

# Image Transforms: A brief introduction

(Week 01- Lectures 3- 4)



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“Image and video processing in the compressed domain”, CRC Press, 2011.



# Image Transform

$$f(x, y) = \sum_j \sum_i \lambda_{ij} b_{ij}(x, y)$$

- Image in continuous form:  $f(x, y)$ : A 2-D function, where  $(x, y)$  in  $R^2$ .
  - Let  $B$  be a set of basis functions:
- Properties of basis functions can be extended in the analysis.

$$B = \{b_i(x, y) \mid i = \dots, -1, 0, 1, 2, 3, \dots\}, \quad b_i(x, y) \text{ in } R \text{ or } C.$$

- Let  $f(x, y)$  be expanded using  $B$  as follows:

$$f(x, y) = \sum_i \lambda_i b_i(x, y)$$

Coefficients of transform

The **transform** of  $f$  w.r.t.  $B$  is given by  $\{\lambda_i \mid i = \dots, -1, 0, 1, 2, 3, \dots\}$ .

Indexing may be multidimensional say,  $\lambda_{ij}$ .



# Orthogonal Expansion and 1-D Transforms

$$f(x) = \sum_i \lambda_i b_i(x)$$

- Inner product:  $\langle f, g \rangle = \int f(x)g^*(x)dx$
- Orthogonal expansion: If  $B$  satisfies :  
 $\langle b_i, b_j \rangle = 0, \text{ for } i \neq j$   
 $= c_i \text{ Otherwise (for } i = j), \text{ where } c_i > 0$
- Transform coefficients in O.E.:  $\lambda_i = \frac{1}{c_i} \langle f, b_i \rangle$
- If  $c_i = 1$ , it becomes orthonormal expansion.  
**Forward transform**  $\lambda_i = \langle f, b_i \rangle$
- **Inverse transform:**  $f(x) = \int_{i=-\infty}^{\infty} \lambda_i b_i(x) di$



# Fourier transform

Complete base

$$B = \{e^{-j\omega x} \mid -\infty < \omega < \infty\}$$

Unit impulse function

Orthogonality:  $\int_{-\infty}^{\infty} e^{j\omega x} dx = \begin{cases} 2\pi\delta(x), & \text{for } \omega = 0 \\ 0, & \text{otherwise.} \end{cases}$

Fourier Transform:  $\mathcal{F}(f(x)) = \hat{f}(j\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx$

Inverse Transform:  $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega) e^{j\omega x} d\omega$

Full reconstruction  $e^{-j\omega x} = \cos(\omega x) - j \sin(\omega x)$

$$\hat{f}(j\omega) = \int_{-\infty}^{\infty} f(x)(\cos(\omega x) - j \sin(\omega x)) dx$$

$$C = \{\cos(\omega x) \mid -\infty < \omega < \infty\} \quad S = \{\sin(\omega x) \mid -\infty < \omega < \infty\}$$

Orthogonal

But not complete!



# Even and odd functions

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

- Even:  $f(-x)=f(x)$  for all  $x$ .
- Odd:  $f(-x)=-f(x)$  for all  $x$ .  $\rightarrow f(0)=0$ .
- For even  $f(x)$  :  $\int_{-\infty}^{\infty} f(x)(\sin(\omega x)) dx = 0$
- For odd  $f(x)$  :  $\int_{-\infty}^{\infty} f(x)(\cos(\omega x)) dx = 0$
- Full reconstruction possible with cosines (sines) only if it is even (odd).



# Discrete representation

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- Discrete representation of a function:

$$f(n) = \{f(nX_0) | n \in \mathbb{Z}\}$$

Set of integers

Sampling interval

- Can be considered as a vector in an infinite dimensional vector space.
- In our context, it is of a finite dimensional space, e.g.  $\{f(n), n=0, 1, \dots, N-1\}$ , or
- $f = [f(0) \ f(1) \ \dots \ f(N-1)]^T$ .



# Discrete Linear Transform: A general form

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- For  $n$ -dimensional vector  $X$  any linear transform,
  - e.g.  $Y_{m \times 1} = B_{m \times n} X_{n \times 1}$
  - $X_{n \times 1}$ : *A column vector of dimension  $n$ .*
  - $Y_{m \times 1}$ : *A column vector of dimension  $m$ .*
  - $B_{m \times n}$ : *A matrix of dimension  $m \times n$ .*
- Has inverse transform if  $B$  is a square matrix and invertible.



# Basis vectors

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- $B$  is the transformation matrix.
- Rows of  $B$  are called basis vectors.

$$B = \begin{bmatrix} \mathbf{b}_0^{*T} \\ \mathbf{b}_1^{*T} \\ \vdots \\ \mathbf{b}_n^{*T} \end{bmatrix}$$

- $Y(i) = \langle \mathbf{b}_i^{*T} \cdot X \rangle$

↖ dot product or inner product.

- Orthogonality condition:

$$\begin{aligned} \langle \mathbf{b}_i^{*T} \cdot \mathbf{b}_j \rangle &= 0 \text{ if } i \neq j \\ &= c_i, \quad \text{otherwise} \end{aligned}$$



# Discrete Fourier Transform (DFT)

$$b_k(n) = \frac{1}{\sqrt{N}} e^{j2\pi \frac{k}{N} n}, \text{ for } 0 \leq n \leq N-1, \text{ and } 0 \leq k \leq N-1.$$

$$\hat{f}(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi \frac{k}{N} n} \text{ for } 0 \leq k \leq N-1. \quad \hat{f}(N+k) = \hat{f}(k)$$

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}(k) e^{j2\pi \frac{k}{N} n} \text{ for } 0 \leq n \leq N-1.$$

k/N: Normalized frequency

A single period



Fundamental  
frequency:  $1/(NX_0)$

$$f(n+N) = f(n)$$

DFT: Fourier series of a periodic function

# DFT: A linear transform

$$F(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi \frac{kn}{N}} \quad \text{for } 0 \leq k \leq N-1$$

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-j2\pi \frac{1}{N}} & \dots & e^{-j2\pi \frac{N-1}{N}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi \frac{N-1}{N}} & \dots & e^{-j2\pi \frac{(N-1)^2}{N}} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

$$\mathcal{F}_N = [e^{-j2\pi \frac{k}{N}n}]_{0 \leq (k,n) \leq N-1}$$

$$\mathbf{F} = \mathcal{F}_N \mathbf{f}$$

$$\mathbf{f} = \mathcal{F}_N^{-1} \mathbf{F}$$

Hermitian transpose

$$\mathcal{F}_N^{-1} = \mathcal{F}_N^H$$

# Generalized Discrete Fourier Transform (GDFT)

$$\mathbf{F}_{\alpha,\beta} = \left[ e^{-j2\pi \frac{k+\alpha}{N}(n+\beta)} \right]_{0 \leq (k,n) \leq N-1}$$

$$\begin{aligned} \mathbf{F}_{0,0}^{-1} &= \frac{1}{N} \mathbf{F}_{0,0}^H = \frac{1}{N} \mathbf{F}_{0,0}^*, \\ \mathbf{F}_{\frac{1}{2},0}^{-1} &= \frac{1}{N} \mathbf{F}_{\frac{1}{2},0}^H = \frac{1}{N} \mathbf{F}_{0,\frac{1}{2}}^*, \\ \mathbf{F}_{0,\frac{1}{2}}^{-1} &= \frac{1}{N} \mathbf{F}_{0,\frac{1}{2}}^H = \frac{1}{N} \mathbf{F}_{\frac{1}{2},0}^*, \text{ and} \\ \mathbf{F}_{\frac{1}{2},\frac{1}{2}}^{-1} &= \frac{1}{N} \mathbf{F}_{\frac{1}{2},\frac{1}{2}}^H = \frac{1}{N} \mathbf{F}_{\frac{1}{2},\frac{1}{2}}^*. \end{aligned}$$

$$b_k^{(\alpha,\beta)}(n) = \frac{1}{\sqrt{N}} e^{j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \leq n \leq N-1, \text{ and } 0 \leq k \leq N-1$$

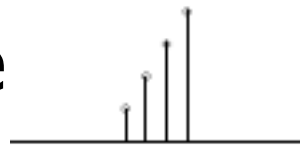
$$\hat{f}_{\alpha,\beta}(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \leq k \leq N-1$$

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \hat{f}_{\alpha,\beta}(k) e^{j2\pi \frac{k+\alpha}{N}(n+\beta)}, \text{ for } 0 \leq n \leq N-1$$

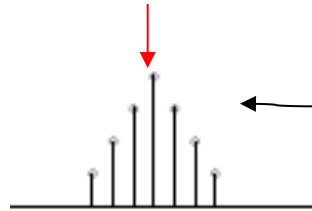
$\alpha$	$\beta$	Transform name	Notation
0	0	Discrete Fourier Transform ( <i>DFT</i> )	$\hat{f}(k)$
0	$\frac{1}{2}$	Odd Time Discrete Fourier Transform ( <i>OTDFT</i> )	$\hat{f}_{0,\frac{1}{2}}(k)$
$\frac{1}{2}$	0	Odd Frequency Discrete Fourier Transform ( <i>OFDFT</i> )	$\hat{f}_{\frac{1}{2},0}(k)$
$\frac{1}{2}$	$\frac{1}{2}$	Odd Frequency Odd Time Discrete Fourier Transform ( <i>O<sup>2</sup>DFT</i> )	$\hat{f}_{\frac{1}{2},\frac{1}{2}}(k)$

# Symmetric / Antisymmetric extension of a finite sequence

Original sequence

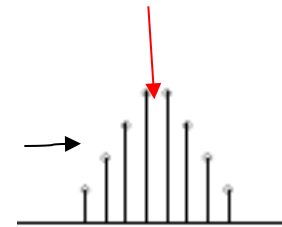


DCTs and DSTs exist for any finite sequence.

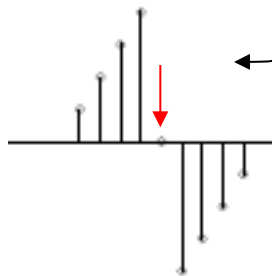


Whole symmetry (WS)

Even function

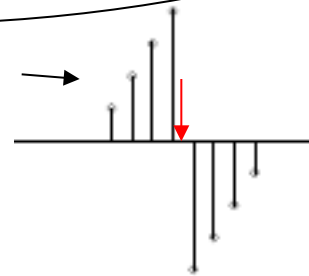


Half symmetry (HS)



Whole antisymmetry (WA)

Odd function



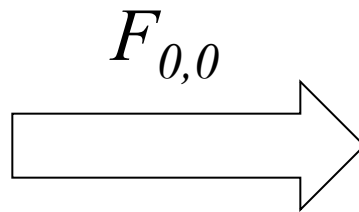
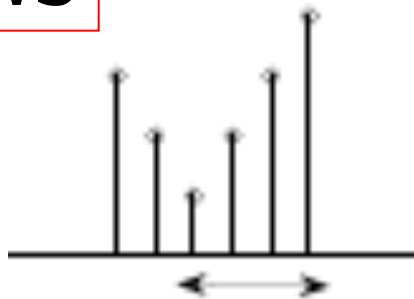
Half antisymmetry (HA)

# Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

- Types of symmetric / antisymmetric extensions at the two ends of a sequence and a type of GDFT  $\rightarrow$  DCTs / DSTs

WSWS



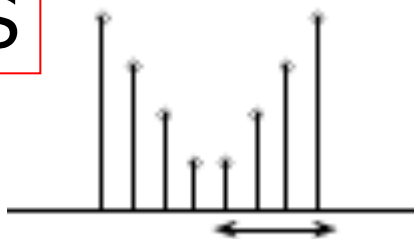
Type-I Even DCT

$$C_{1e}(x(n)) = X_{Ie}(k) = \sqrt{\frac{2}{N}} \alpha^2(k) \sum_{n=0}^N x(n) \cos\left(\frac{2\pi nk}{2N}\right), \quad 0 \leq k \leq N,$$

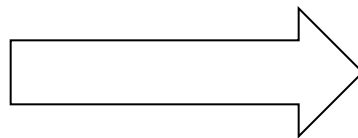
# Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

HSHS



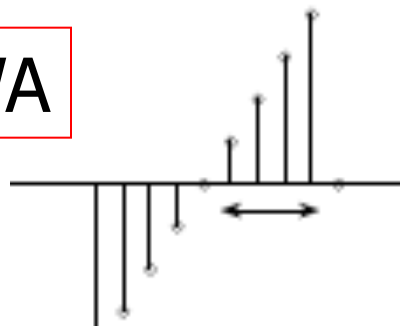
$F_{0,1/2}$



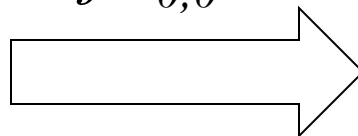
Type-2 Even DCT

$$C_{2e}(x(n)) = X_{IIe}(k) = \sqrt{\frac{2}{N}} \alpha(k) \sum_{n=0}^{N-1} x(n) \cos \left( \frac{2\pi k(n + \frac{1}{2})}{2N} \right), \quad 0 \leq k \leq N-1$$

WAWA



$jF_{0,0}$



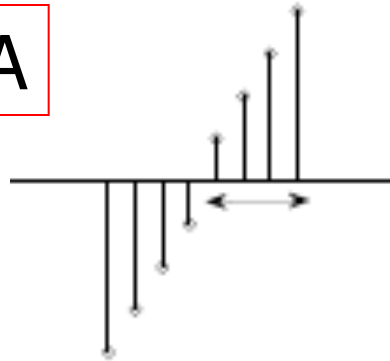
Type-1 Even DST

$$S_{1e}(x(n)) = X_{sIe}(k) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N-1} x(n) \sin \left( \frac{2\pi kn}{2N} \right), \quad 1 \leq k \leq N-1$$

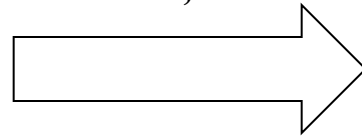
# Discrete Cosine / Sine Transforms

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

HAHA



$jF_{0,1/2}$



Type-2 Even DST

$$S_{2e}(x(n)) = X_{sIIe}(k) = \sqrt{\frac{2}{N}} \alpha(k) \sum_{n=0}^{N-1} x(n) \sin \left( \frac{2\pi k(n + \frac{1}{2})}{2N} \right), \quad 1 \leq k \leq N-1$$

There exist 16 different types of DCTs and DSTs. Type-II Even DCT is used in signal, image, and video compression.



# Matrix form of Type-II DCT

$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

- Matrix form

N-point DCT

$$C_N = \left[ \sqrt{\frac{2}{N}} \cdot \alpha(k) \cos\left(\frac{\pi k(2n+1)}{2N}\right) \right]_{0 \leq (k,n) \leq N-1}.$$

- Each row is either symmetric (even row) or antisymmetric (odd row).

$$C_N(k, N-1-n) = \begin{cases} C_N(k, n) & \text{for } k \text{ even} \\ -C_N(k, n) & \text{for } k \text{ odd} \end{cases}$$

$$X = C_N \cdot x$$

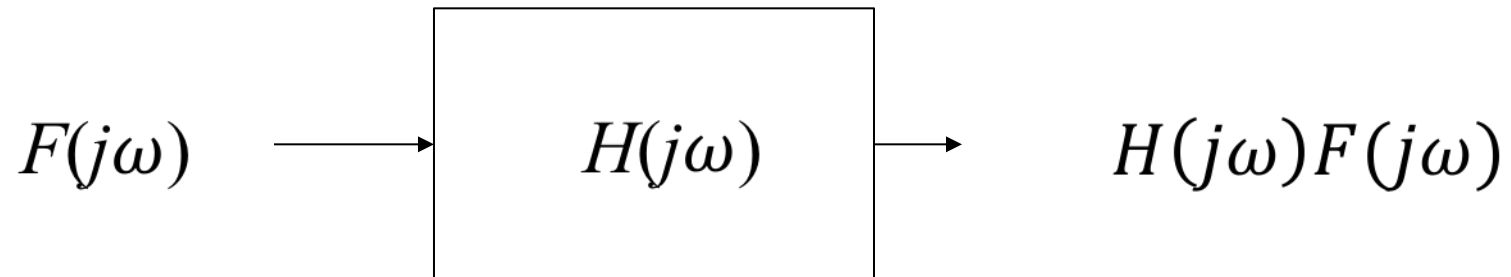
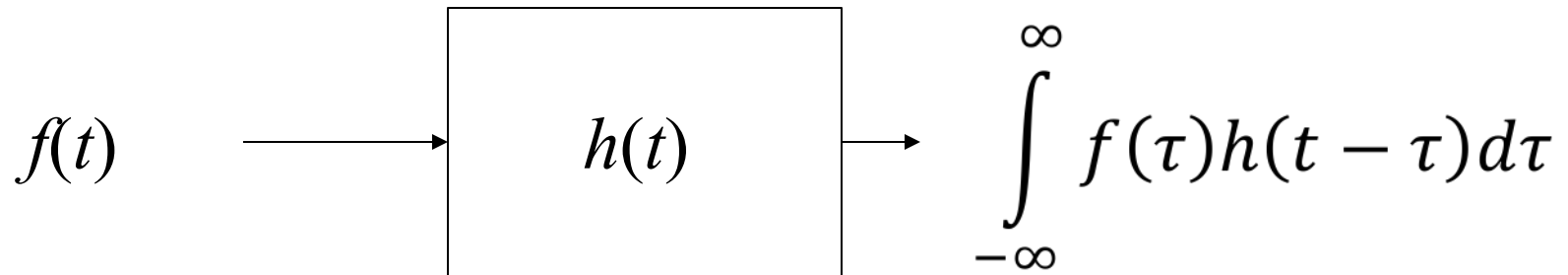
$$C_N^{-1} = C_N^T$$





# Convolution Multiplication Property (CMP)

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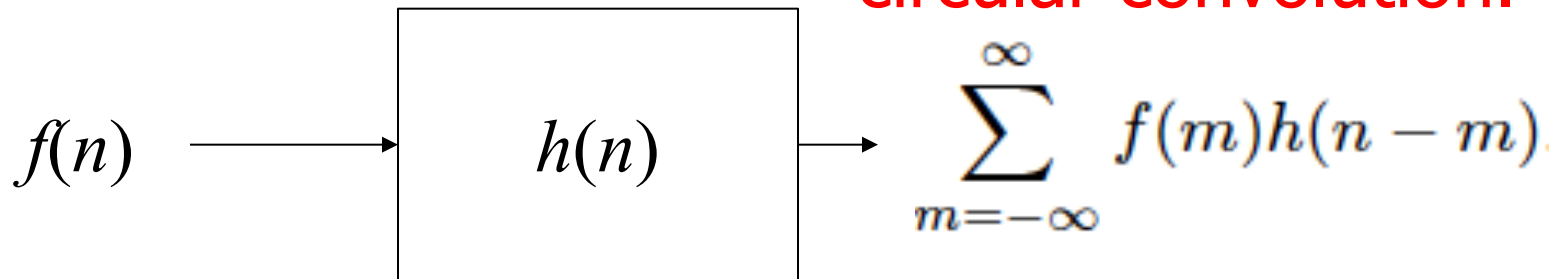


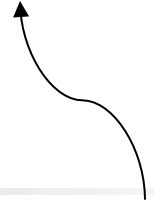
CMP for Fourier Transform



# CMP for DFT

## Linear convolution



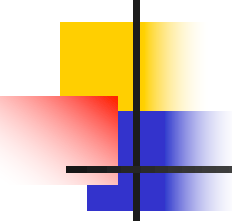
$$\widehat{f \otimes h}(k) = \hat{f}(k)\hat{h}(k)$$


CMP for DFT holds for circular convolution.

- Periodic convolution: Convolution between two finite sequences with periodic extension.
- It is defined if both have the same period, providing a periodic sequence with the same period.

## Circular Convolution

$$\begin{aligned} f \circledast h(n) &= \sum_{m=0}^{N-1} f(m)h(n-m), \\ &= \sum_{m=0}^n f(m)h(n-m) + \sum_{m=n+1}^{N-1} f(m)h(n-m+N). \end{aligned}$$



# Antiperiodic extension and skew-circular convolution

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- Antiperiodic function with an antiperiod  $N$ , if  $f(x+N)=-f(x)$ .
- An antiperiodic function of antiperiod  $N \rightarrow$  a periodic function of period  $2N$ .
- Skew-circular convolution: convolution between two antiperiodic extended sequences of the same antiperiod.

$$\begin{aligned} f \circledast h(n) &= \sum_{m=0}^{N-1} f(m)h(n-m), \\ &= \sum_{m=0}^n f(m)h(n-m) - \sum_{m=n+1}^{N-1} f(m)h(n-m+N) \end{aligned}$$



## CMPs for DCTs

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$$u(n) = x(n) \circledast y(n)$$

$$w(n) = x(n) \circledcirc y(n)$$

$$C_{1e}(u(n)) = \sqrt{2N} C_{1e}(x(l)) C_{1e}(y(m))$$

$$C_{2e}(u(n)) = \sqrt{2N} C_{2e}(x(l)) C_{1e}(y(m))$$

$$C_{3e}(w(n)) = \sqrt{2N} C_{3e}(x(l)) C_{3e}(y(m))$$

$$f(x, y) = \sum_j \sum_i \lambda_{ij} b_{ij}(x, y)$$

## 2-D Transforms

- Easily extendable if basis functions are separable, i.e.  $B = \{ b_{ij}(x, y) = g_i(x) \cdot g_j(y) \}$ .

They could be from two different sets, say  $b(x, y) = g(x) \cdot h(y)$ .

1-D basis function

- $B$ : Orthogonal if  $G = \{ g_i(x), i=1, 2, \dots \}$  is orthogonal.
- $B$ : Orthogonal and complete if  $G$  is so.
- Reuse of 1-D transform computation.

$$\lambda_{ij} = \sum_j g_j^*(y) \left( \sum_i f(x, y) g_i^*(x) \right)$$



# 2D Discrete Transform

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$$Y_{m \times n} = B_{m \times m} X_{m \times n} B_{n \times n}^T$$

- Use of separability:
  - Transform columns.
  - Transform rows.
- Input:  $X_{m \times n}$       1-D Transform Matrix:  $B$
- Transform columns:  $[Y_1]_{m \times n} = B_{m \times m} X_{m \times n}$
- Transform rows:  $Y_{m \times n} = [B_{n \times n} Y_1^T]^T$ 
$$= Y_1 B_{n \times n}^T$$
$$= B_{m \times m} X_{m \times n} B_{n \times n}^T$$



# Image Transform: DFT

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Image:  $f(m, n)$ , of size  $M \times N$

$$F(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi \frac{km}{M}} e^{-j2\pi \frac{ln}{N}}$$

$$f(m, n) = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F(k, l) e^{j2\pi \frac{km}{M}} e^{j2\pi \frac{ln}{N}}$$

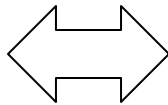
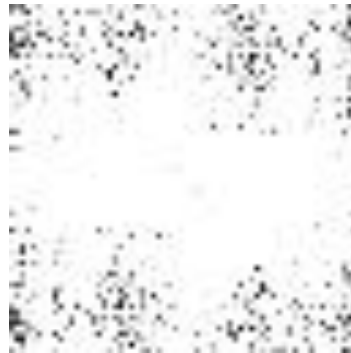
Property of separability

$$\mathbf{F} = \mathcal{F}_m \mathbf{f} \mathcal{F}_N^T$$

$$F(k, l) = \sum_{m=0}^{M-1} e^{-j2\pi \frac{km}{M}} \sum_{n=0}^{N-1} f(m, n) e^{-j2\pi \frac{ln}{N}}$$

# DFT Examples:

Magnitude



Phase



Magnitudes and phases are shown by bringing them into displayable range, and shifting the origin at the center of image.





## 2D DCT

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$$\alpha(p) = \begin{cases} \sqrt{\frac{1}{2}}, & \text{for } p = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases}$$

- Type-I:

$$X_I(k, l) = \frac{2}{N} \cdot \alpha^2(k) \cdot \alpha^2(l) \cdot \sum_{m=0}^M \sum_{n=0}^N (x(m, n) \cos(\frac{m\pi k}{M}) \cos(\frac{n\pi l}{N})), \\ 0 \leq k \leq M, 0 \leq l \leq N.$$

- Type-II

$$X_{II}(k, l) = \frac{2}{N} \cdot \alpha(k) \cdot \alpha(l) \cdot \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (x(m, n) \cos(\frac{(2m+1)\pi k}{2M}) \cos(\frac{(2n+1)\pi l}{2N})), \\ 0 \leq k \leq M-1, 0 \leq l \leq N-1.$$

- Matrix Representation:

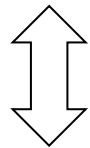
$$X = DCT(x) = C_M \cdot x \cdot C_N^T$$



# An example:

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Input image



Discrete Cosine Transform



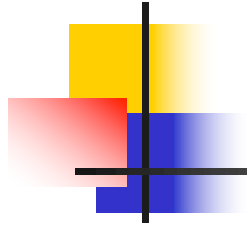
There are 16 different types of DCTs and DSTs.



# Why image transforms?

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- Alternative representation provides other insights of structure of images.
  - low frequency and high frequency components.
- May become useful for providing more compact representation.
  - A few transform coefficients.
  - Selective quantization of components, considering their effect on our perception.
    - Image compression.
- Sometimes convenient for processing.
  - Filtering, enhancement, ....



Thank you!