Projective Geometry- Part II
Week 02 Lecture # 6

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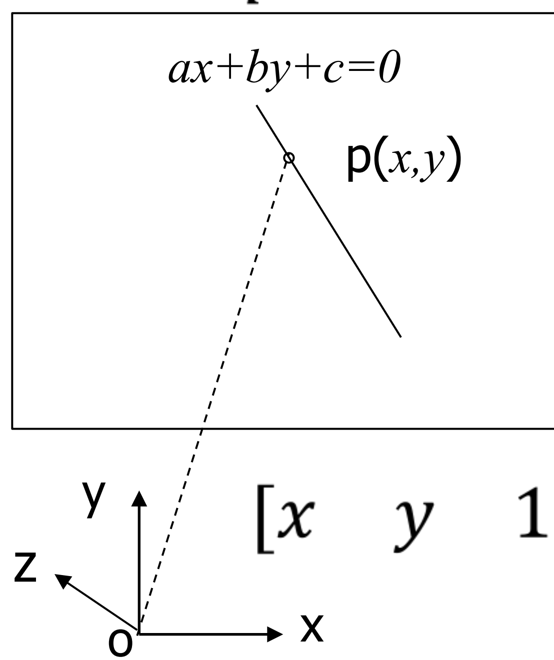
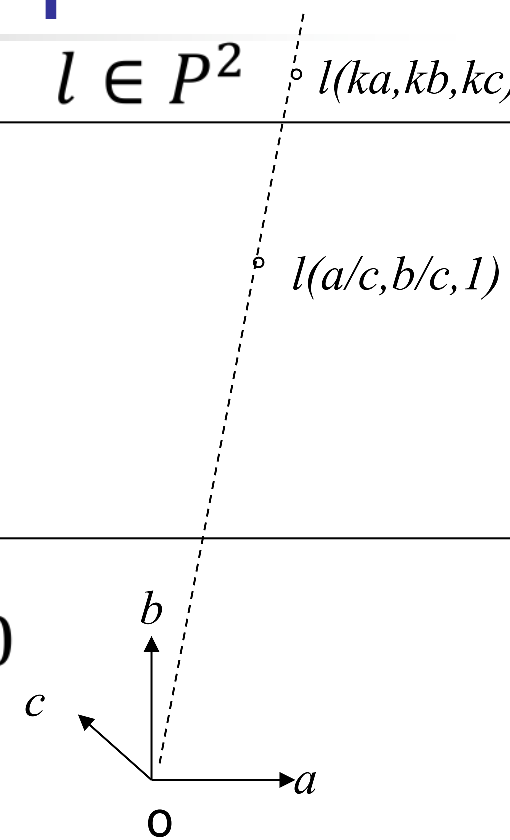
Points and lines in a plane

$p \in P^2$



$$[x \quad y \quad 1]\begin{bmatrix}a\\ b\\ c\end{bmatrix} = 0$$

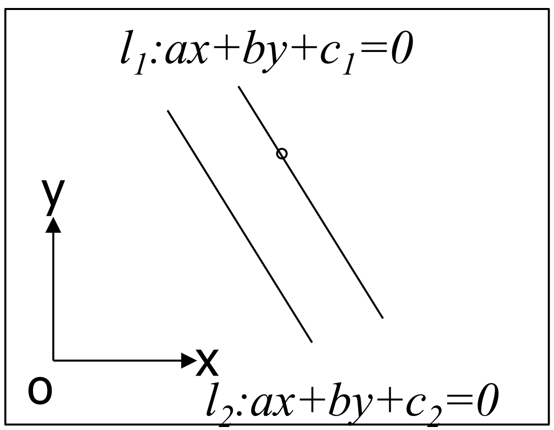
$l \in P^2$



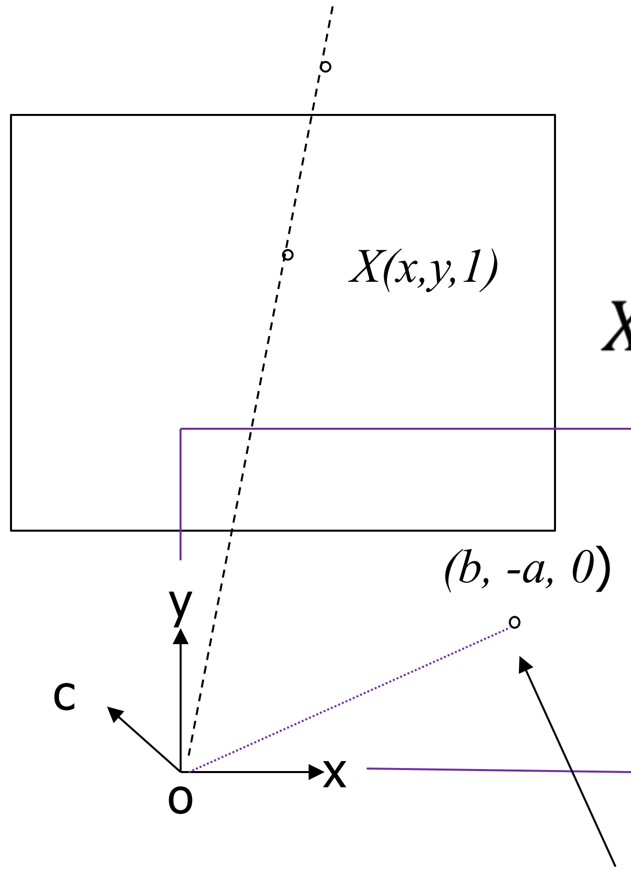
$$\mathrm{x}^T\mathrm{l} = 0 \longleftrightarrow \mathrm{l}^T\mathrm{x} = 0$$

$$\mathrm{x} = \mathrm{l} \times \mathrm{l}' \longleftrightarrow \mathrm{l} = \mathrm{x} \times \mathrm{x}'$$

Intersection of parallel lines



$$\vec{l}_1 \times \vec{l}_2 = (c_2 - c_1)\begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}$$



$X \in P^2$

Ideal Point

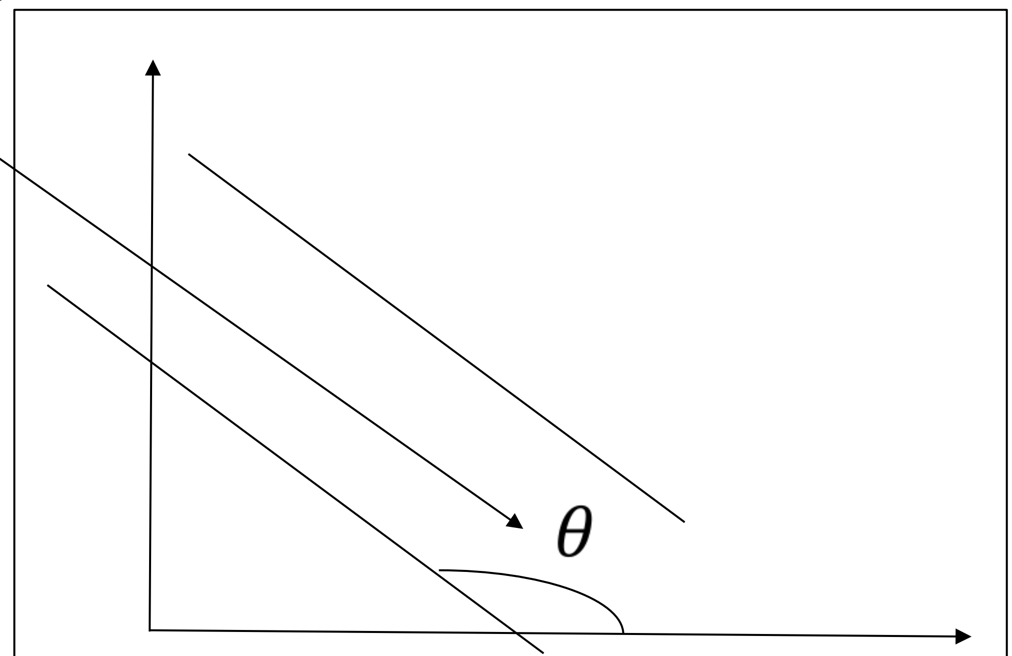
Meaning of an ideal point

$$ax + by + c = 0 \quad \Rightarrow \quad y = -\frac{a}{b}x + \frac{c}{b} = \tan(\theta)\, x + c'$$

Intersection point

$\downarrow$

$(b, -a, 0)$



A direction !

Ideal points

Ideal points: Points on the X-Y plane or principal plane parallel to projection plane.

For canonical coordinate system, they are of the form: $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

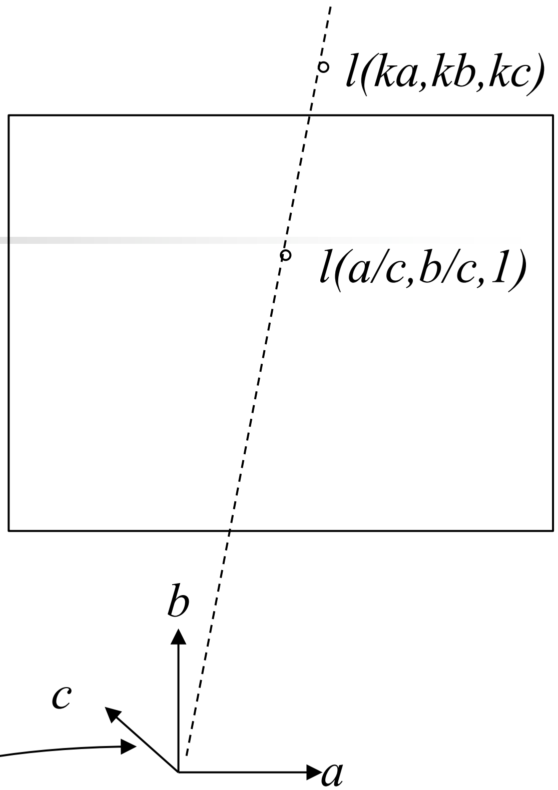
An ideal point denotes a direction toward infinity!

Line at infinity

$$[x \quad y \quad 0]\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$$

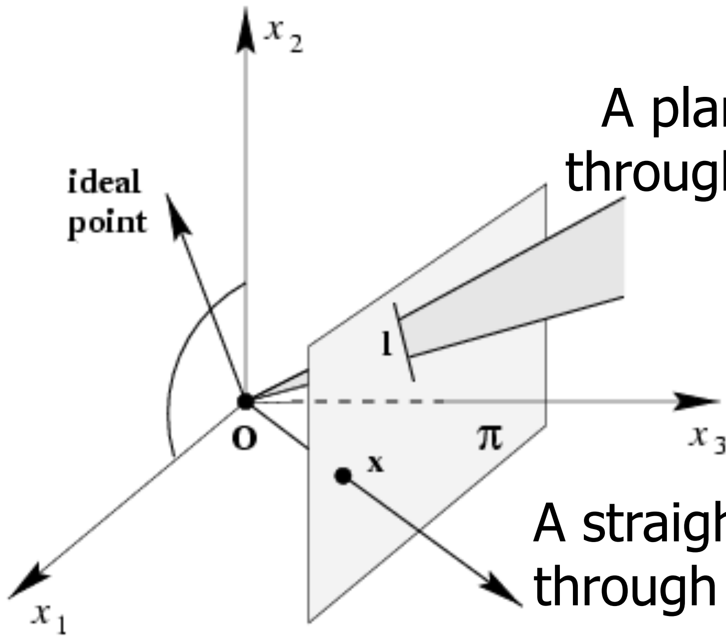
Any ideal point

$$\begin{bmatrix}0\\0\\1\end{bmatrix}$$



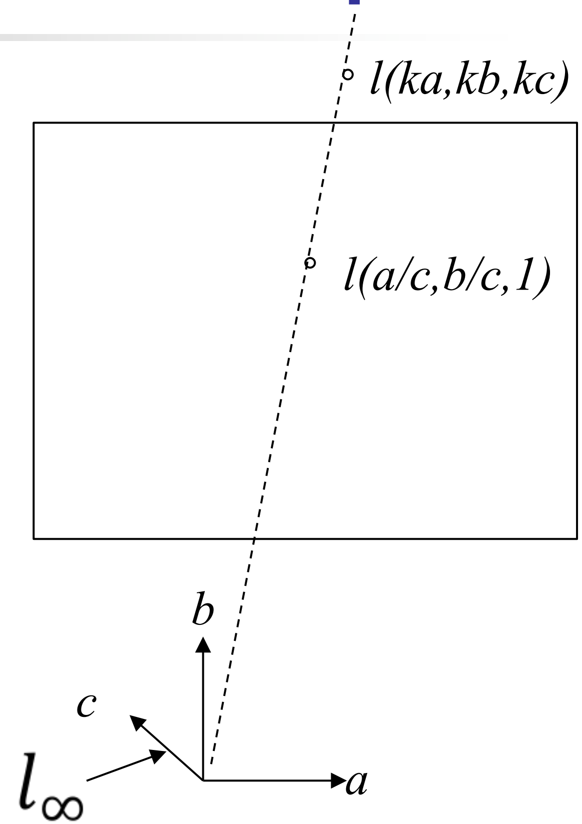
Line at infinity (l_∞): Line containing every ideal point.
In canonical system, it is $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

A model for the projective plane



A plane passing through the origin.

A straight line passing through the origin.



$$P^2 = R^3 - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = R^2 \cup l_\infty$$

From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press. (2000)

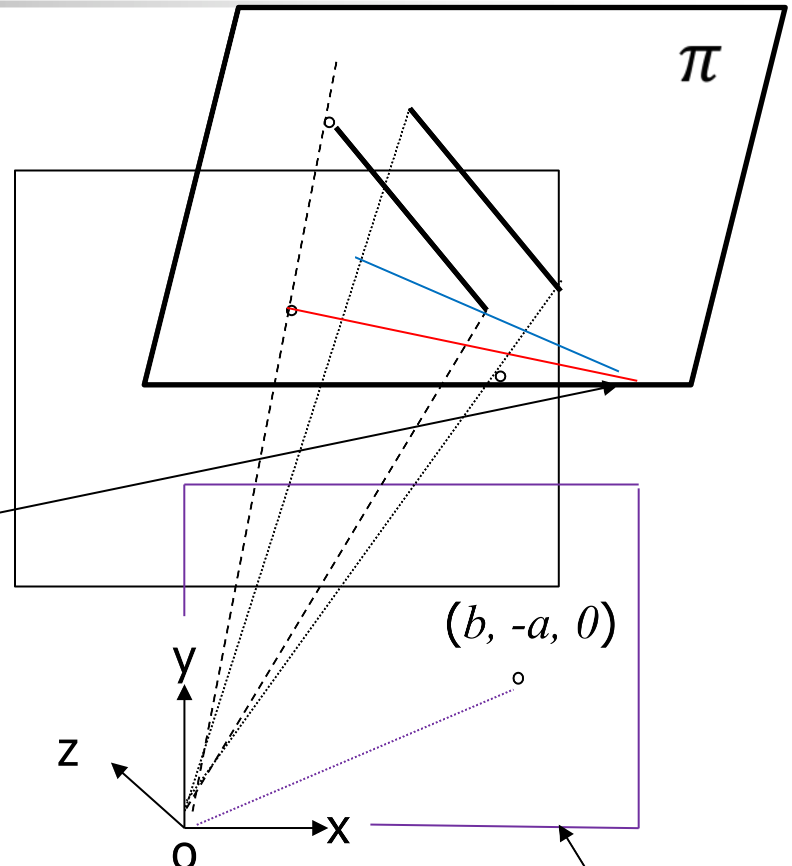
Projection of parallel lines from any arbitrary plane

Canonical projection plane (CPP)

Vanishing Point

⇩

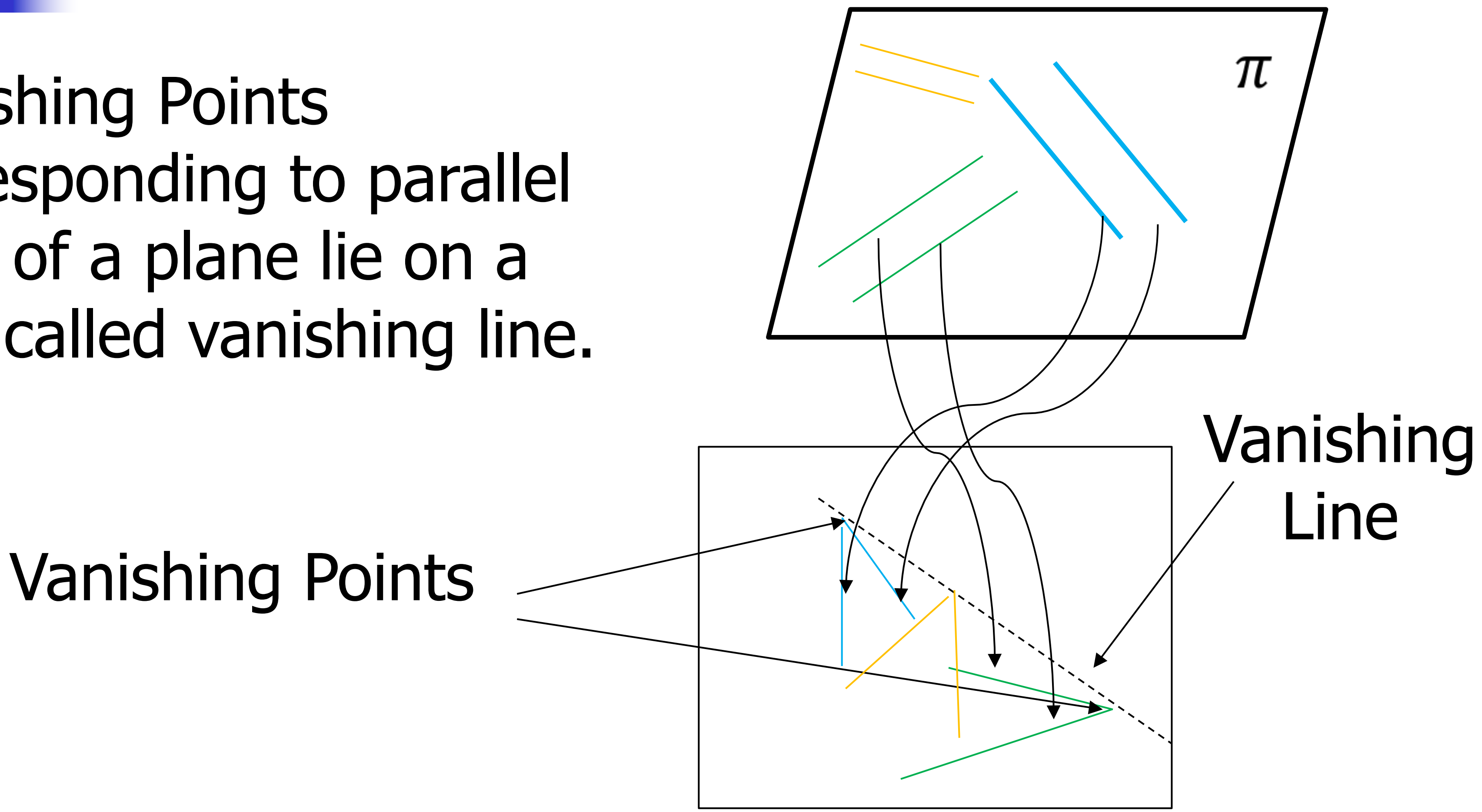
Point of intersection of parallel lines on π.



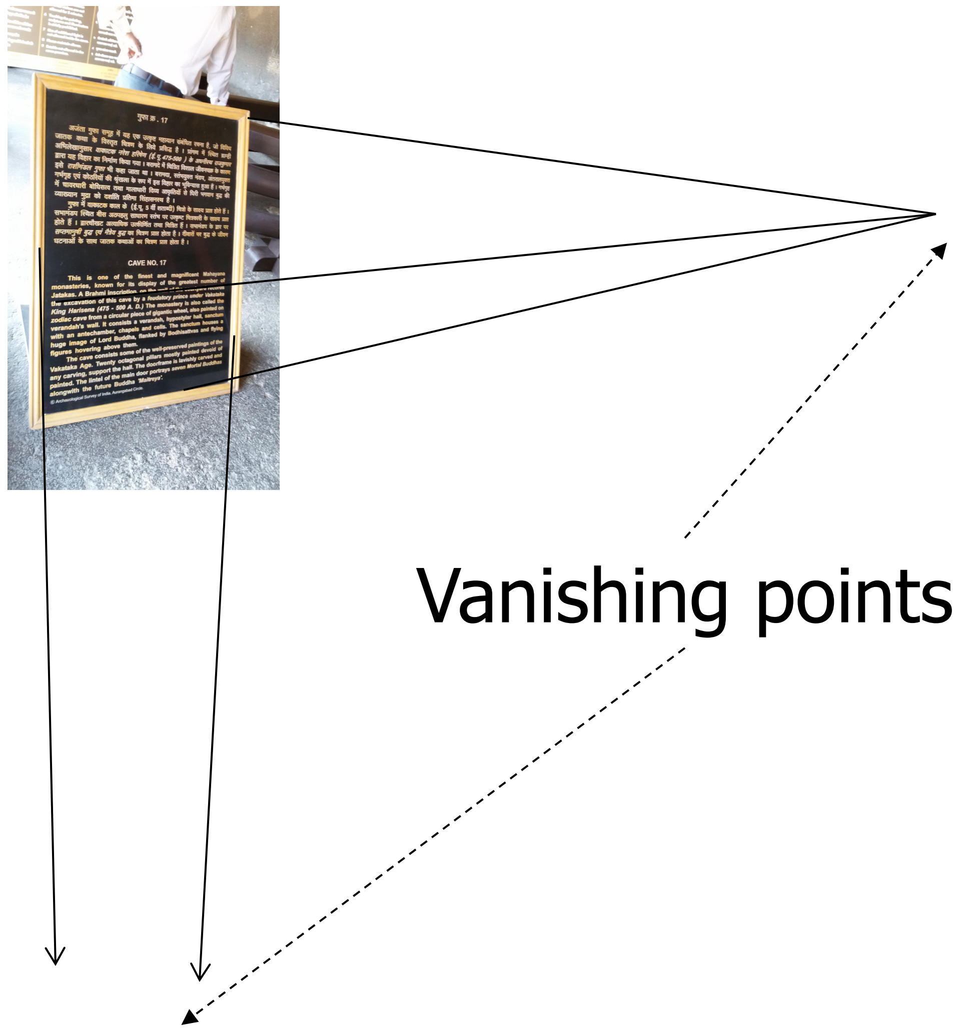
Ideal Plane

Vanishing points

Vanishing Points corresponding to parallel lines of a plane lie on a line, called vanishing line.



A real life example



Vanishing points

A journey toward infinity ….

Conics in P^2

- Curves described by 2nd degree equation in the plane.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

- A point in homogeneous coordinate: (x_1, x_2, x_3)

$$\rightarrow (x_1/x_3,\ x_2/x_3)$$

$$a\left(\frac{x_1}{x_3}\right)^2 + \mathrm{b}\left(\frac{x_1}{x_3}\right)\left(\frac{x_2}{x_3}\right) + \mathrm{c}\left(\frac{x_2}{x_3}\right)^2 + \mathrm{d}\left(\frac{x_1}{x_3}\right) + \mathrm{e}\left(\frac{x_2}{x_3}\right) + \mathrm{f} = 0$$

$$\Longrightarrow ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

Conics in P^2

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$
$$\Rightarrow X^T C X = 0$$

Where

$$C = \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix}$$

Conics identified by C with 5 d.o.f. ($a{:}b{:}c{:}d{:}e{:}f$)

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

$$\Rightarrow \quad \left(x_i^2, x_iy_i, y_i^2, x_i, y_i, f\right)\mathbf{c} = 0$$

$$\mathbf{c} = \left(a, b, c, d, e, f\right)^{\mathsf{T}}$$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

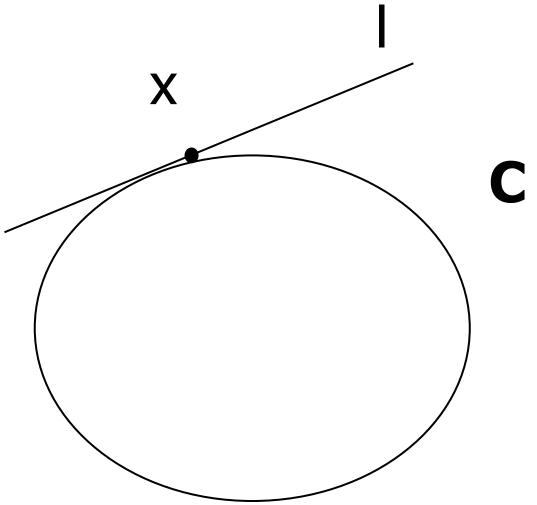
Stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

- Rank deficient **C**
  → degenerate conic
  two lines (of rank 2)
  a repeated line (of rank 1).

Tangent lines to conics

The line **l** tangent to **C** at point x on **C** is given by **l**=**C**x



From Hartley and Zisserman, “Multiple view geometry in computer vision”, Cambridge Univ. Press (2000)

Dual conics

A line tangent to the conic **C** satisfies $\mathrm{l}^{\mathrm{T}}\, \mathrm{C}^{*}\, \mathrm{l} = 0$

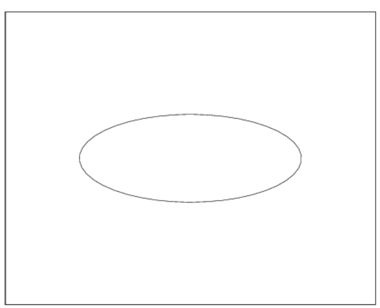
As **C** is symmetric, $\mathbf{C}^{-1}$

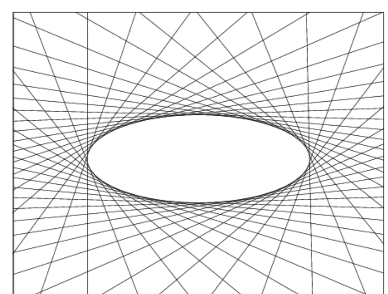
$$\mathbf{l} = \mathbf{C}\mathbf{x} \Rightarrow \mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{C}\mathbf{x} = \mathbf{0} \Rightarrow \left(\mathbf{C}^{-1}\mathbf{l}\right)^{\mathrm{T}}\mathbf{C}\left(\mathbf{C}^{-1}\mathbf{l}\right) = \mathbf{0}$$

$$\Rightarrow \mathbf{l}^{\mathrm{T}}\left(\mathbf{C}^{-1}\right)^{\mathrm{T}}\mathbf{C}\mathbf{C}^{-1}\mathbf{l} = \mathbf{0} \Rightarrow \mathbf{l}^{\mathrm{T}}\mathbf{C}^{-\mathrm{T}}\mathbf{l} = \mathbf{0}$$

Dual conics = line conics = conic envelopes





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Degenerate Conics

- Rank of C <3
- Rank 2 → Two lines / points
- Rank 1 → One repeated lines / points
- Degenerate point conic:

  $\mathbf{C}=\mathbf{l}.\mathbf{m}^T+\mathbf{m}.\mathbf{l}^T$ rank 2, if $\mathbf{l} <> \mathbf{m}$
- Degenerate dual line conic:

  $\mathbf{C}^*=\mathbf{x}.\mathbf{y}^T+\mathbf{y}.\mathbf{x}^T$ rank 2, if $\mathbf{x} <> \mathbf{y}$

- $\mathbf{x}^T\mathbf{l}=0$, and $\mathbf{l}^T\mathbf{x}=0$
- $\mathbf{x}=\mathbf{l} \times \mathbf{l}'$, and $\mathbf{l}=\mathbf{x} \times \mathbf{x}'$

Summary

- A point in a 2-D projective space represents a ray passing through origin of an implicit 3D space.
  - Requires additional dimension for representation.
    - Homogeneous Coordinate Representation
- Straight lines in R^2 are elements of a 2D projective space.
- Points and lines hold duality theorem.
- Conics are represented by a 3x3 symmetric matrix.
  - Every conic has a dual conic or line conic as an envelop of its tangents.