Homography: Properties
Week 02 Lectures 8-10

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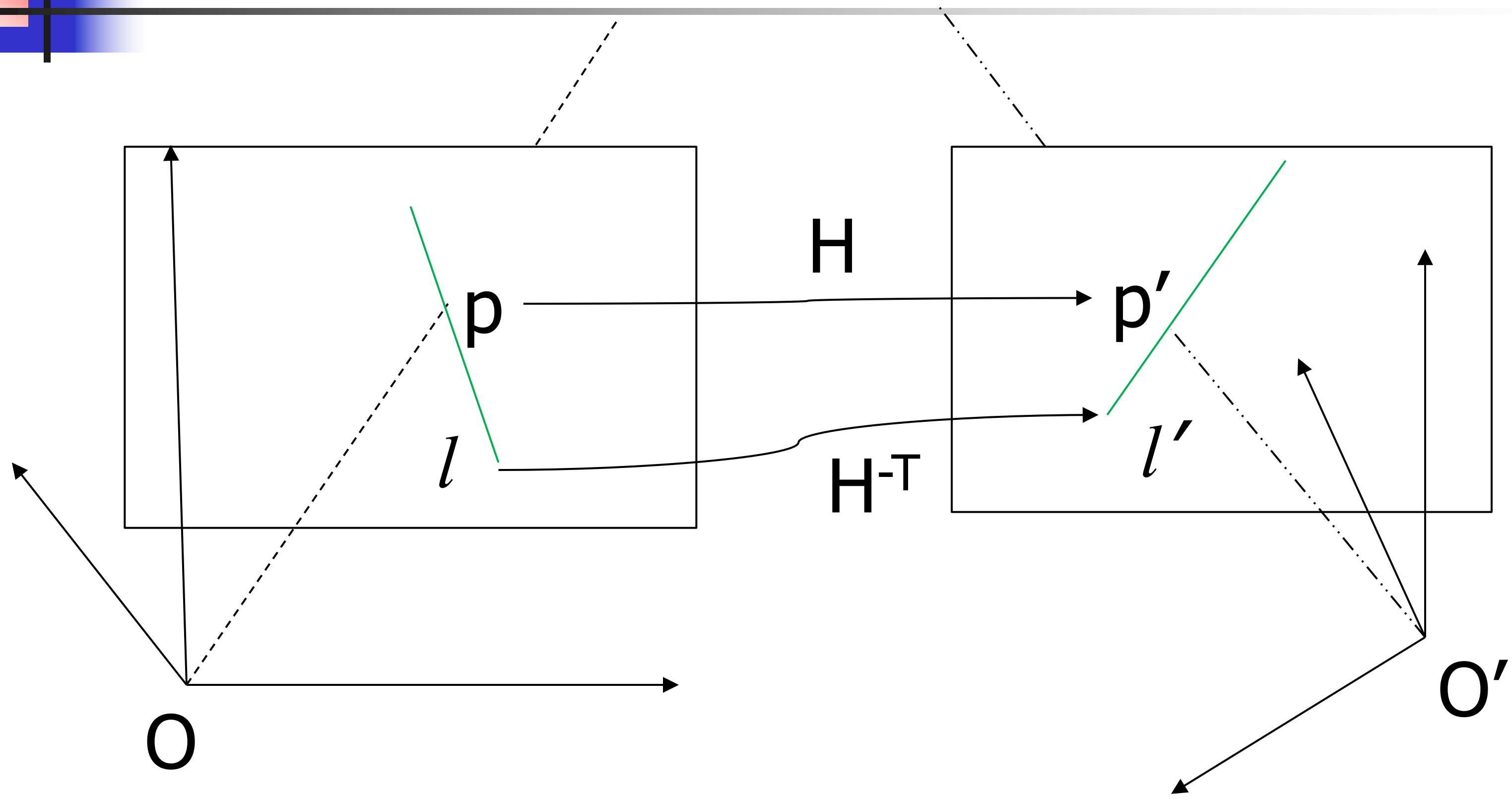
Projective transformation

- h: $P^2 \rightarrow P^2$.
- Invertible.
- Collinearity of every three points to be preserved, i.e. three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.
- Only in the form of non-singular 3x3 matrix.

$P^2 \rightarrow P^2$

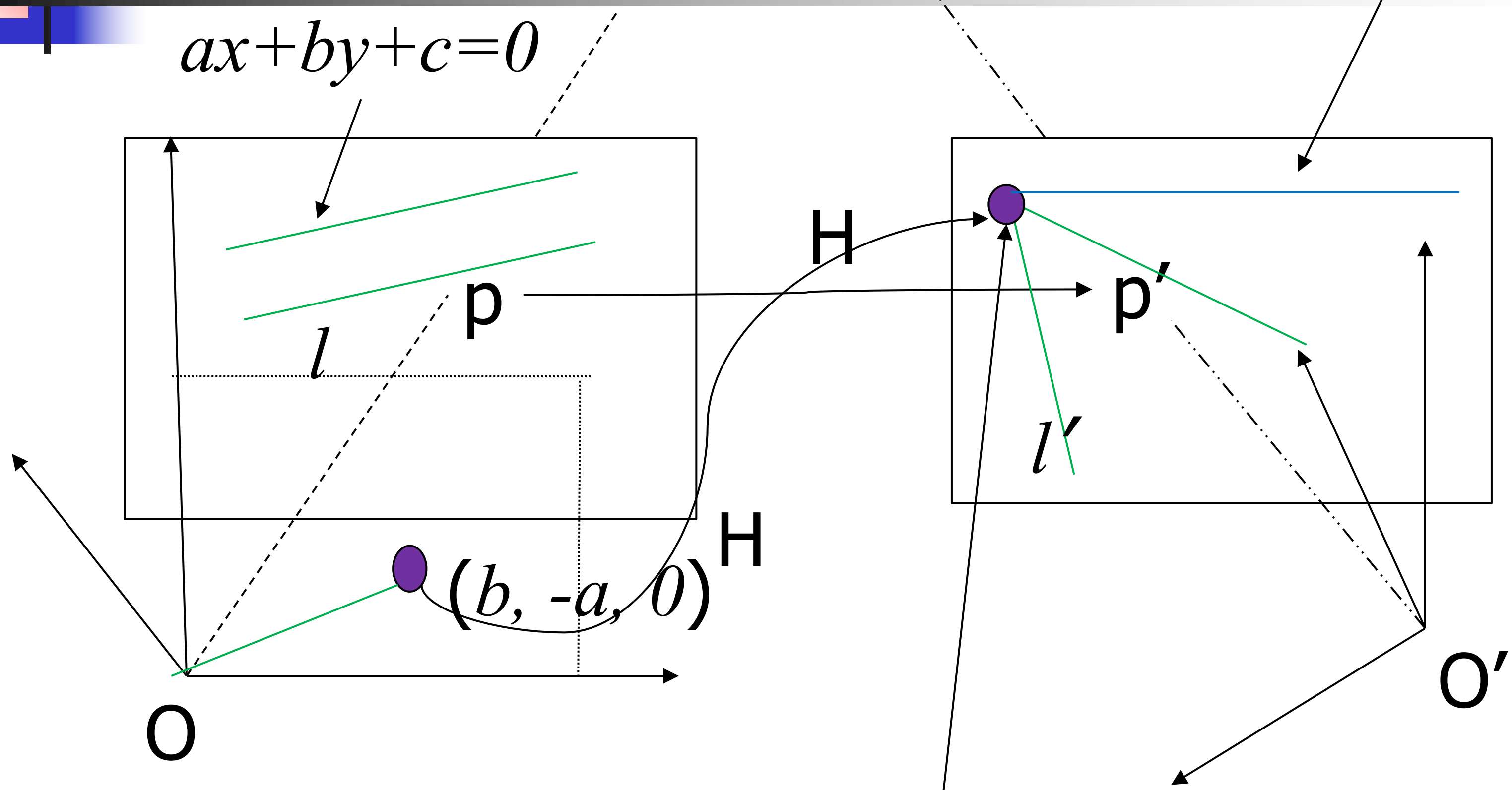


Point: $\mathbf{x}' = \mathbf{H}\mathbf{x}$

Line: $\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$

Vanishing line=$H^{-T}\mathbf{l}_\alpha = H^{-T}(0, 0,1)^T$

Vanishing point and line



Vanishing point: $\mathbf{v_l}=H\,(b, -a, 0)^T$

Point and line transformation

- Point transformation:
  - $\mathbf{x}' = \mathbf{H}\mathbf{x}$
- Line transformation:
  - $\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$
- Vanishing point for lines parallel to $\mathbf{l} = (a, b, c)^T$:
  - $\mathbf{v}_\mathbf{l} = \mathbf{H}\,(b, -a, 0)^T$
- Vanishing line:
  - $\mathbf{l}_\mathbf{H} = \mathbf{H}^{-T}\mathbf{l}_\alpha$

    $= \mathbf{H}^{-T}\,(0, 0, 1)^T$

Examples

- Consider the following homography H between two projective spaces.

$$H = \begin{bmatrix} 3 & 4 & -6 \\ 1 & 3 & -8 \\ 0 & 5 & 1 \end{bmatrix}$$

Compute the transformation of the line formed by two points (2,4,2) and (6,9,3) in P^2.

$$H=\begin{bmatrix}3 & 4 & -6\\ 1 & 3 & -8\\ 0 & 5 & 1\end{bmatrix}$$

Method-I

- Compute transformed points of (2,4,2) and (6,9,3).
- Take their cross product to compute the transformed line.

$$\begin{bmatrix}2\\4\\2\end{bmatrix} \equiv \begin{bmatrix}1\\2\\1\end{bmatrix} \qquad \begin{bmatrix}6\\9\\3\end{bmatrix} \equiv \begin{bmatrix}2\\3\\1\end{bmatrix}$$

$$H.\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}5\\-1\\11\end{bmatrix} \qquad H.\begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}12\\3\\16\end{bmatrix}$$

$$H=\begin{bmatrix}3 & 4 & -6\\ 1 & 3 & -8\\ 0 & 5 & 1\end{bmatrix}$$

Method-I

- Take their cross product to compute the transformed line.

$$H.\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}5\\-1\\11\end{bmatrix} \qquad H.\begin{bmatrix}2\\3\\1\end{bmatrix}=\begin{bmatrix}12\\3\\16\end{bmatrix}$$

$$\begin{bmatrix}5\\-1\\11\end{bmatrix}\times\begin{bmatrix}12\\3\\16\end{bmatrix}=\begin{bmatrix}-49\\52\\27\end{bmatrix}$$

Method-II

Compute the line and transform it.

The line between the points l:
(1,2,1) x (2,3,1)→ (-1,1-1)

Transformed line: $l' = H^{-T} l$

$$H^{-1} = \frac{1}{95}\begin{bmatrix} 43 & -34 & -14 \\ -1 & 3 & 18 \\ 5 & -15 & 5 \end{bmatrix} \qquad l' = \frac{1}{95}\begin{bmatrix} -49 \\ 52 \\ 27 \end{bmatrix}$$

$$H=\begin{bmatrix}3 & 4 & -6\\1 & 3 & -8\\0 & 5 & 1\end{bmatrix}$$

Example: Vanishing line

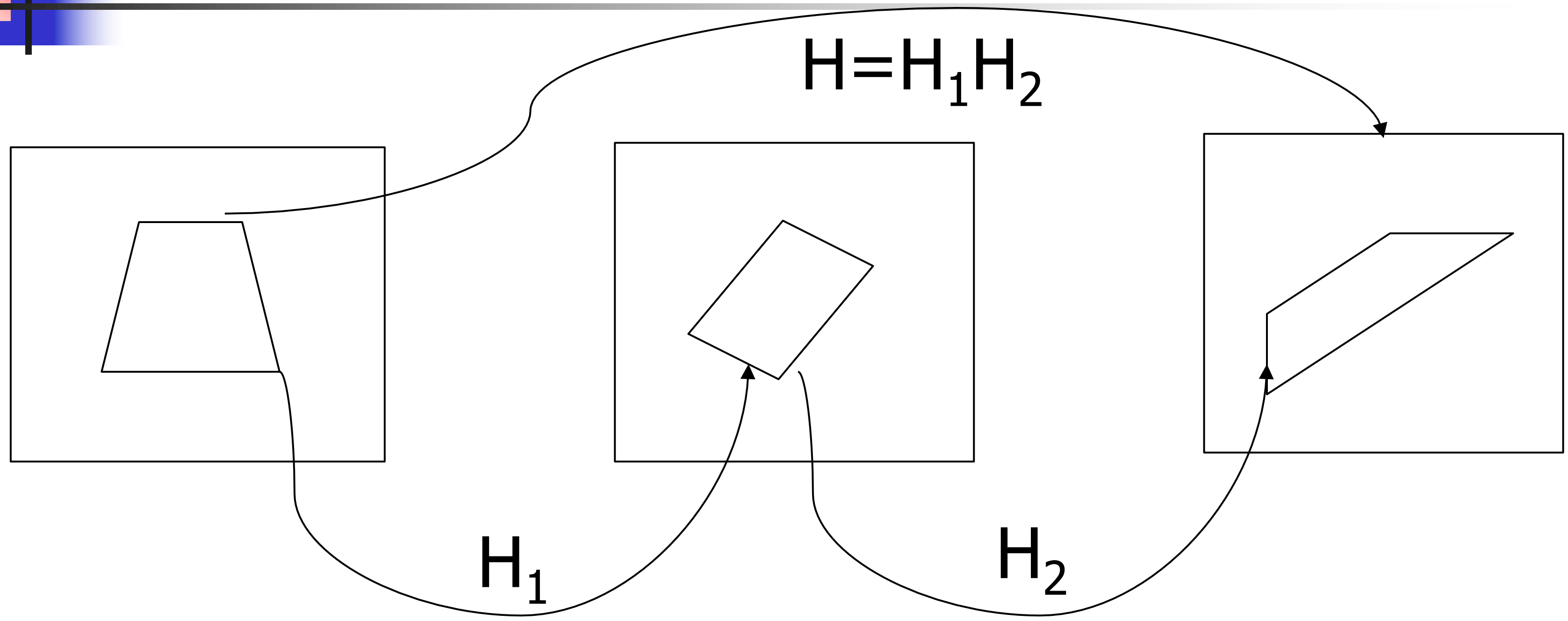
- Compute the vanishing line in the transformed space.

Transformed line: $l'_v = H^{-T}\, l_\alpha$

$$H^{-T} l_\infty = \frac{1}{95}\begin{bmatrix}43 & -1 & 5\\-34 & 3 & -15\\-14 & 18 & 5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

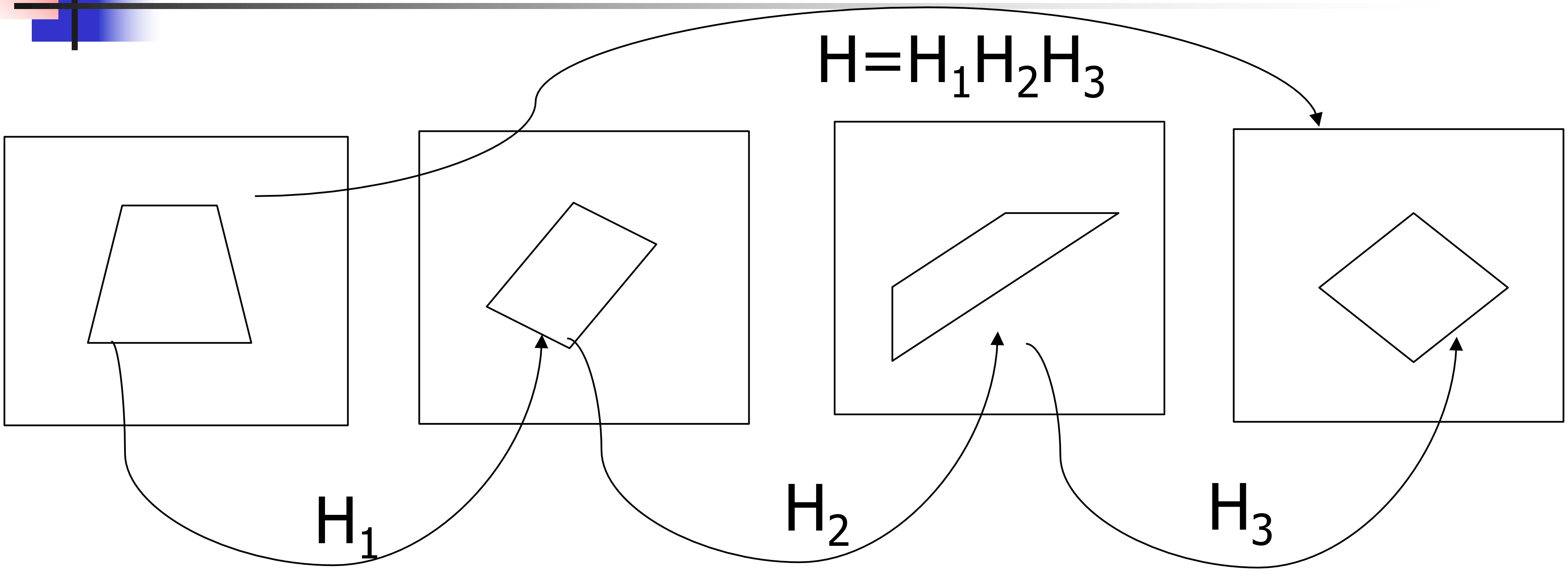
$$l'_v = \begin{bmatrix}5\\-15\\5\end{bmatrix}$$

Projective linear group



A cascade of transformation can be replaced by a single transformation.

Different compositions



A series of transformation could be performed in different composition.

$$H=H_1(H_2H_3)=(H_1H_2)H_3$$

Subgroups and hierarchy

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Projective linear group

$\Uparrow$

Affine group (last row (0,0,1))

$\Uparrow$

Euclidean group (upper left 2x2 orthogonal)

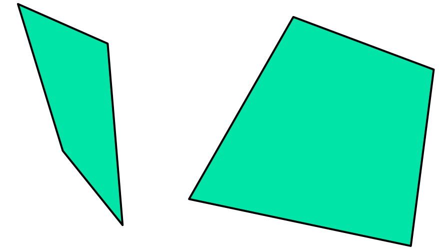
$\Uparrow$

Oriented Euclidean group (upper left 2x2 det 1)

From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

Projective Group

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$



$$\mathrm{x}' = \mathbf{H}_P\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \mathrm{x}$$

$$\mathrm{v} = (v_1, v_2)^{\mathsf{T}}$$

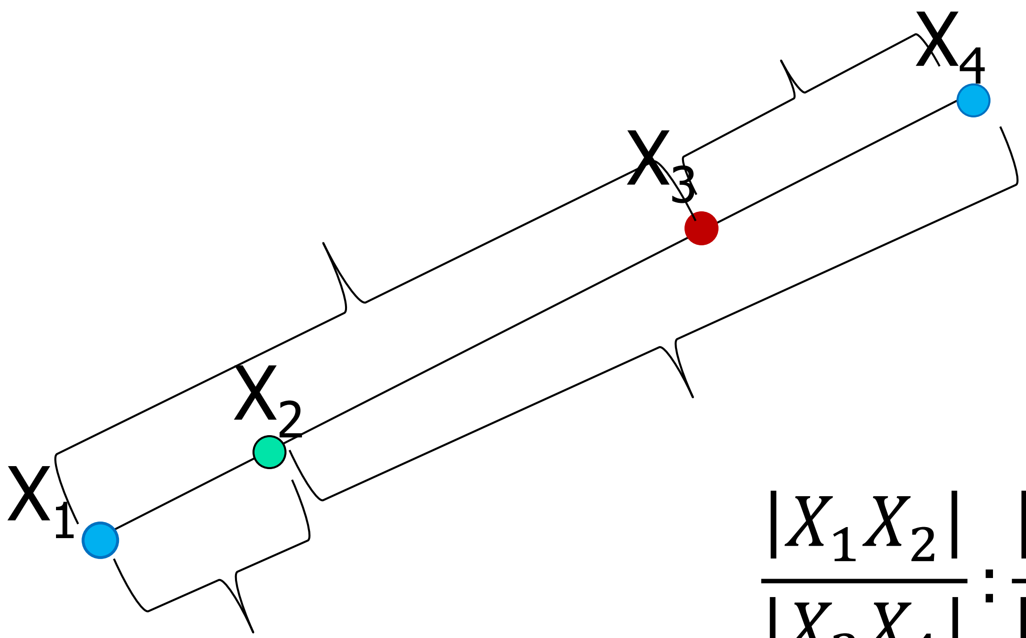
dof=8: 2 scale, 2 rotation, 2 translation, 2 line at infinity)

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathrm{v}^{\mathsf{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite, allows to observe vanishing points, horizon.

Concurrency, collinearity, order of contacts, cross ratio (ratio of ratio).

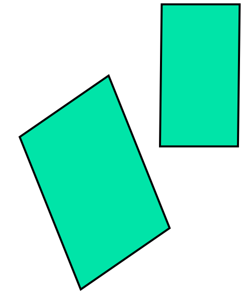
Cross Ratio



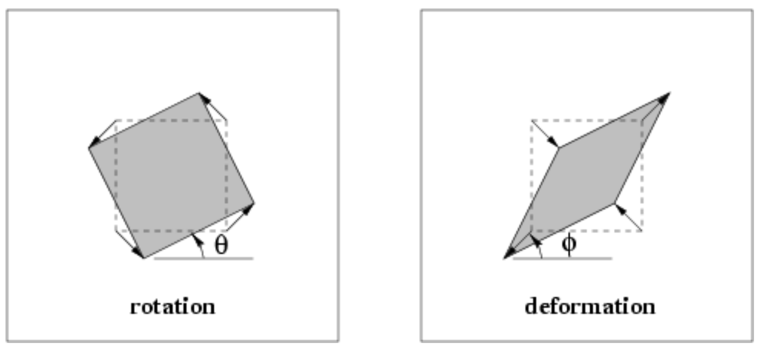
$$\frac{|X_1X_2|}{|X_2X_4|} : \frac{|X_1X_3|}{|X_3X_4|}$$

Affine group

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{x}' = \mathbf{H}_A \mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathbf{x}$$



dof=6

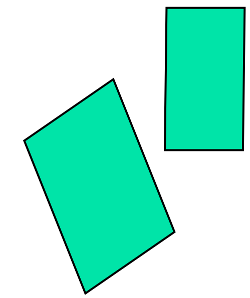
$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

Affine group

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{x}' = \mathbf{H}_A\,\mathbf{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathbf{x}$$

dof=6

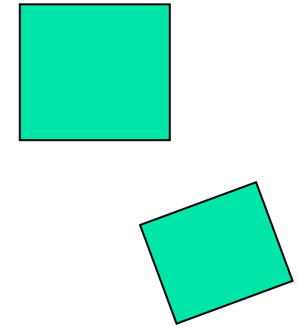
$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity, but points move along line.

Parallelism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity $\mathbf{l}_\infty$.**

Similarity Group

$$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathrm{x}' = \mathbf{H}_S\,\mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix}\mathrm{x}$$

$$\mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

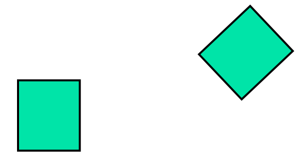
dof=4 (1 scale, 1 rotation, 2 translation)

$$\mathbf{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$\mathrm{I}' = \mathbf{H}_S\,\mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

Ratios of lengths, angles. **The circular points I,J.**

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Isometry

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

Orientation preserving: $\varepsilon = 1$

Orientation reversing: $\varepsilon = -1$

dof=3 (1 rotation, 2 translation)

Invariants: length, angle, area

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$$

$\mathbf{K}$ Upper-triangular

$\det \mathbf{K} = 1$ $\quad v \neq 0$

Decomposition unique (if chosen s>0)

QR decomposition:
Any square matrix decomposed as a product of an orthogonal matrix (Q) and an upper triangular matrix (R)

Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$$v \neq 0$$

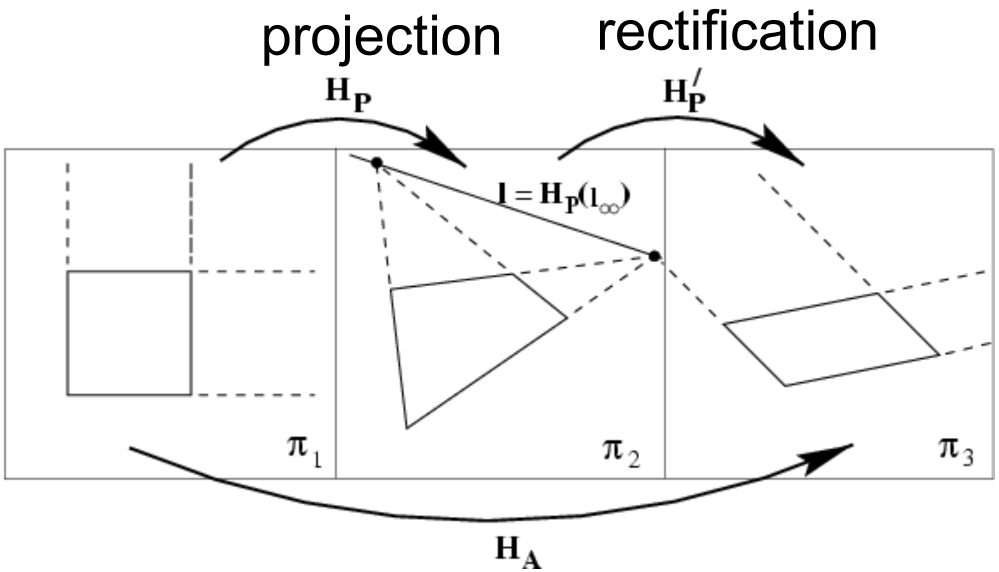
$$\mathbf{A} = s\mathbf{RK} + \mathrm{tv}^\mathsf{T} \qquad \det \mathbf{K} = 1$$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Affine properties from images



$$\mathbf{l}_\infty = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\mathsf{T}, l_3 \neq 0$$

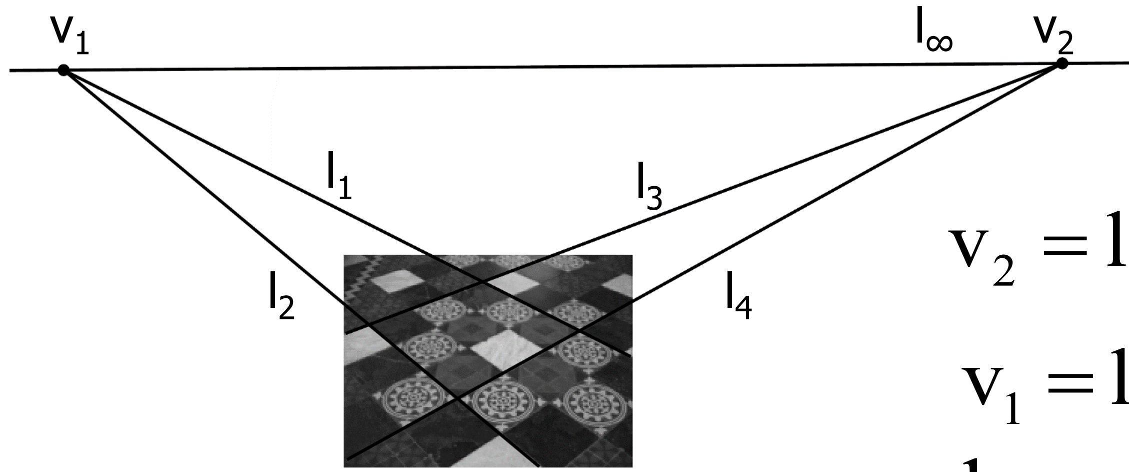
$$H'_p = H_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$

For any affine H_A.

$$H'^{-T}_p \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

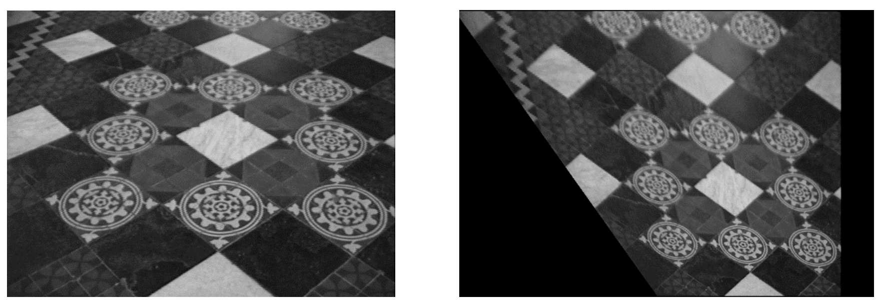
Affine rectification



$$\mathrm{v}_2 = \mathrm{l}_3 \times \mathrm{l}_4$$

$$\mathrm{v}_1 = \mathrm{l}_1 \times \mathrm{l}_2$$

$$\mathrm{l}_\infty = \mathrm{v}_1 \times \mathrm{v}_2$$



From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

An example

$(104,69,1) \times (380, 71,1)$

$l_1=(-2, 276, -18836)$

$(122,226,1) \times (366,228,1)$

$l_2=(-2, 244 , -54900)$

$(88,254,1) \times (62,49,1)$

$l_3=(205,-26,-11436)$

$(390,250,1) \times (406,53,1)$

$l_4=(197,16,-80830)$

$v_1=l_1 \times l_2 =(-10556416, -72128, 64)$

$v_2=l_3 \times l_4= (2284556,14317258,8402)$

Vanishing Line: $l_v= 1.0e+14 * (0,0.0009, -1.5097)$

An example

Vanishing Line: l_v= 1.0e+14 * (0,0.0009, -1.5097)

Scaled l_v= (0, -0.0006,1)

$Y=10^4/6$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -.0006 & 1 \end{bmatrix}$$

Conics in P^2

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$X^T C X = 0$$

$$C = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

Conics identified by C with 5 d.o.f. $(a:b:c:d:e:f)$

A line tangent to the conic C satisfies

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$

Dual conic

C^{-1}

Transformation of conics under homography **H**

- $\mathbf{X}' = \mathbf{H}\mathbf{X}$
- $\mathbf{X}^{T}\mathbf{C}\mathbf{X} = 0$

  $\rightarrow (\mathbf{H}^{-1}\mathbf{X}')^{T}\mathbf{C}(\mathbf{H}^{-1}\mathbf{X}') = 0$

  $\rightarrow \mathbf{X}'^{T}\,\mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}\,\mathbf{X}' = 0$

  $\rightarrow \mathbf{X}'^{T}\,\mathbf{C}'\,\mathbf{X}' = 0$

A conic remains a conic under homography.

where transformed conics $\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$

- $\mathbf{C}'^{*} = \mathbf{C}'^{-1} = (\mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1})^{-1} = \mathbf{H}\mathbf{C}^{-1}\mathbf{H}^{T}$

The circular points

$$\mathrm{I} = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \mathrm{J} = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \qquad \mathrm{I}' = \mathbf{H}_S \, \mathrm{I} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = se^{i\theta} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = \mathrm{I}$$

The circular points I, J are fixed points under the projective transformation **H** iff **H** is a similarity. They are also on $\mathbf{l}_\alpha$.

Every circle intersects $\mathbf{l}_\alpha$ at I and J.

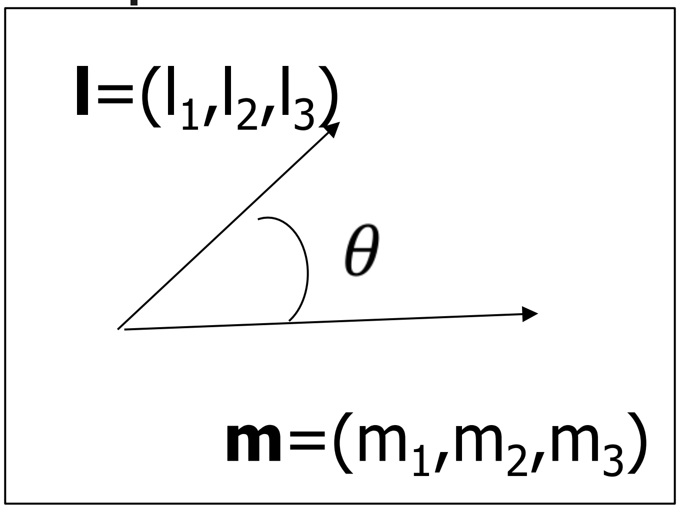
Circle: $x_1^2 + x_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$

Setting x_3=0, $x_1^2 + x_2^2$=0. (I and J satisfies it)

Conic dual to the circular points (C_α^*)

- $C_\alpha^* = I.J^T + J.I^T$ (line conic)
- $C_\alpha^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- As I and J are fixed under similarity C_α^* is also fixed, i.e. $C_\alpha^{*\prime} = H_s\, C_\alpha^* H_s^T = C_\alpha^*$
- C_α^* is fixed iff H is a similarity.
- D.o.f. of transformed C_α^* is 4 and det. =0.
- l_α is the NULL vector of C_α^*.

Measurement of angle under homography



$$cos(\theta) = \frac{l_1 m_1 + l_2 m_2}{\sqrt{(l_1^2 + l_2^2)(m_1^2 + m_2^2)}}$$

Invariant under homography

$$\cos(\theta) = \frac{\boldsymbol{l}^T C_\infty^* \boldsymbol{m}}{\sqrt{(\boldsymbol{l}^T C_\infty^* \boldsymbol{l})(\boldsymbol{m}^T C_\infty^* \boldsymbol{m})}}$$

$C_\infty^{*\prime} = H C_\infty^* H^T$ and $\boldsymbol{l}' = H^{-T} \boldsymbol{l}$

$$\boldsymbol{l}'^T C_\infty^{*\prime} \boldsymbol{m}'$$
$$= \boldsymbol{l}^T H^{-1} H C_\infty^* H^T H^{-T} \boldsymbol{m}$$
$$= \boldsymbol{l}^T C_\infty^* \boldsymbol{m}$$

Once $C_\alpha^{*\prime}$ is obtained Euclidean angle could be recovered.

If $\boldsymbol{l}$ and $\boldsymbol{m}$ orthogonal, $\boldsymbol{l}^T C_\alpha^{*\prime} \boldsymbol{m} = 0$.

Estimation of $C_\alpha^{*\prime}$

- Use the property of orthogonal lines.
  - $\mathbf{l}^T \mathbf{C}_\alpha^{*\prime} \mathbf{m} = 0$
- Minimum 5 such orthogonal pairs needed.
  - A typical equation

$$\left[l_1 m_1 \quad \frac{1}{2}(l_1 m_2 + l_2 m_1) \quad l_2 m_2 \quad \frac{1}{2}(l_1 m_3 + l_3 m_1) \quad \frac{1}{2}(l_2 m_3 + l_3 m_2) \quad l_3 m_3 \right] C = 0$$

  - Where C represented by $(a,b,c,d,e,f)^T$.
- Apply direct linear transform (LSE method) to solve a set of homogeneous equations to get C.
- Make it ($C_\alpha^{*\prime}$) a rank 2 matrix using SVD on C.

Recovery of metric properties

- Compute H from $C_\alpha^{*\prime}$ upto similarity.
  - Matrix decomposition method $C_\propto^{*\prime} = H C_\propto^{*} H^T$

$$C_\propto^{*\prime} = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T$$

H

- Apply H^{-1} to the image.

Summary

- Various transformation under homography
  - $\mathbf{x}' = \mathbf{H}\mathbf{x}$
  - $\mathbf{l}' = \mathbf{H}^{-T}\,\mathbf{l}$
  - $\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$
  - $\mathbf{C}'^{*} = \mathbf{C}'^{-1} = \mathbf{H}\mathbf{C}^{-1}\mathbf{H}^{T}$
- Projective linear group, its subgroup and hierarchy
  - Projective linear group (8 D.O.F)
  - Affine group (6 D.O.F)
  - Euclidean group (4 D.O.F.)
  - Oriented Euclidean group (3 D.O.F)

Summary (contd.)

- Conic dual to circular points (C_α^*)
  - Invariant under similarity transform.
  - l_α is the zero (NULL) vector.
  - Preserves cosine of angle of two lines under transformation

$$\cos(\theta) = \frac{\boldsymbol{l}^T C_\infty^* \boldsymbol{m}}{\sqrt{(\boldsymbol{l}^T C_\infty^* \boldsymbol{l})(\boldsymbol{m}^T C_\infty^* \boldsymbol{m})}}$$

- Use of homography
  - Affine rectification
  - Stratification (recovery of metric properties)