delta w}{\partial s} + n_y \frac{\partial \delta w}{\partial n} \right) n_x ds + M_{xy} \left(-n_y \frac{\partial \delta w}{\partial s} + n_x \frac{\partial \delta w}{\partial n} \right) (-n_y) ds \\
& + \oint Q_x \delta w n_x ds - \oint Q_y \delta w (-n_y) ds = 0, \\
\text{or, } & \int_A (D\nabla^4 w - q) \delta w dA + \oint (M_x n_y n_x - M_y n_x n_y - M_{xy} n_x^2 + M_{xy} n_y^2) \frac{\partial \delta w}{\partial s} ds \\
& + \oint (-M_x n_x^2 - M_y n_y^2 - 2M_{xy} n_x n_y) \frac{\partial \delta w}{\partial n} ds \\
& + \oint (Q_x n_x + Q_y n_y) \delta w ds = 0. \tag{8}
\end{aligned}$$

From stress-transformation we have

$$\sigma_{nn} = n_x^2 \sigma_{xx} + 2n_x n_y \sigma_{xy} + n_y^2 \sigma_{yy}, \tag{9a}$$

$$\sigma_{ns} = n_x n_y (\sigma_{yy} - \sigma_{xx}) + (n_x^2 - n_y^2) \sigma_{xy}, \tag{9b}$$

from which we obtain

$$M_n = n_x^2 M_x + 2n_x n_y M_{xy} + n_y^2 M_y, \tag{10a}$$

$$M_{ns} = n_x n_y (M_y - M_x) + (n_x^2 - n_y^2) M_{xy}. \tag{10b}$$

Using Eqns (10a) and (10b) in Eq. (8), we obtain

$$\int_A (D\nabla^4 w - q) \delta w dA - \oint M_{ns} \frac{\partial \delta w}{\partial s} ds - \oint M_n \frac{\partial \delta w}{\partial n} ds + \oint Q_n \delta w ds = 0. \tag{11}$$

Now, $\oint M_{ns} \frac{\partial \delta w}{\partial s} ds = [M_{ns} \delta w]_1^2 - \oint \frac{\partial M_{ns}}{\partial s} \delta w ds$. For a closed contour, $[M_{ns} \delta w]_1^2 = 0$; therefore

$$\begin{aligned}
& \int_A (D\nabla^4 w - q) \delta w dA + \oint \frac{\partial M_{ns}}{\partial s} \delta w ds - \oint M_n \frac{\partial \delta w}{\partial n} ds + \oint Q_n \delta w ds = 0, \\
\text{or, } & \int_A (D\nabla^4 w - q) \delta w dA + \oint \left(\frac{\partial M_{ns}}{\partial s} + Q_n \right) \delta w ds - \oint M_n \frac{\partial \delta w}{\partial n} ds = 0. \tag{12}
\end{aligned}$$

So, the governing equation is

$$D\nabla^4 w = q, \tag{13}$$

and the boundary conditions are given by

$$\text{Either } \frac{\partial M_{ns}}{\partial s} + Q_n = 0 \quad \text{or, } w \text{ is specified,} \tag{14a}$$

$$\text{Either } M_n = 0 \quad \text{or, } \frac{\partial w}{\partial n} \text{ is specified.} \tag{14b}$$