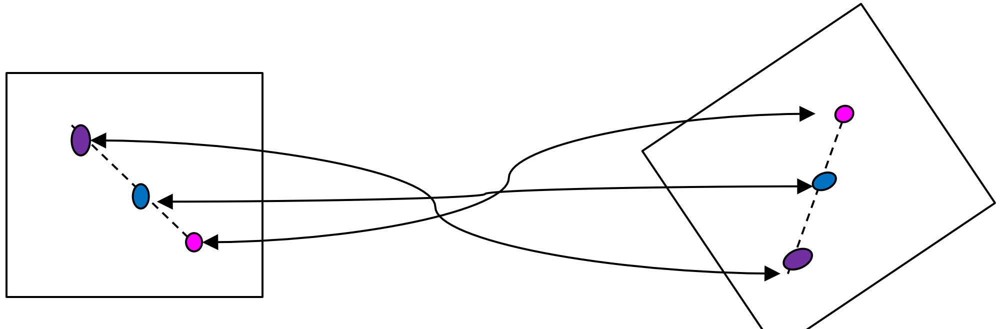
Projective Transformation Week-02 Lecture # 7

Jayanta Mukhopadhyay
Dept. of Computer Science and Engg.

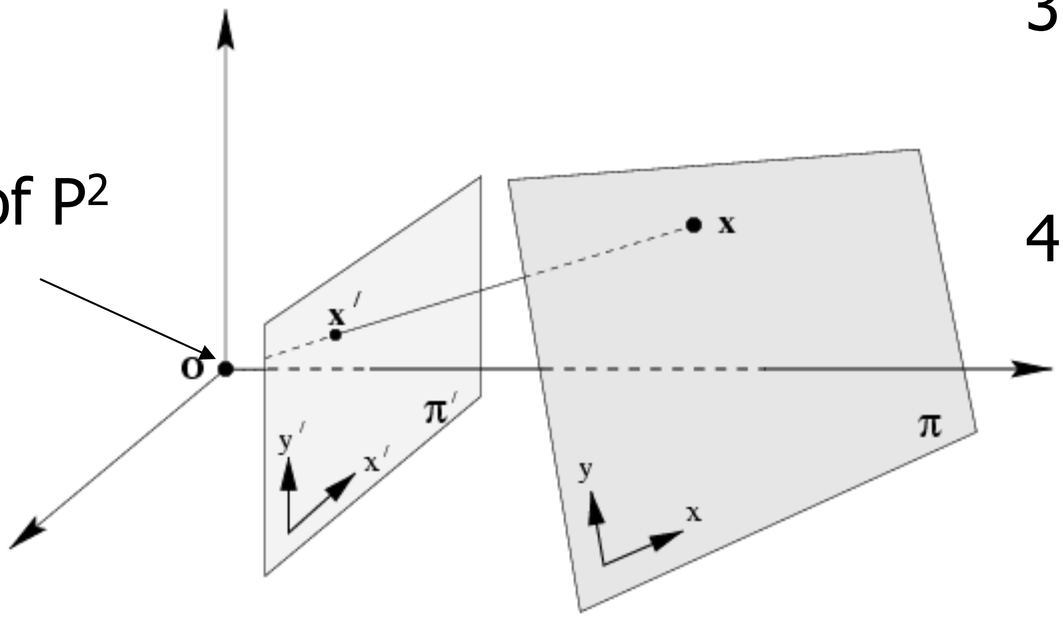
Projective transformation

- h: $P^2 \rightarrow P^2$
- Invertible
- Collinearity of every three points to be preserved, i.e. three points x_1, x_2, x_3 lie on the same line if and only if $h(x_1), h(x_2), h(x_3)$ do.



An example: change of coordinate convention

Origin of P^2 fixed.

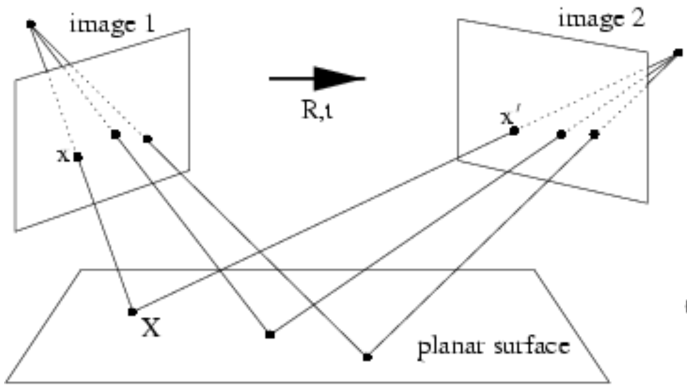


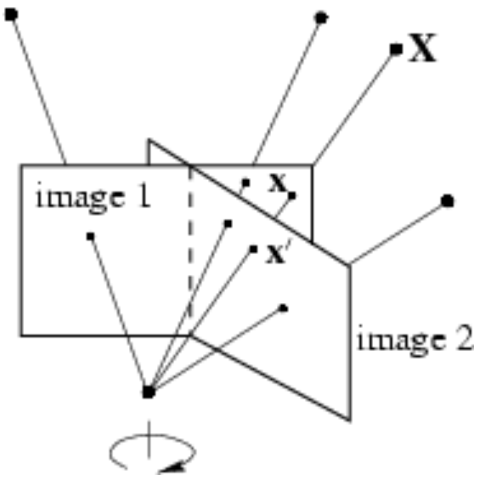
1. Rotation of axes.
2. Change of scale.
3. Translation of origin in planar coordinate system.
4. Use of Affine coordinate system in plane.

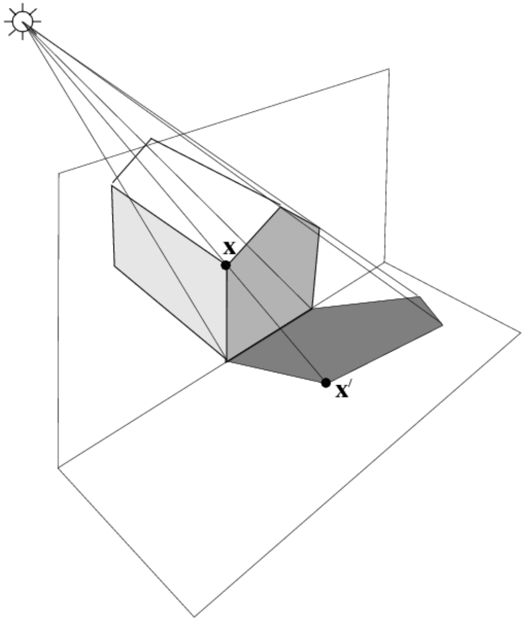
From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

More examples

Shift of origin of P^2







From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

Form of h

- Only one form possible.
- It is linear and invertible.

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

8DOF

$$\mathbf{X}' = \mathbf{H}\mathbf{X} \equiv k\mathbf{H}\mathbf{X}$$

Also called homography and **H** is the homography matrix.

Hx preserves collinearity.

- Let l be a line in P^2.
- A point $\mathbf{x}$ on l satisfies

$$l^T\mathbf{x}=\mathbf{0}$$

$$\rightarrow l^T\mathbf{H}^{-1}\mathbf{H}\mathbf{x}=\mathbf{0}$$

$$\rightarrow (\mathbf{H}^{-T}l)^T\mathbf{x}=\mathbf{0}$$

- $\mathbf{H}^{-T}l$ is the transformed line of l.

Harder to show that **H** is the only form of homography.

Implications

- If there is a homography, there exists a unique **H**, which is a 3x3 invertible matrix.
- Functional form known, so easier to estimate.
- **H** and k**H** are equivalent, where k is a scalar constant.
- Number of unknowns in **H** = 8.

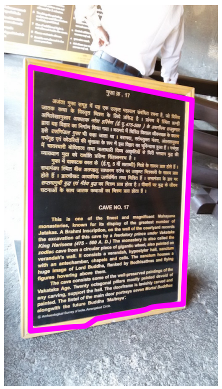
Estimation of $\mathbf{H}$

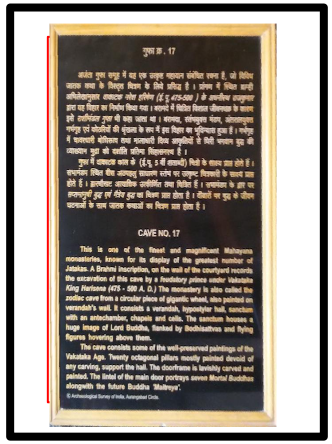
- Given point correspondences ($\mathbf{x_i}$, $\mathbf{x_i'}$) estimate $\mathbf{H}$ such that $\mathbf{x_i'}=\mathbf{Hx_i}$.
- There are 8 unknowns.
- $\mathbf{x'}=\mathbf{Hx}$ → Two independent equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

- Minimum 4 point correspondences needed.

Removing projective distortion





1. Select four points in a plane with known coordinates.
2. Form equations.

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

(linear in h_{ij})

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$

$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$

3. Setting h_{33} at *1* solve them.

Remark: no calibration at all necessary.
Does not work if $h_{33}=0$ in **H**.

Form equations

$$x'\left(h_{31}x+h_{32}y+h_{33}\right)=h_{11}x+h_{12}y+h_{13}$$

$$y'\left(h_{31}x+h_{32}y+h_{33}\right)=h_{21}x+h_{22}y+h_{23}$$

(51,791) → (1, 900)
(63, 143) → (1,1)
(444 ,211) → (501,1)
(426, 719) → (501,900)

$-51\ h_{11}\ -791\ h_{12} - h_{13} + 51h_{31} + 791h_{32} = -1$

$-51\ h_{21}\ -791\ h_{22} - h_{23} + 45900\ h_{31} + 711900\ h_{32} = -900$

$-63\ h_{11}\ -143\ h_{12} - h_{13} + 63h_{31} + 43\ h_{32} = -1$

$-63\ h_{21}\ -143\ h_{22} - h_{23} + 63\ h_{31} + 143\ h_{32} = -1$

$-444h_{11}\ -211h_{12} - h_{13} + 222444\ h_{31} + 105711h_{32} = -501$

$-444h_{21}\ -211\ h_{22} - h_{23} - 444\ h_{31} + 211\ h_{32} = -1$

$-426h_{11} -719\ h_{12} - h_{13} + 213426\ h_{31} + 360219\ h_{32} = -501$

$-426h_{21}\ -719\ h_{22} - h_{23} + 383400\ h_{31} + 647100\ h_{32} = -900$

In matrix form

$$\begin{bmatrix} -51 & -791 & -1 & 0 & 0 & 0 & 51 & 791 \\ 0 & 0 & 0 & -51 & -791 & 1 & 45900 & 711900 \\ -63 & -143 & -1 & 0 & 0 & 0 & 63 & 143 \\ 0 & 0 & 0 & -63 & -143 & -1 & 63 & 143 \\ -444 & -211 & -1 & 0 & 0 & 0 & 222444 & 105711 \\ 0 & 0 & 0 & -444 & -211 & -1 & 444 & 211 \\ -426 & -719 & -1 & 0 & 0 & 0 & 213426 & 360219 \\ 0 & 0 & 0 & -426 & -719 & -1 & 383400 & 647100 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} -1 \\ -900 \\ -1 \\ -1 \\ -501 \\ -1 \\ -501 \\ -900 \end{bmatrix}$$

Solve by matrix inversion.

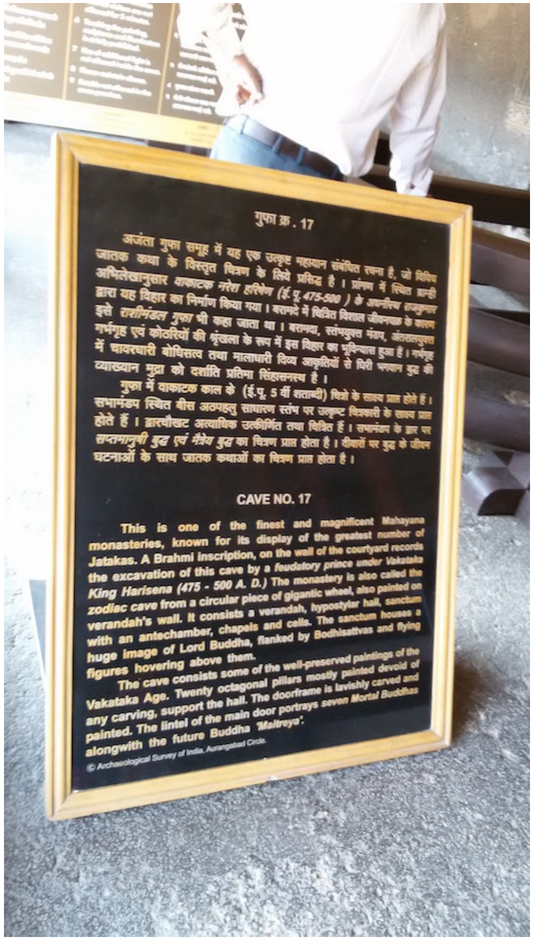
$h_{33}=1$

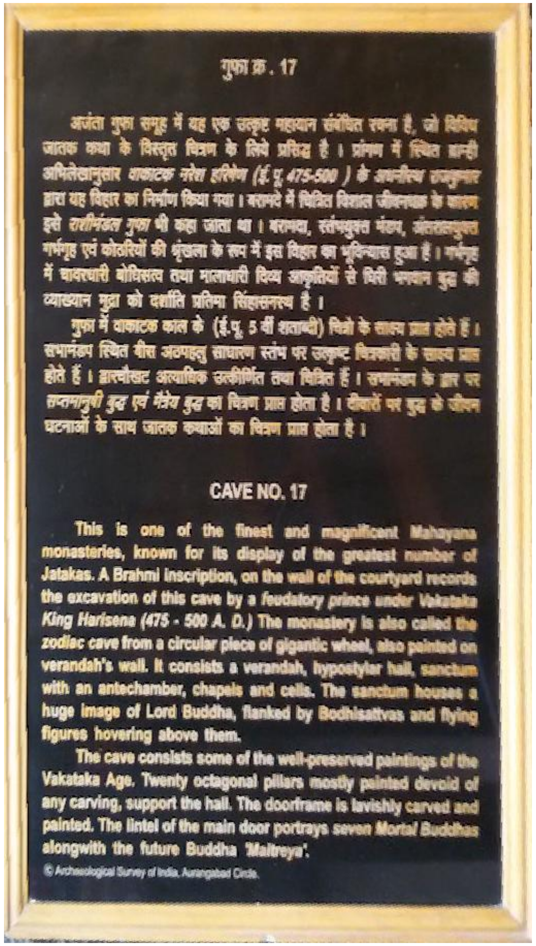
$$H = \begin{bmatrix} 0.9791 & 0.0181 & -63.3104 \\ -0.2303 & 1.2874 & -168.6295 \\ -0.0005 & -0.0001 & 1.0000 \end{bmatrix}$$

Solve equations

$$H = \begin{bmatrix} 0.9791 & 0.0181 & -63.3104 \\ -0.2303 & 1.2874 & -168.62955 \\ -0.0005 & -0.0001 & 1.0000 \end{bmatrix}$$

Apply homography





Direct Linear Transformation (DLT)

$$\mathbf{x}'_i = (x'_i, y'_i, w'_i)^\mathsf{T}$$

$$\mathbf{x}'_i = \mathrm{H}\mathbf{x}_i$$

$$\mathbf{x}'_i \times \mathrm{H}\mathbf{x}_i = 0$$

$$\mathrm{H}\mathbf{x}_i = \begin{pmatrix} \mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ \mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \end{pmatrix}$$

$$H = \begin{bmatrix} h^{1^T} \\ h^{2^T} \\ h^{3^T} \end{bmatrix}$$

$$\mathbf{x}'_i \times \mathrm{H}\mathbf{x}_i = \begin{pmatrix} y'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i - w'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i \\ w'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i - x'_i\mathbf{h}^{3\mathsf{T}}\mathbf{x}_i \\ x'_i\mathbf{h}^{2\mathsf{T}}\mathbf{x}_i - y'_i\mathbf{h}^{1\mathsf{T}}\mathbf{x}_i \end{pmatrix} = 0$$

Redundant: $x_i'(1)+y_i'(2)=(3)$

Direct Linear Transformation (DLT)

$$\begin{bmatrix} 0^{\mathsf{T}} & -w'_i \mathbf{x}_i^{\mathsf{T}} & y'_i \mathbf{x}_i^{\mathsf{T}} \\ w'_i \mathbf{x}_i^{\mathsf{T}} & 0^{\mathsf{T}} & -x'_i \mathbf{x}_i^{\mathsf{T}} \\ -y'_i \mathbf{x}_i^{\mathsf{T}} & x'_i \mathbf{x}_i^{\mathsf{T}} & 0^{\mathsf{T}} \end{bmatrix} \begin{pmatrix} \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \end{pmatrix} = 0$$

$\mathrm{A}_i \mathrm{h} = 0$ where $A_i = \begin{bmatrix} 0^T & -w'_i x_i^T & y'_i x_i^T \\ w'_i x_i^T & 0^T & -x'_i x_i^T \end{bmatrix}$

Dimension of A_i : 2 x 9.

Direct Linear Transformation (DLT): Non-homogeneous Equations

- Solving for H by setting h_{33}=1.

$$h = \begin{bmatrix} \tilde{h} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & -x_i w'_i & -y_i w'_i & -w_i w'_i & x_i y'_i & y_i y'_i \\ x_i w'_i & y_i w'_i & w_i w'_i & 0 & 0 & 0 & -x_i x'_i & -y_i x'_i \end{bmatrix} \tilde{h} = \begin{bmatrix} -w_i y'_i \\ w_i x'_i \end{bmatrix}$$

$$\tilde{A}_i \tilde{h} = b_i$$

$$A\tilde{h} = b$$

$$Minimize \left\| A\tilde{h} - b \right\|$$

Solution: $\tilde{h} = (A^T A)^{-1} A^T b$

Dimension of A: 2nx8

Rank: 8

Dimension of h: 8x1

Dimension of b: 2nx1

Caution: If h_{33}=0, no multiplication scale exists, and no solution obtained.

Direct Linear Transformation (DLT): Homogeneous Equations

- Solving for **H**: $\mathrm{Ah} = 0$

$$\mathrm{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

Dimension of A: 2nx9

Rank: 8

Dimension of h: 9x1

Dimension of Ah: 2nx1

Minimize $||Ah||$ such that $\|\mathrm{h}\| = 1$

Solution: Unit eigen vector of smallest eigen value of A^TA.

Other error criteria

- Algebraic error: Error term in DLT.
- Geometric error: $\sum d_e^2\left(\mathbf{x}', \mathbf{Hx}\right)$ Euclidean distance
- Geometric error with reprojection:

$$\sum\left(d_e^2\left(\mathbf{x}', \mathbf{Hx}\right) + d_e^2\left(\mathbf{H}^{-1}\mathbf{x}', \mathbf{x}\right)\right)$$

- Use of nonlinear iterative optimization techniques such as Newton iteration, Levenberg-Marquardt (LM) method, etc.

Transformation invariance and normalization

- Problem: To estimate **H** given a set of ($\mathbf{x_i}$, $\mathbf{x_i'}$) .
- Consider, $\mathbf{y_i}=\mathbf{Tx_i}$ and $\mathbf{y'_i}=\mathbf{T'x_i'}$ for known **T** and **T'**, which are invertible.
- Now estimate homography **G** from ($\mathbf{y_i}$, $\mathbf{y'_i}$).
- Can you estimate **H** from **G**?

$$
\begin{aligned}
\mathbf{x}' &= \mathbf{Hx} \\
\Rightarrow \mathbf{T}'^{-1}\mathbf{y}' &= \mathbf{HT}^{-1}\mathbf{y} \\
\Rightarrow \mathbf{y}' &= \underbrace{\mathbf{T}'\mathbf{HT}^{-1}}_{\mathbf{G}}\,\mathbf{y}
\end{aligned}
$$

Caution: For DLT it is not equivalent.
As the constraint ||g||=1 is not equivalent to ||h||=1.

Robust computation through Normalization of data

- Transform the point set so that its center becomes origin (in the plane) and avg. distance from it is $\sqrt{2}$.

$$x_i^{(n)} = \frac{x_i - \bar{x}}{\sigma_x} \qquad y_i^{(n)} = \frac{y_i - \bar{y}}{\sigma_y}$$

- Apply DLT on transformed point.
- Recover homography from the homography of transformed point sets.

Summary

- A projective transformation, invertible and preserving collinearity, is always in a linear form.
  - $\mathbf{x}'=\mathbf{H}\mathbf{x}$
  - $\mathbf{l}'=\mathbf{H}^{-T}\mathbf{l}$
- A computational problem: estimation of homography given a set of point correspondences.
  - Minimum 4 point correspondences needed.
  - Direct linear transformation (using LSE).
  - Use of linear transformation of points to make the computation robust.