



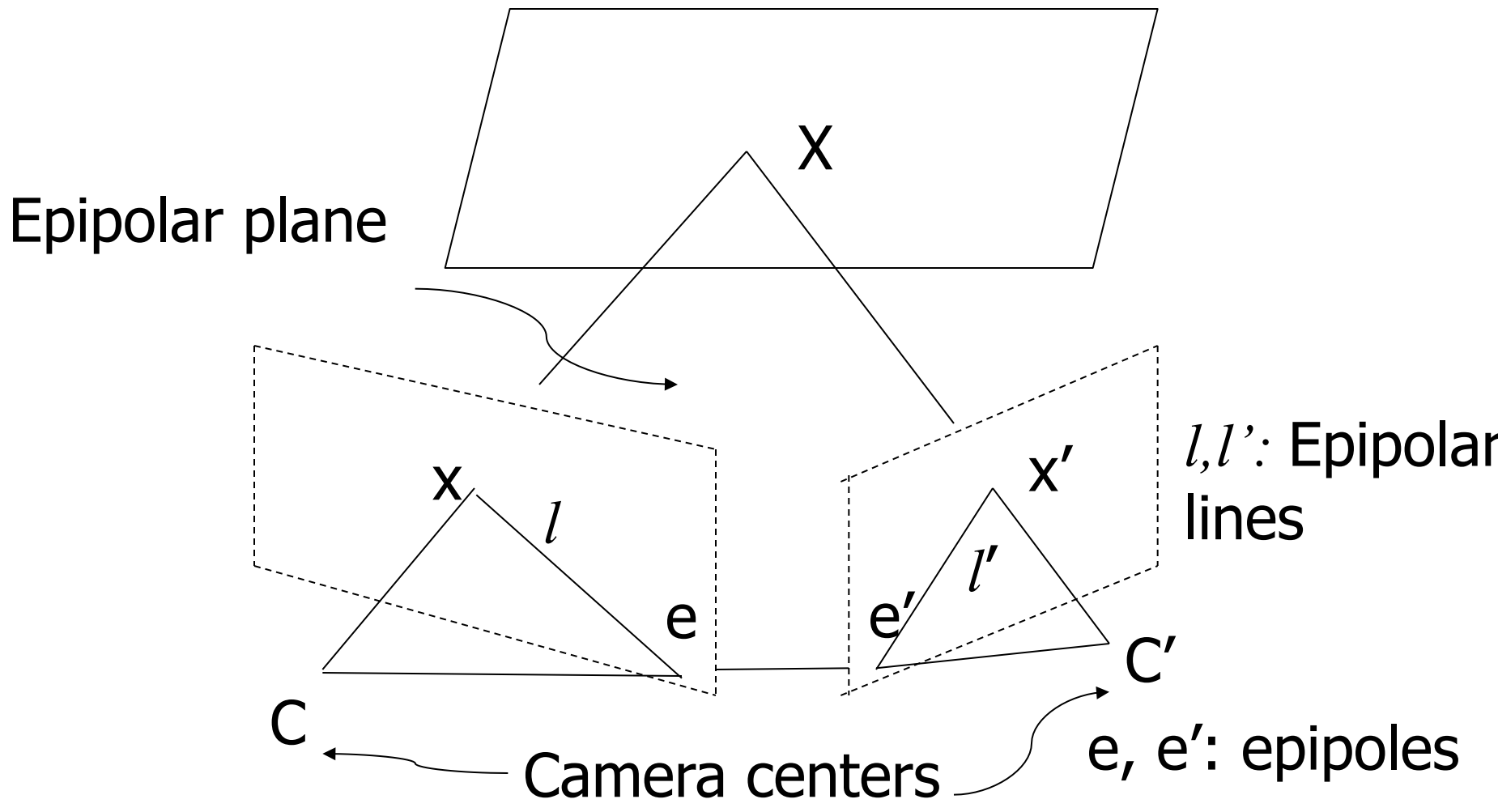
# Stereo Geometry

## (Week 04-05 Lectures: 16-23)

---

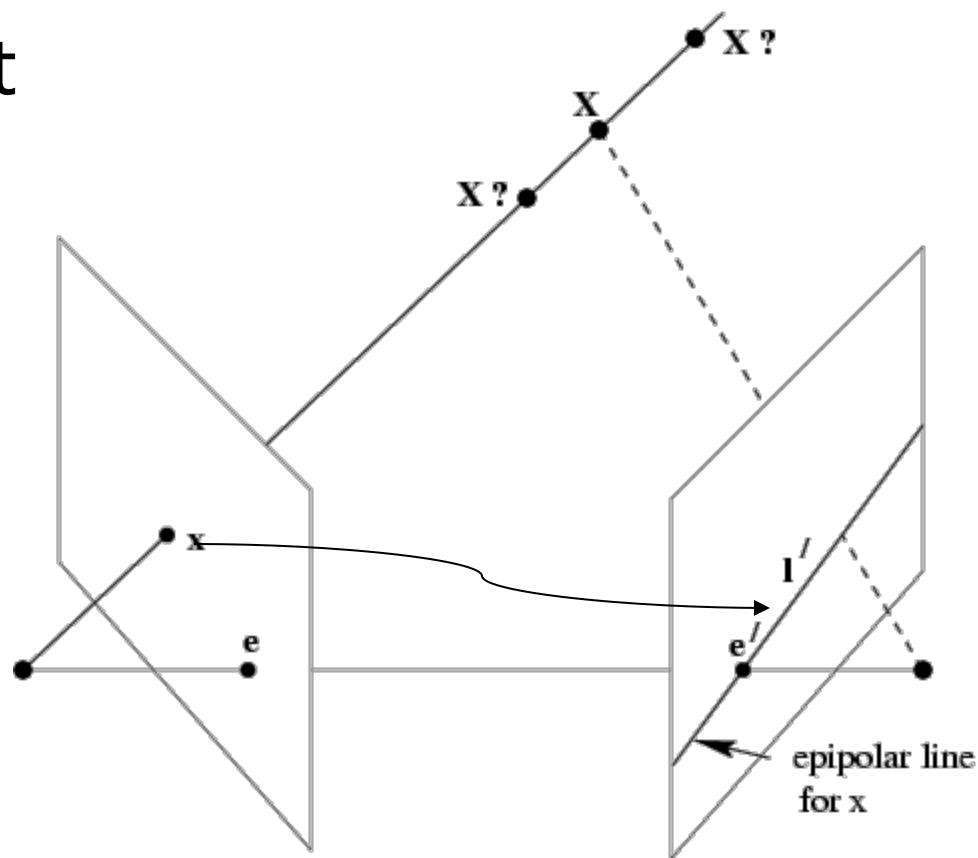
**Jayanta Mukhopadhyay**  
**Dept. of Computer Science and Engg.**

# Stereo Set-up



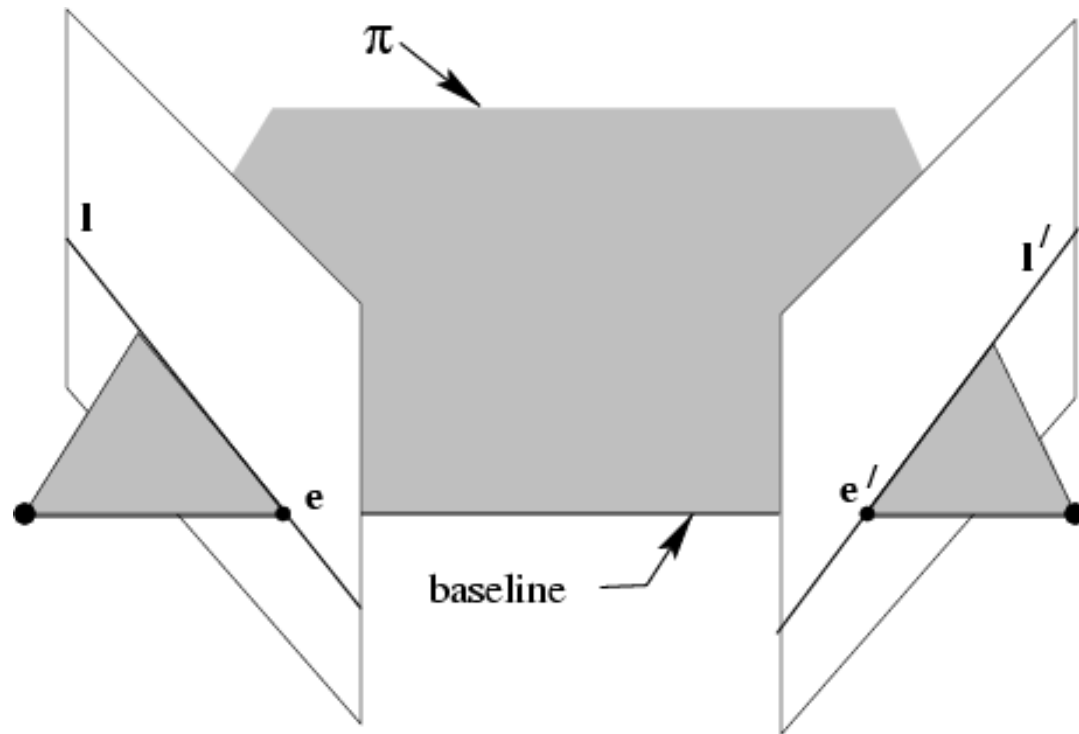
# Epipolar geometry

Corresponding point of  $x$  in the right image lies on  $l'$ .



From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

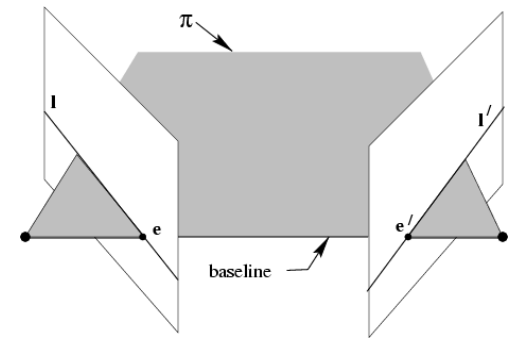
# Epipolar geometry



All points on  $\pi$  project on  $l$  and  $l'$

From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

# Epipolar geometry



Epipoles  $e, e'$

= intersection of baseline with image plane

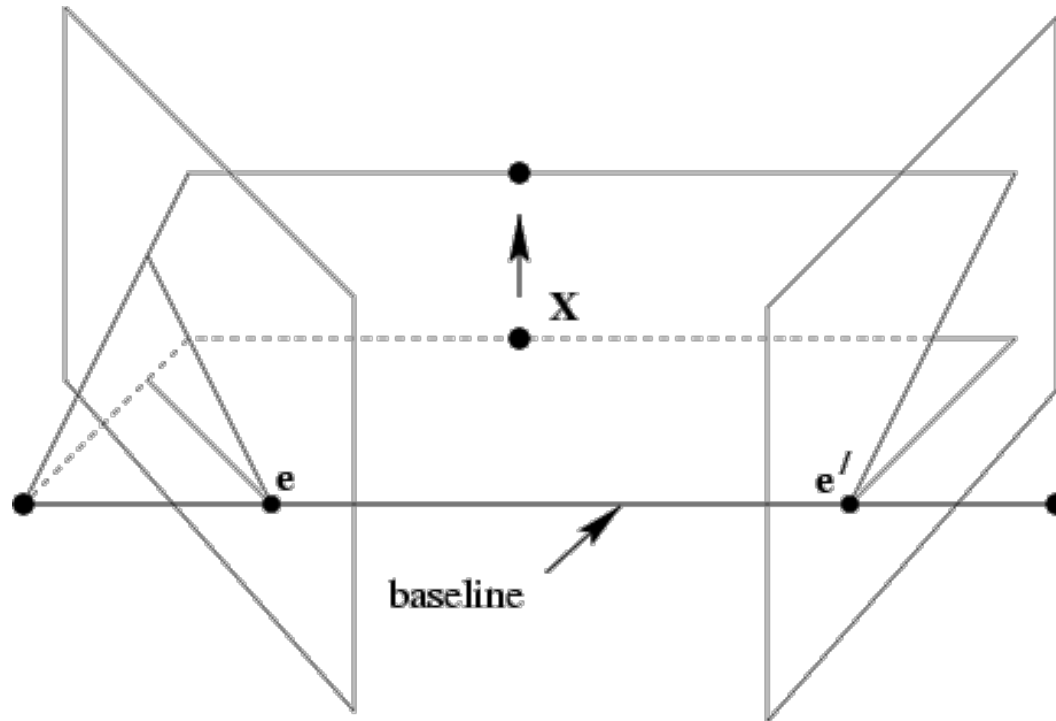
= projection of projection center in other image

= vanishing point of camera motion direction

An epipolar plane: plane containing baseline (1-D family)

An epipolar line: intersection of epipolar plane with image (always come in corresponding pairs)

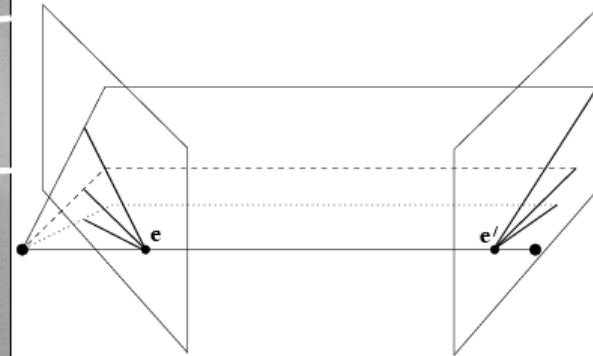
# Epipolar geometry



Family of planes  $\pi$  and lines  $l$  and  $l'$   
Intersection in  $e$  and  $e'$

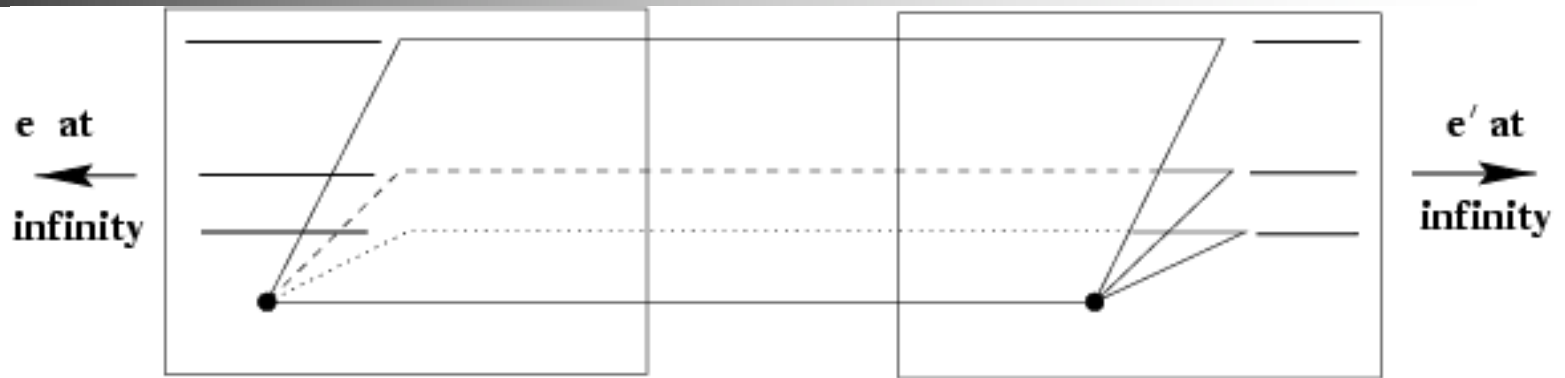
From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

# Example: converging cameras



From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

# Example: motion parallel with image plane



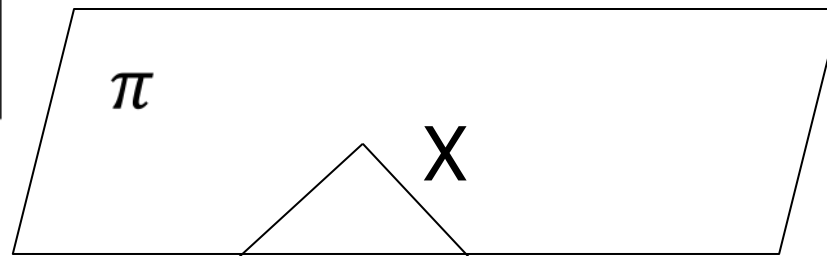
From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)



# Epipolar Geometry

$$[e']_{\times} = \begin{bmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{bmatrix}$$

$$H_{\pi} = [K'R | Kt] \begin{bmatrix} K^{-1} \\ 0^T \end{bmatrix} \\ = K'RK^{-1}$$

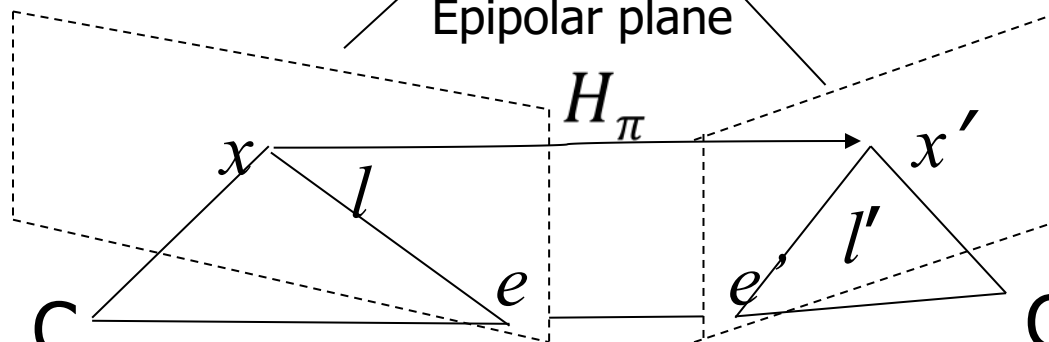


$$x' = P'X \\ = P'P^+x$$

$$P^+ = \begin{bmatrix} K^{-1} \\ 0^T \end{bmatrix}$$

For scene points lying on plane at infinity.

$$l' = e' \times x' \\ = [e']_{\times} x' \\ = [e']_{\times} H_{\pi} x \\ = Fx$$



$$P = K[I | 0]$$

$$P' = K'[R | t]$$

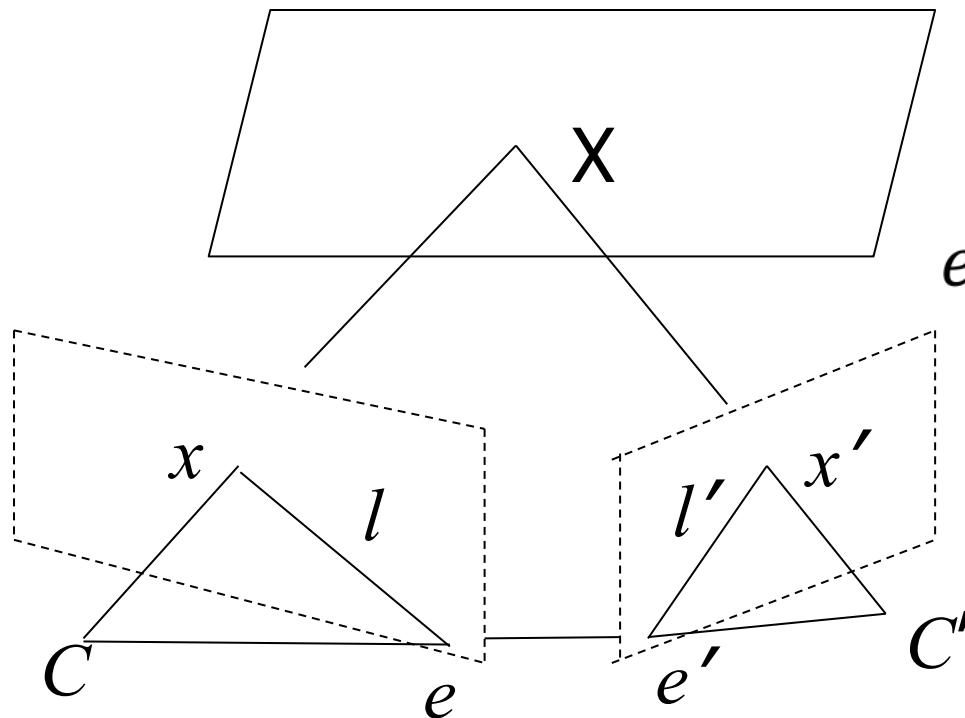
$$F = [e']_{\times} K'RK^{-1}$$

Coplanar:  $X, x, x', C, C', e, e', l, l'$

# Epipolar Geometry

Fundamental  
Matrix:  $F$

$$l' = Fx$$



$$e = PC'$$

$$e' = P'C$$

$$e = -KR^T t \equiv KR^T t$$

$$e' = K't$$

$$l' = Fx$$

$$\Rightarrow x'^T l' = 0$$

$$\Rightarrow x'^T Fx = 0$$

$$\Rightarrow x^T F^T x' = 0$$

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = K[I \mid 0]$$

$$P' = K'[R \mid t]$$

$$C' = \begin{bmatrix} -R^T t \\ 1 \end{bmatrix}$$

# Fundamental and Essential Matrices

$$P = K[I \mid 0]$$

$$P' = K'[R \mid t]$$

$$F = [e']_{\times} P' P^+ \\ = [K' t]_{\times} K' R K^{-1}$$

Say,  $P = [I \mid 0]$ , i.e.  $K = I$

$$P' = K'[R \mid t] = [K'R \mid K't] = [M \mid m]$$

$$F = [m]_{\times} M$$

for  $K = I$  and  $K' = I$ ,

$$F = [t]_{\times} R$$

Essential Matrix ( $E$ )



## Ex. 1

---

- Consider the following stereo imaging matrices given by  $P = [I|0]$  (ref. camera) and  $P'$  as follows.

$$P' = \begin{bmatrix} 3 & 4 & 6 & 4 \\ 8 & 7 & 2 & 8 \\ 1 & 5 & 2 & 1 \end{bmatrix}$$

Compute the fundamental matrix of the system.



Ans. 1

---

$$P' = \begin{bmatrix} 3 & 4 & 6 & 4 \\ 8 & 7 & 2 & 8 \\ 1 & 5 & 2 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 3 & 4 & 6 \\ 8 & 7 & 2 \\ 1 & 5 & 2 \end{bmatrix} \quad m = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} \quad [m]_{\times} = \begin{bmatrix} 0 & -1 & 8 \\ 1 & 0 & -4 \\ -8 & 4 & 0 \end{bmatrix}$$

$$\begin{aligned} F &= [m]_{\times} M \\ &= \begin{bmatrix} 0 & 33 & 14 \\ -1 & -16 & -2 \\ 8 & -4 & -40 \end{bmatrix} \end{aligned}$$



## Ex. 2

---

- Consider the following stereo imaging matrices given by  $P$  (ref. camera) and  $P'$ .

$$P = \begin{bmatrix} 3 & 2 & 4 & -2 \\ 8 & 6 & 0 & 4 \\ 9 & 5 & 7 & 3 \end{bmatrix}$$

$$P' = \begin{bmatrix} 3 & 8 & 5 & 2 \\ 2 & 7 & 6 & -3 \\ 6 & 4 & 9 & 8 \end{bmatrix}$$

- Compute the fundamental matrix of the system.
- Given an image point (15,20) of the reference camera ( $P$ ), compute the epipolar line and its two end image points of  $P'$ .

Ans. 2 (a)

$$P = \begin{bmatrix} 3 & 2 & 4 & -2 \\ 8 & 6 & 0 & 4 \\ 9 & 5 & 7 & 3 \end{bmatrix} \quad P' = \begin{bmatrix} 3 & 8 & 5 & 2 \\ 2 & 7 & 6 & -3 \\ 6 & 4 & 9 & 8 \end{bmatrix}$$

$\underbrace{\quad\quad\quad}_M \quad \underbrace{\quad}_p_4 \qquad \underbrace{\quad\quad\quad}_{M'}$

$$\tilde{C} = -M^{-1}p_4$$

$$M^{-1} = -\frac{1}{42} \begin{bmatrix} 42 & 6 & -24 \\ -56 & -15 & 32 \\ -14 & 3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} \tilde{C} \\ 1 \end{bmatrix} = \frac{1}{42} \begin{bmatrix} -132 \\ 148 \\ 46 \\ 1 \end{bmatrix}$$

$$e' = P'C = \begin{bmatrix} 26.23 \\ 21.94 \\ 13.09 \end{bmatrix} = 13.09 \begin{bmatrix} 2 \\ 1.67 \\ 1 \end{bmatrix}$$

$$H_{\alpha} = M'M^{-1} \Rightarrow H_{\alpha} = \begin{bmatrix} 9.33 & 2.07 & -4.62 \\ 9.33 & 1.78 & -4.48 \\ 2.33 & -0.07 & -0.05 \end{bmatrix}$$

$$F = [e']_{\times} H_{\alpha}$$

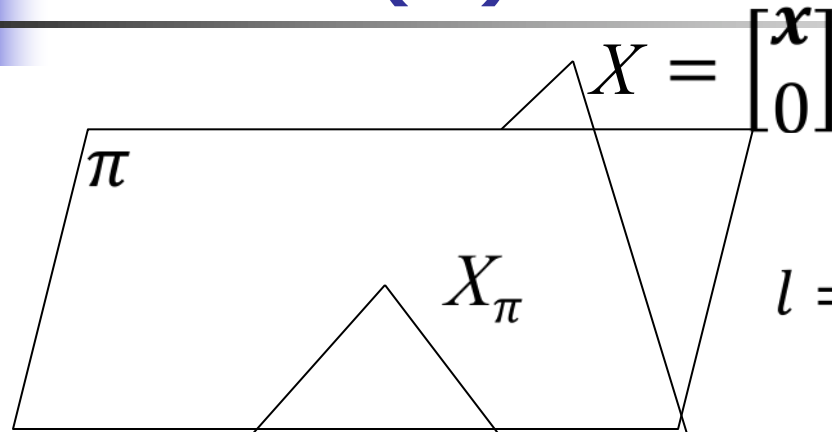
$$[e']_{\times} = \begin{bmatrix} 0 & -1 & 1.67 \\ 1 & 0 & -2 \\ -1.67 & 2 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -5.41 & -1.91 & 4.40 \\ 4.67 & 2.21 & -4.52 \\ 2.99 & 0.09 & -1.19 \end{bmatrix}$$

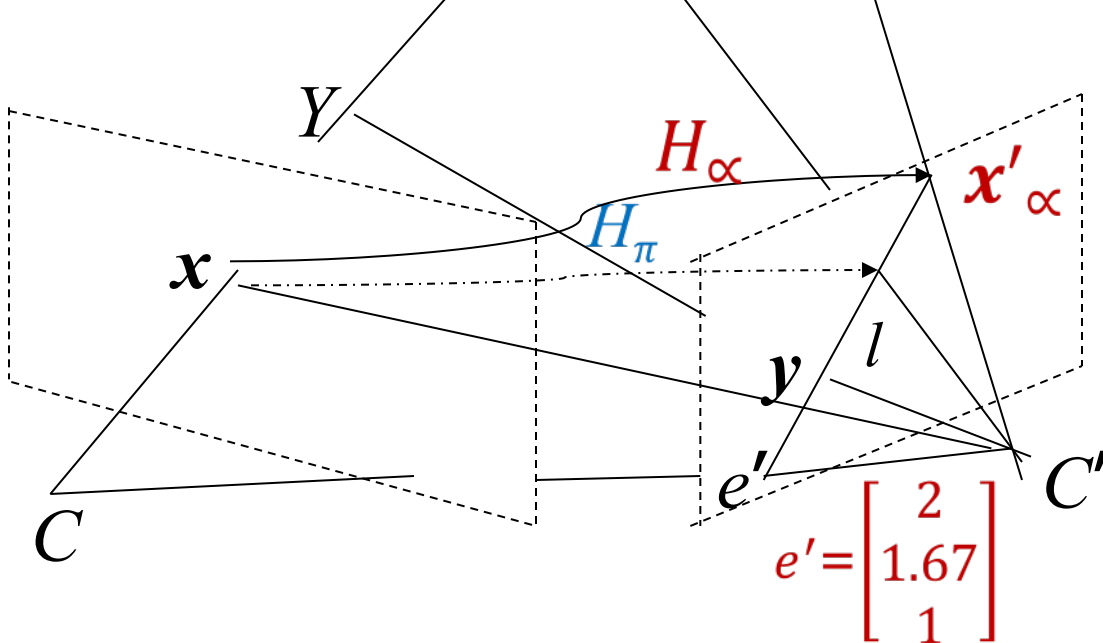
(b) Given an image point (15,20) of the reference camera (P), compute the epipolar line and its two end image points of P'.

Ans. 2(b)

$$F = \begin{bmatrix} -5.41 & -1.91 & 4.40 \\ 4.67 & 2.21 & -4.52 \\ 2.99 & 0.09 & -1.19 \end{bmatrix}$$



$$l = F \begin{bmatrix} 15 \\ 20 \\ 1 \end{bmatrix} = \begin{bmatrix} -114.92 \\ 109.76 \\ 45.44 \end{bmatrix} \equiv \begin{bmatrix} -2.53 \\ 2.42 \\ 1 \end{bmatrix}$$



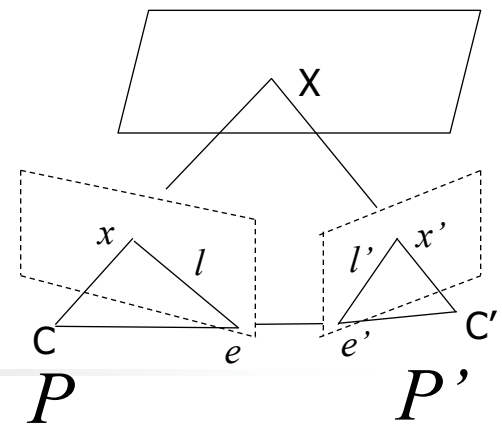
$$x'_\alpha = H_\alpha \begin{bmatrix} 15 \\ 20 \\ 1 \end{bmatrix}$$

$$H_\alpha = \begin{bmatrix} 9.33 & 2.07 & -4.62 \\ 9.33 & 1.78 & -4.48 \\ 2.33 & -0.07 & -0.05 \end{bmatrix}$$

$$x'_\alpha = \begin{bmatrix} 176.81 \\ 171.24 \\ 33.52 \end{bmatrix} \equiv \begin{bmatrix} 5.27 \\ 5.11 \\ 1 \end{bmatrix}$$



# Fundamental Matrix: Properties



$$x'^T F x = x^T F^T x' = 0, \forall (x', x)$$

## Transpose:

If  $F$  is fundamental matrix of  $(P, P')$ ,  $F^T$  for  $(P', P)$ .

**Epipolar lines:** For  $x$ , epipolar line  $l' = Fx$ .  
For  $x'$ , epipolar line  $l = F^T x'$ .

$F$  is a correlation.  
→ Rank deficient.  
→ Inverse does not exist.

**Epipoles:**  $e'^T (Fx) = e^T (F^T x') = (Fe)^T x' = 0$

$e'^T F = 0 \rightarrow e'$  is **left NULL vector** of  $F$ .

$Fe = 0 \rightarrow e$  is the **right NULL vector** of  $F$ .

## Rank deficient:

$\det(F) = 0$  and  $F$  is a projective element  $\rightarrow 7$  d.o.f.



## Ex. 3

---

- Consider the following fundamental matrix.

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

- Given two points (5,8) and (7,-5) in the left image compute the corresponding epipolar lines in the right image.
- Compute the right epipole.
- Compute the left epipole.

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

Ans.3

- $l_1 = F \cdot [5 \ 6 \ 1]^T = [172 \ 57 \ 1]^T$
- $l_2 = F \cdot [7 \ -5 \ 1]^T = [80 \ 150 \ -370]^T$
- $e_R = l_1 \times l_2 = [-21240 \ 63720 \ 21240]^T$   
 $= [-1 \ 3 \ 1]^T$
- $e_L = \text{Right zero of } F$

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

## Ans.3 (contd.)

□  $e_L$  = Right zero of  $F$

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \end{bmatrix} \begin{bmatrix} e_{L1} \\ e_{L2} \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} e_{L1} \\ e_{L2} \end{bmatrix} = - \begin{bmatrix} 20 & 12 \\ 8 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 59 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

## Ans.3 (contd.)

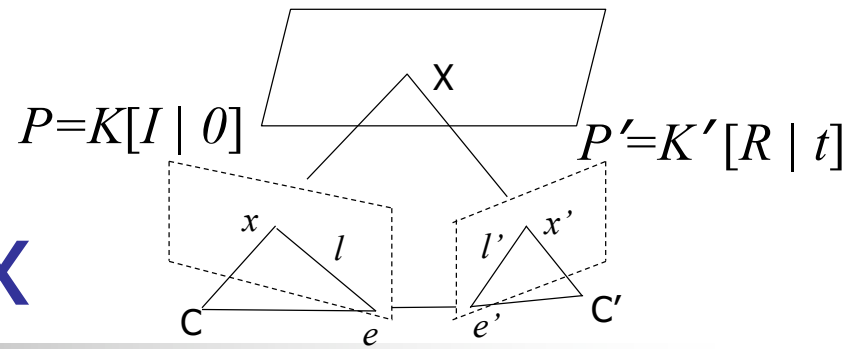
□  $e_R$  = Right zero of  $F^T$

$$F^T = \begin{bmatrix} 20 & 8 & -4 \\ 12 & -7 & 33 \\ 0 & 59 & -177 \end{bmatrix}$$

$$\begin{bmatrix} 20 & 8 & -4 \\ 12 & -7 & 33 \end{bmatrix} \begin{bmatrix} e_{R1} \\ e_{R2} \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} e_{R1} \\ e_{R2} \end{bmatrix} = - \begin{bmatrix} 20 & 8 \\ 12 & -7 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ 33 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

# Essential Matrix



Stereo geometry for calibrated cameras.

$$x_c = K^{-1}x \quad x_c'^T E x_c = 0$$

Coordinates in calibrated image planes.

$$E = K^T F K$$

$$F = K'^T E K^{-1}$$

$$= K'^T [t]_x R K^{-1} \quad \text{6 parameters}$$

$$\text{d.o.f.} = 5$$

$$E = [t]_x R$$

$$x_c' = K'^{-1}x'$$

$$x^T F x = 0$$

$$\rightarrow (K' x_c')^T F (K x_c) = 0$$

$$\rightarrow x_c'^T K'^T F K x_c = 0$$

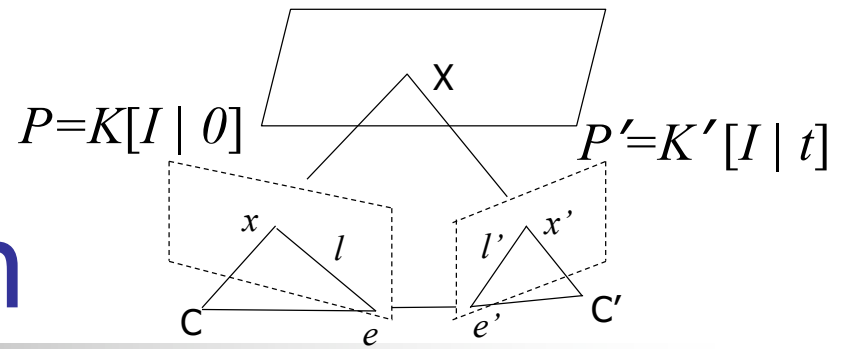
$$\rightarrow x_c'^T (K'^T F K) x_c = 0$$

$$E e_c = e_c' E = 0$$

$$\text{Rank: } 2$$

$$\det(E) = 0$$

# Pure translation



$$F = [e']_x K' I K^{-1} \\ = [e']_x K' K^{-1}$$

camera translation  $\parallel l$   
to x-axis,  $e' = [1 \ 0 \ 0]^T \rightarrow F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

For  $K = K'$ ,  $F = [e']_x$

$$[e']_x = \begin{bmatrix} 0 & -e_z & e_y \\ e_z & 0 & -e_x \\ -e_y & e_x & 0 \end{bmatrix}$$

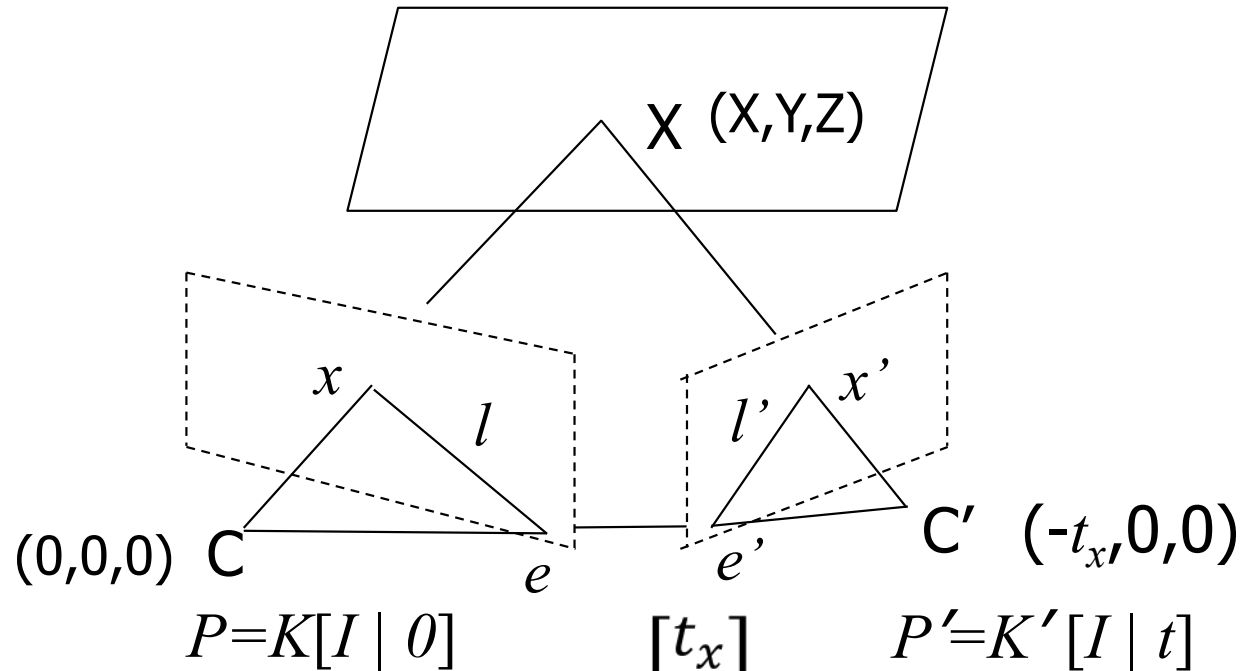
$\rightarrow x'^T F x = 0$

$\rightarrow y' = y$

# Pure Translation: Computing Depth

$$\tilde{X} = ZK^{-1}x$$

$$\tilde{X} + t = ZK'^{-1}x'$$



For  $K=K'$

$$ZK^{-1}(x' - x) = (\tilde{X} + t) - \tilde{X} = t \quad t = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x' = x + \frac{Kt}{Z} \quad \Rightarrow Z = \frac{Kt}{x' - x}$$

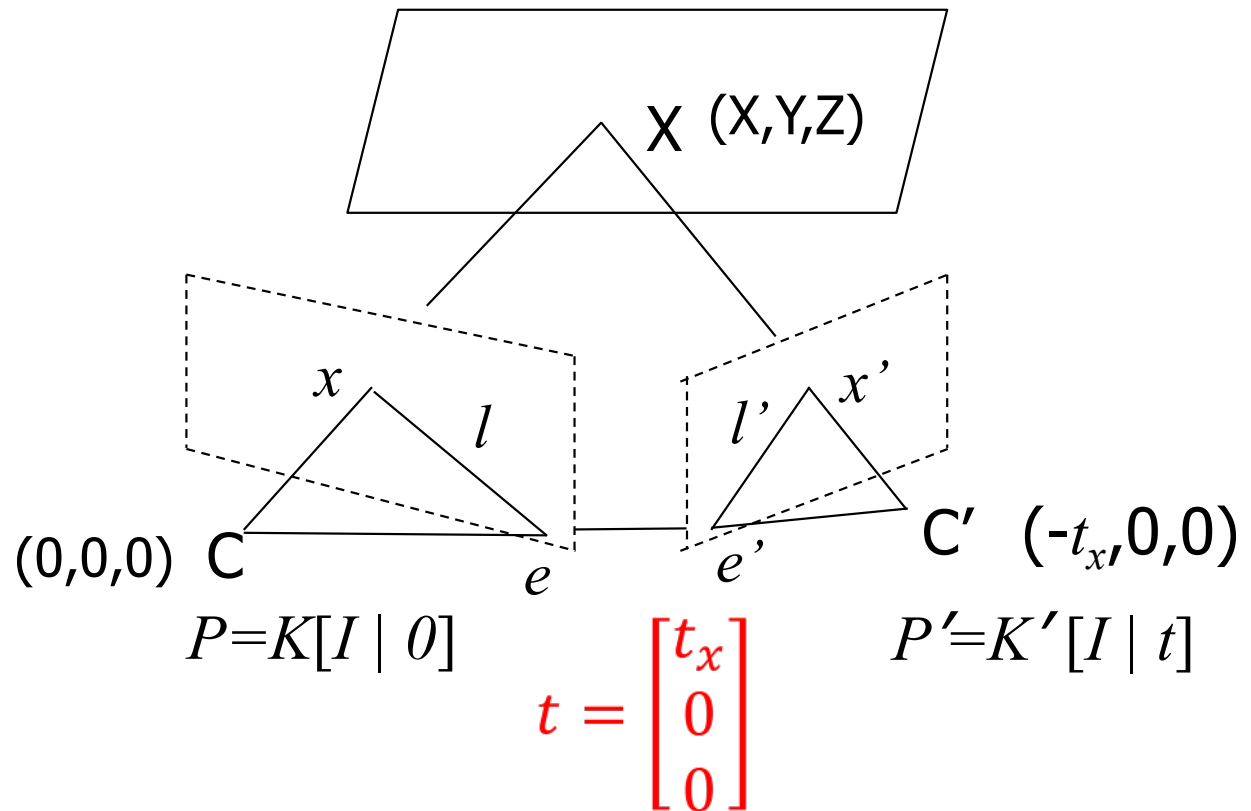


# Pure Translation: Computing Depth

$$K = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Z = \frac{Kt}{x' - x}$$

$$Z = \frac{f t_x}{x' - x}$$



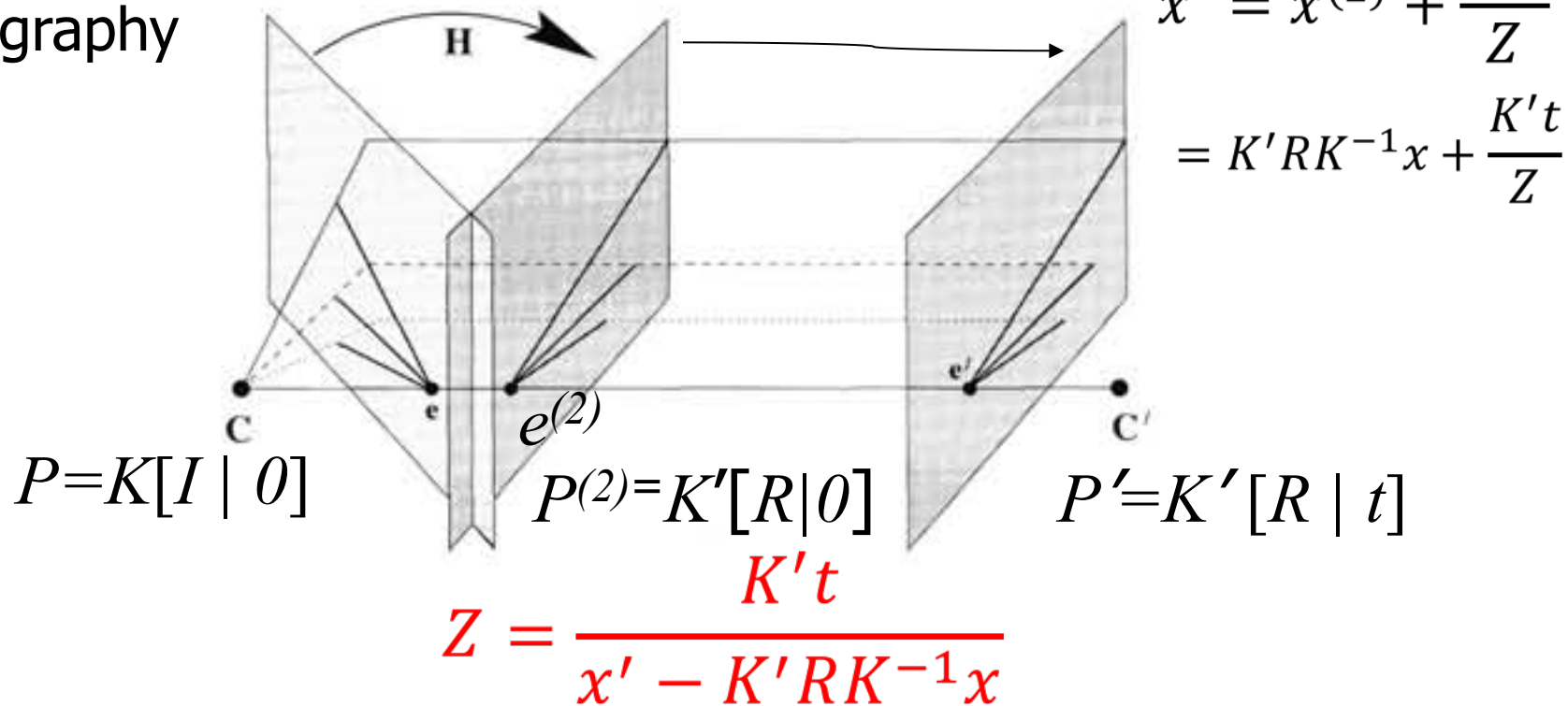
# General Motion of Camera

$$\begin{aligned} x^{(2)} &= K [R | 0] X \\ &= K' R [I | 0] X \\ &= K' R K^{-1} K [I | 0] X \\ &= K' R K^{-1} x \end{aligned}$$

Rotational  
Homography

$$H = K' R K^{-1}$$

|| stereo



From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

# General Motion of Camera

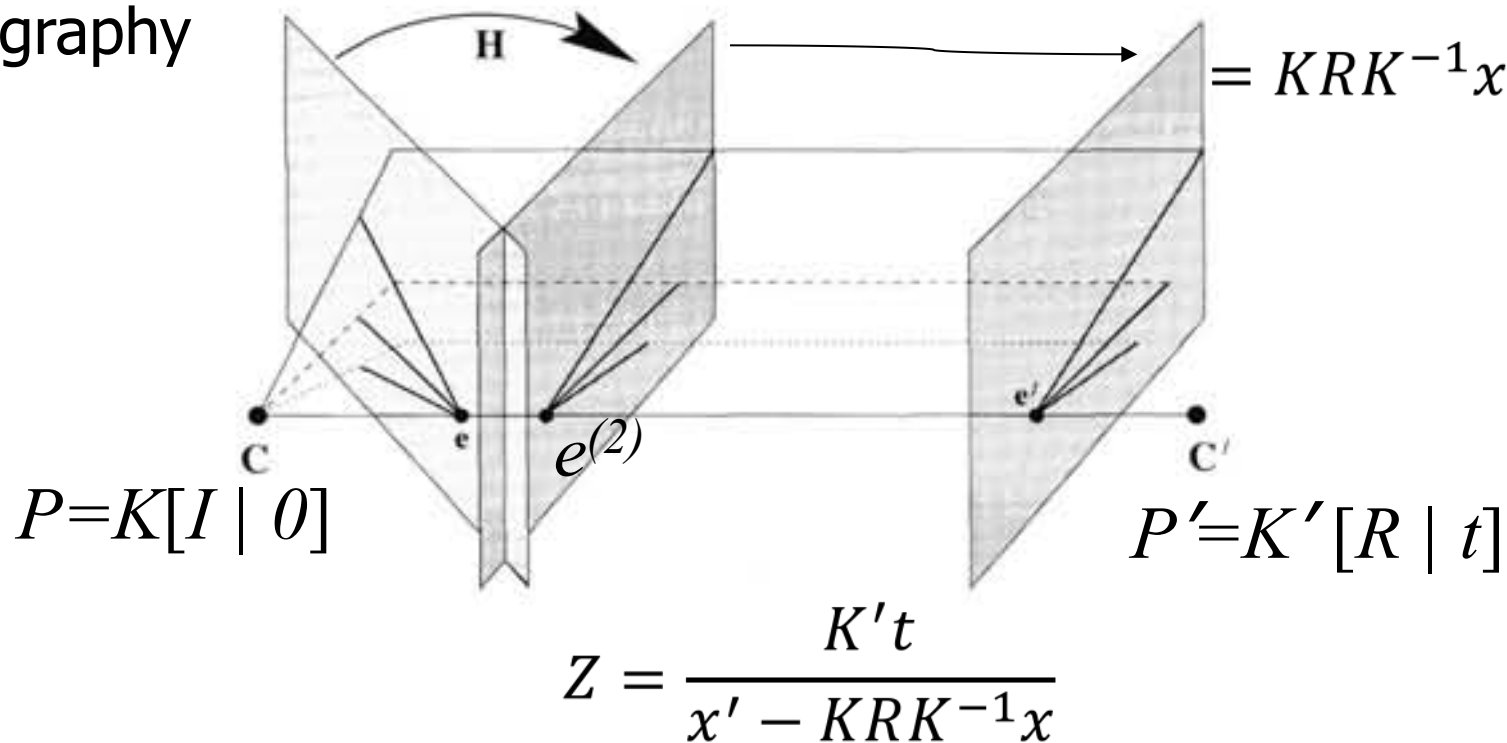
$$\begin{aligned} x^{(2)} &= K[R|0]X \\ &= KR[I|0]X \\ &= KRK^{-1}K[I|0]X \\ &= KRK^{-1}x \end{aligned}$$

Rotational  
Homography

$$H = KRK^{-1}$$

||<sup>l</sup> stereo

$$\begin{aligned} x' &= x^{(2)} + \frac{K't}{Z} \\ &= KRK^{-1}x + \frac{K't}{Z} \end{aligned}$$



From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)



## Ex. 4

---

- Consider a stereo set-up with  $P$  and  $P'$  (camera matrices for left and right camera) as given below.

$$P = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P' = \begin{bmatrix} 6 & 0 & 0 & 10 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If the image coordinates of a 3-D point are (6,8) and (9.33,8) in left and right cameras, compute its depth (z-coordinate) in the 3D.



## Ans. 4

---

- $P=K[I|0]$  and  $P'=K[I|t]$

$$K = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad t = \begin{bmatrix} 10/6 \\ 0 \\ 0 \end{bmatrix}$$

- $\mathbf{x}' = \mathbf{x} + (\mathbf{Kt})/Z$
- $Z = (6 \times 10/6) / (x' - x)$   
 $= 10 / (9.33 - 6) = 3$

# Estimation of Fundamental Matrix

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

$$\mathbf{x}'^T F \mathbf{x} = 0$$

$$x' x f_{11} + x' y f_{12} + x' f_{13} + y' x f_{21} + y' y f_{22} + y' f_{23} + x f_{31} + y f_{32} + f_{33} = 0$$

$$[x' x, x' y, x', y' x, y' y, y', x, y, 1] [f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{32}, f_{33}]^T = 0$$

$$\begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_n & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} \mathbf{f} = 0$$

- Solution up to scale.
- Minimum 8 point correspondences.
- Use of DLT (for 7 point correspondences from linear combination of smallest and second smallest eigen vectors.

$$A \mathbf{f} = 0$$

# Estimation of Fundamental Matrix

$$\begin{bmatrix}
 x_1 x_1' & y_1 x_1' & x_1' & x_1 y_1' & y_1 y_1' & y_1' & x_1 & y_1 & 1 \\
 x_2 x_2' & y_2 x_2' & x_2' & x_2 y_2' & y_2 y_2' & y_2' & x_2 & y_2 & 1 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 x_n x_n' & y_n x_n' & x_n' & x_n y_n' & y_n y_n' & y_n' & x_n & y_n & 1
 \end{bmatrix}
 \begin{bmatrix}
 f_{11} \\
 f_{12} \\
 f_{13} \\
 f_{21} \\
 f_{22} \\
 f_{23} \\
 f_{31} \\
 f_{32} \\
 f_{33}
 \end{bmatrix}
 = 0$$

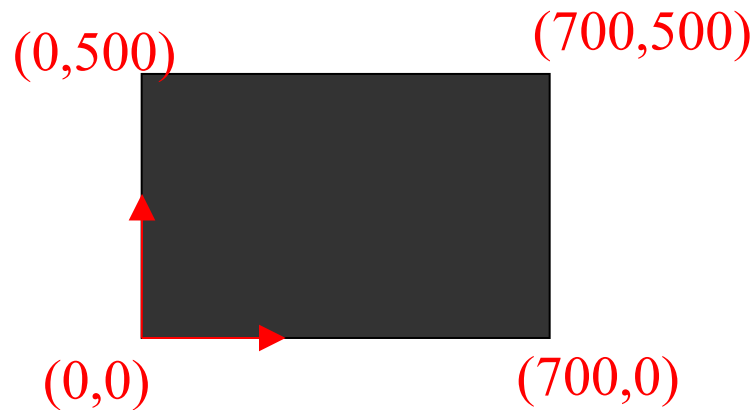
$\sim 10000 \quad \sim 10000 \quad \sim 100 \quad \sim 10000 \quad \sim 10000 \quad \sim 100 \quad \sim 100 \quad \sim 100 \quad 1$



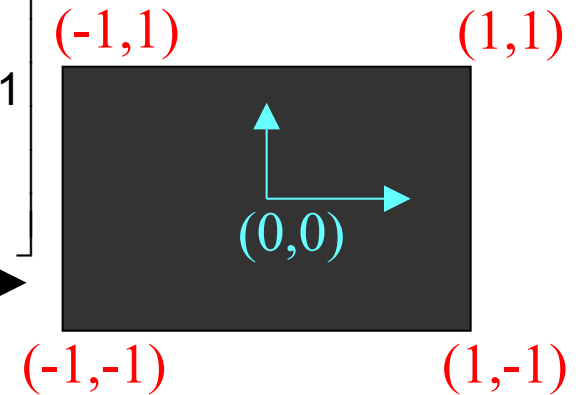
Orders of magnitude difference  
 Between column of data matrix  
 → least-squares yield poor results

# The normalized 8-point algorithm

Transform image to  
 $\sim [-1,1] \times [-1,1]$

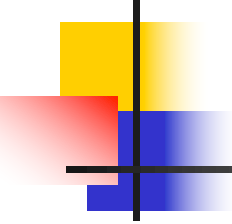


$$\begin{bmatrix} \frac{2}{700} & 0 & -1 \\ \frac{2}{500} & -1 & 1 \end{bmatrix}$$



Least squares yields good results  
(Hartley, PAMI '97)





# The singularity constraint

---

$$\det F = 0 \quad \text{rank } F = 2$$

SVD from linearly computed F matrix (rank 3)

$$F = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} V^T = U_1 \sigma_1 V_1^T + U_2 \sigma_2 V_2^T + U_3 \sigma_3 V_3^T$$

$$\min \|F - F'\|_F$$

Compute closest rank-2 approximation

$$F' = U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & 0 \end{bmatrix} V^T = U_1 \sigma_1 V_1^T + U_2 \sigma_2 V_2^T$$

# The singularity constraint

Nonsingular  
 $F$



Singular  
 $F$



Non-singular  $F$  causes epipolar lines not converging.

From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)



# The singularity constraint for Essential Matrix

---

$$\det(E)=0$$

Estimate  $\hat{E}$  by any of the techniques used for  $F$ .

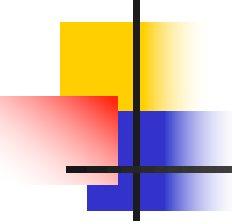
Perform SVD of  $\hat{E}$ .

$$\hat{E}=UDV^T \quad \text{Where } D=\text{diag}(a,b,c) \quad a \geq b \geq c$$

For essential matrix, two singular values are the same.

$$\Rightarrow \hat{E} = U\hat{D}V^T \quad \text{where } \hat{D} = \left( \frac{a+b}{2}, \frac{a+b}{2}, 0 \right)$$

# The minimum case – 7 point correspondences


$$\begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_7 x_7 & x'_7 y_7 & x'_7 & y'_7 x_7 & y'_7 y_7 & y'_7 & x_7 & y_7 & 1 \end{bmatrix} \mathbf{f} = 0$$

$$\mathbf{A} = \mathbf{U}_{7 \times 7} \text{diag}(\sigma_1, \dots, \sigma_7, 0, 0) \mathbf{V}_{9 \times 9}^T$$

Last two  
column  
vectors of  $\mathbf{V}$   
also provide  
 $\mathbf{F}_1$  and  $\mathbf{F}_2$ .

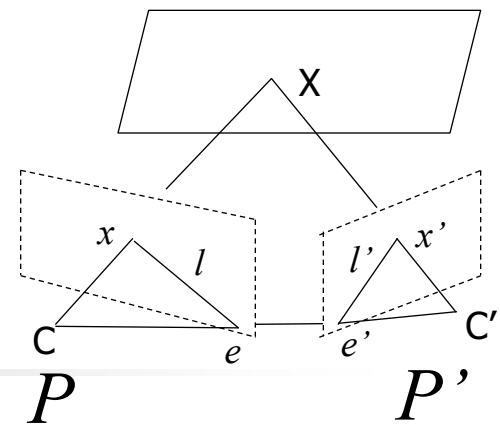
$\mathbf{F}_1, \mathbf{F}_2 \rightarrow$  Eigen vectors corresponding  
to two zero's. The solution is  $\mathbf{F}_1 + \lambda \mathbf{F}_2$ .

But  $\mathbf{F}_1 + \lambda \mathbf{F}_2$  not automatically rank 2.

Solve for  $\lambda$  from  $\det(\mathbf{F}_1 + \lambda \mathbf{F}_2) = 0$ .

As it is a cubic polynomial, there are 1 or 3 solutions.

# Parametric representation of F



Over parameterization:  $F = [t]_x M \rightarrow \{t, M\} \rightarrow 12$  params.

Epipolar parameterization:

$$F = \begin{bmatrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ e & f & \alpha e + \beta f \end{bmatrix} \quad \{a, b, c, d, e, f, \alpha, \beta\}$$

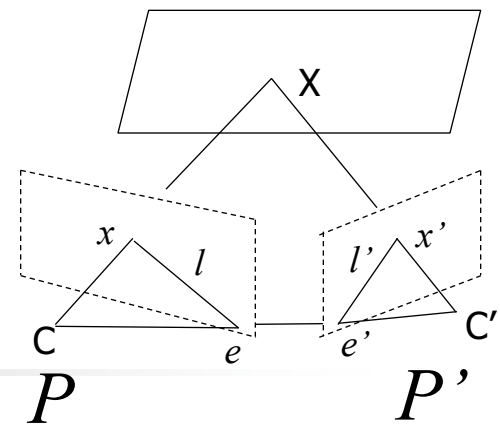
Left epipole  $e = [\alpha \quad \beta \quad -1]^T$

Both epipoles as parameters  $\{a, b, c, d, \alpha, \beta, \alpha', \beta'\}$

$$F = \begin{bmatrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ \alpha' a + \beta' c & \alpha' b + \beta' d & \alpha \alpha' a + \alpha \beta' c + \alpha' \beta b + \beta \beta' d \end{bmatrix}$$

Epipoles  $e = [\alpha \quad \beta \quad -1]^T$   $e' = [\alpha' \quad \beta' \quad -1]^T$

# Retrieving the camera matrices from $F$



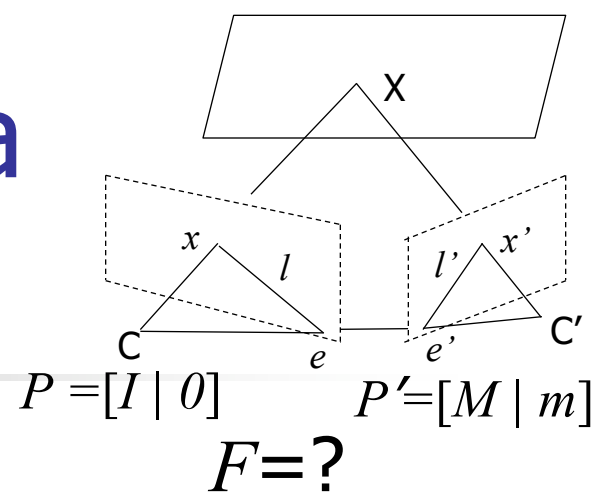
- $F$  only depends on projective properties of  $P$  and  $P'$ .
- Independent of choice of world frame.
- $(P, P') \rightarrow F$  (unique)
- $F \rightarrow (P, P')$  (?)
- Given a homography  $H$  (4x4 non-singular matrix) in  $P^3$ , if  $(P, P') \rightarrow F$ , then  $(PH, P'H) \rightarrow F$ .

Proof:  $PX \leftrightarrow P'X$

$$\rightarrow (PH)(H^{-1}X) \leftrightarrow (P'H)(H^{-1}X)$$

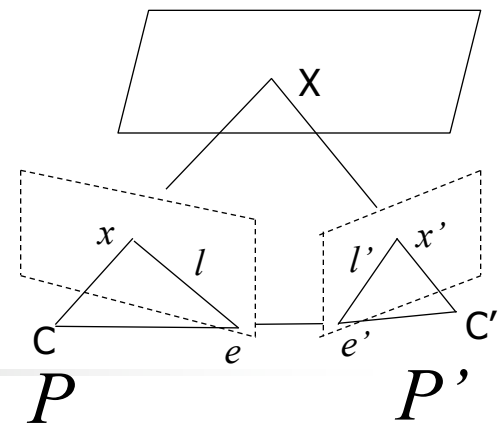
- $F$  does not uniquely map to  $(P, P')$ .

# Retrieving the camera matrices from $F$



- $P = [I \mid 0]$  &  $P' = [M \mid m] \rightarrow F = [m]_x M$ .
- If  $F$  derived from both  $(P_1, P_1')$  and  $(P_2, P_2')$ , there exists  $4 \times 4$   $H$  s.t.  $P_2 = P_1 H$  &  $P_2' = P_1' H$ .
- d.o.f. of  $P$  + d.o.f. of  $P' = 22$
- d.o.f. of  $H = 15$
- d.o.f. of  $F = 22 - 15 = 7$

# Retrieving the camera matrices from $F$



- $F$  corresponds to  $(P, P')$ , iff  $P'^T F P$  is *skew symmetric*.

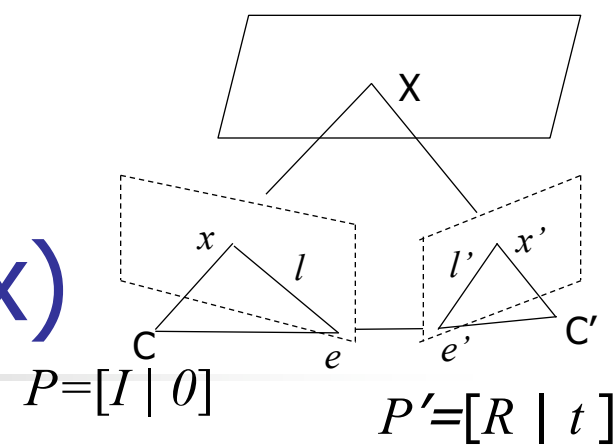
Proof: For a skew symmetric matrix  $S$ ,  $X^T S X = 0$ , for all  $X$ .

Now,  $X^T P'^T F P X = (P' X)^T F (P X) = x'^T F x = 0$  (for any  $X$  in  $P^3$ , as  $F$  is the fundamental matrix). ...

- $F$  corresponds to  $P = [I \mid 0]$  &  $P' = [S F \mid e']$ , where  $e'$  is the right epipole of  $F$  s.t.  $e'^T F = 0$ .
- A good choice of  $S = [e']_x$ .
- $F \rightarrow ([I \mid 0], [[e']_x F \mid e'])$   
 $\leftrightarrow ([I \mid 0], [[e']_x F + e' v^T \mid k e'])$

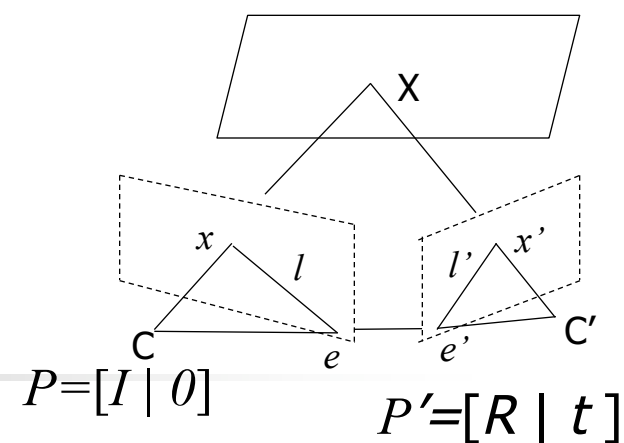


# The camera matrices from E (Essential matrix)



- $E$  is an essential matrix iff two of its singular values are equal and the third one is zero.
- $E = [t]_x R$
- $[t]_x$  and  $R$  can be computed through decomposition of  $E$  s.t.  $E = SR$ , where  $S$  is a skew symmetric matrix and  $R$  is orthogonal.

# Decomposition of E (Essential matrix)



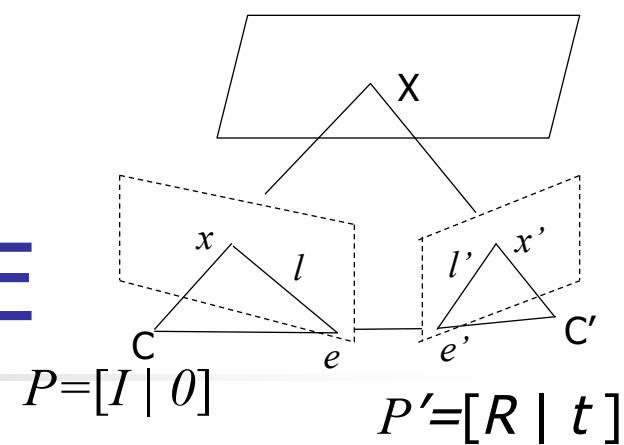
- SVD of  $E = U \text{diag}(1, 1, 0) V^T$
- Two possible decomposition of  $E = SR$
- $S = UZU^T$  and  $R = UWV^T$  or  $UW^TV^T$

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Any skew symmetric matrix  $S$  can be decomposed as  $S = kUZU^T$
- $W$  is orthogonal and  $Z = \text{diag}(1, 1, 0)W$ .

# Camera matrices from E



- SVD of  $E = U \text{diag}(1, 1, 0) V^T$
- Two possible decomposition of  $E = SR$
- $S = UZU^T$  and  $R = UWV^T$  or  $UW^TV^T$

$$Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P' = \begin{bmatrix} UWV^T & +u_3 \\ UW^TV^T & +u_3 \end{bmatrix} \text{ or } \begin{bmatrix} UWV^T & -u_3 \\ UW^TV^T & -u_3 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↑ Last column of  $U$ .

Out of the four only one is valid for viewing a point from both the cameras. It is sufficient to test a single point for the above.



## Ex. 5

---

- Check whether the following fundamental matrix and the camera matrices  $P$  and  $P'$  (for left and right cameras) are compatible.

$$F = \begin{bmatrix} 20 & 12 & 0 \\ 8 & -7 & 59 \\ -4 & 33 & -177 \end{bmatrix}$$

$$P = \begin{bmatrix} 7 & 4 & -6 & 3 \\ 8 & -1 & 2 & -5 \\ 9 & -10 & 4 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} 6 & 4 & -6 & 10 \\ 8 & -5 & 2 & -7 \\ 9 & -10 & 6 & 2 \end{bmatrix}$$



# Ans. 5

---

■  $P^T F P =$

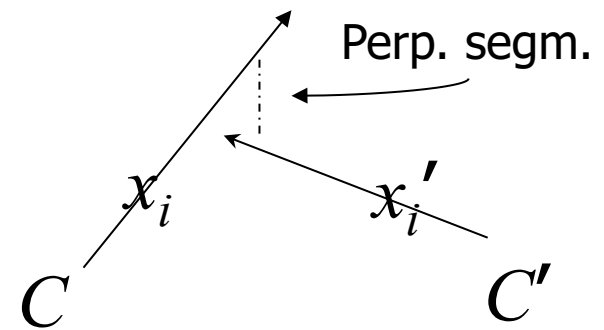
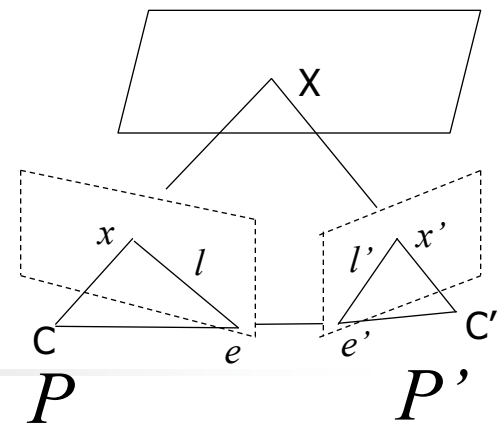
$$\begin{bmatrix} -6549 & 11489 & -4746 & -2242 \\ 11859 & -14183 & 4926 & 2950 \\ -8496 & 8816 & -2784 & -1888 \\ -4071 & 7979 & -3414 & -1534 \end{bmatrix}$$

- Not a skew symmetric matrix.
- Not compatible.

# Computing scene points (structure)

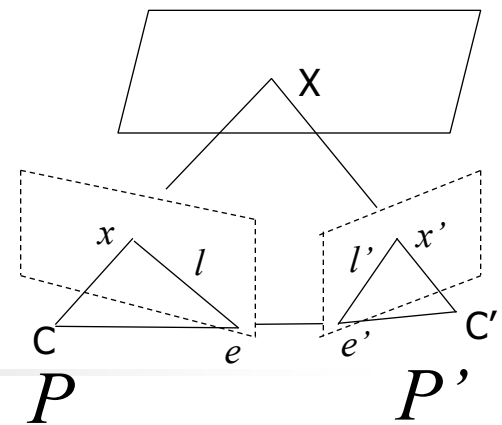
Given  $x_i \leftrightarrow x_i'$ , compute  $X$ .

1. Compute  $F$ .
2. Compute  $P$  and  $P'$ .
3. For each  $(x_i, x_i')$  compute  $X$  by triangulation.
  - i. Compute intersection of  $Cx_i$  and  $C'x_i'$ .
  - ii. Compute segment perpendicular to both.
  - iii. Get the mid-point.



Not projective invariant, i.e.  $(PH, P'H)$  does not give  $H^{-1}X$ .

# Minimizing Reprojection Error



Given  $x_i \leftrightarrow x_i'$ , compute  $X$ .

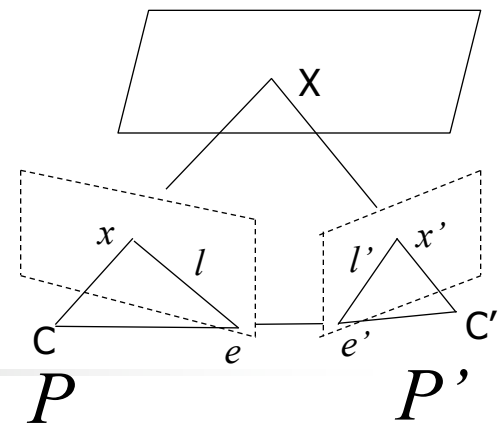
1. Estimate  $\hat{X}$  s.t.  $P\hat{X} = \hat{x}$  and  $P'\hat{X} = \hat{x}'$ .
2. Minimize the reprojection error ( $E_{rp}$ ).

$$E_{rp} = d(x, \hat{x})^2 + d(x', \hat{x}')^2$$

subject to  $x'^T F x = 0$

Projective invariant.

# Linear triangulation methods



Given  $x_i \leftrightarrow x_i'$ , compute  $X$ .

$$x \times PX = 0$$

$$x' \times P'X = 0$$

4 equations,  
3 unknowns.

$$[A]_{4 \times 4} X = 0$$

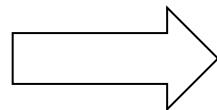
Minimize  $\|AX\|$   
subject to  $\|X\|=1$ .

Use DLT.

Not projective invariant.

Generalize to multiview correspondences.

$$\begin{matrix} x_1 & \leftrightarrow & x_2 & \leftrightarrow & x_3 \\ P_1 & & P_2 & & P_3 \end{matrix}$$



$$x_1 \times P_1 X = 0$$

$$x_2 \times P_2 X = 0$$

$$x_3 \times P_3 X = 0$$

6 equations  
3 unknowns.





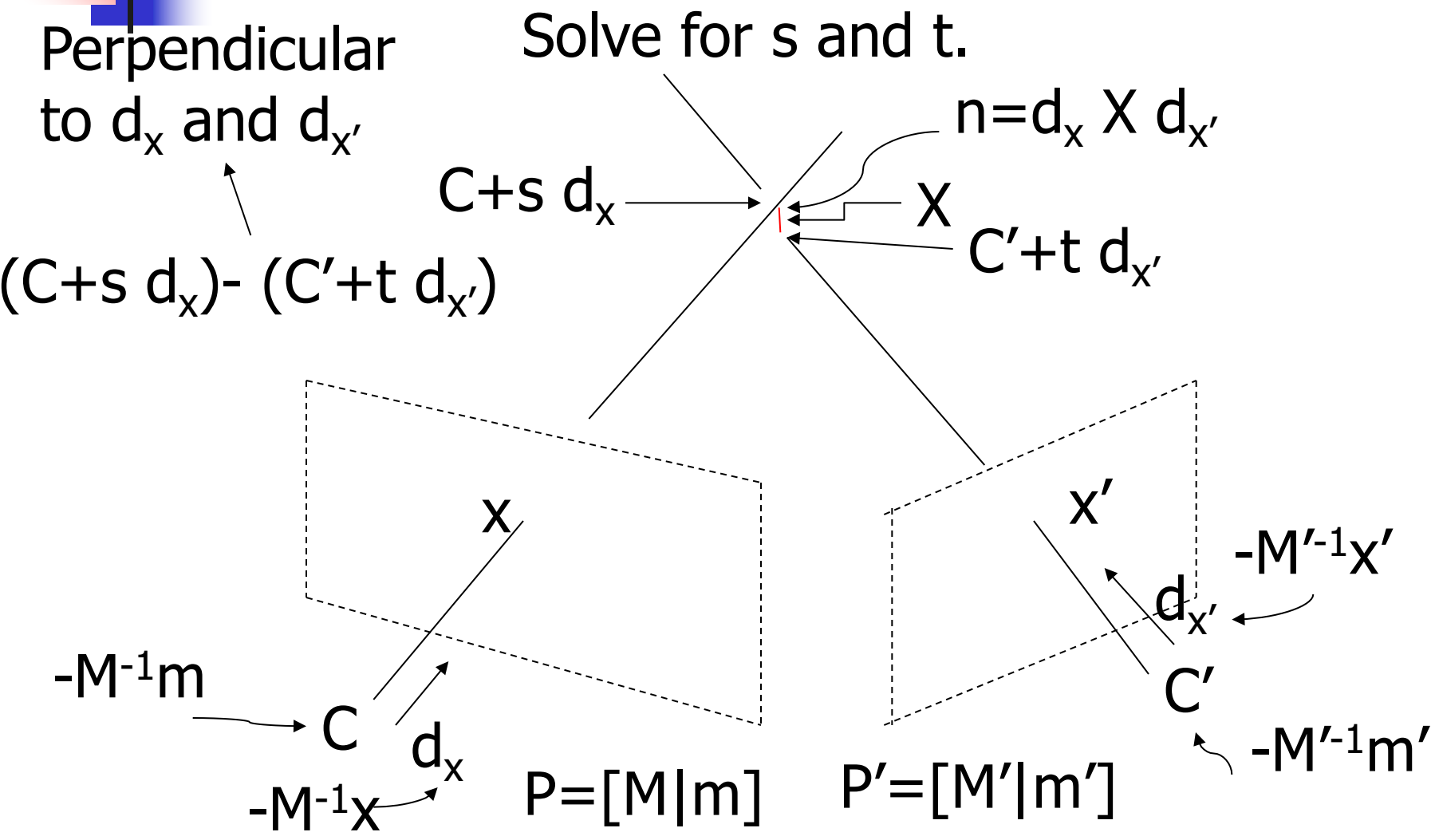
## Ex. 4

---

- Suppose  $P$  and  $P'$  (for left and right cameras). Images of a scene point are formed at  $(0, 3.5)$ , and  $(2/3, -1/3)$ , respectively. Find the 3D coordinate of the scene point.

$$P = \begin{bmatrix} 7 & 4 & -6 & 3 \\ 8 & -1 & 2 & -5 \\ 9 & -10 & 4 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} 6 & 4 & -6 & 10 \\ 8 & -5 & 2 & -7 \\ 9 & -10 & 6 & 2 \end{bmatrix}$$

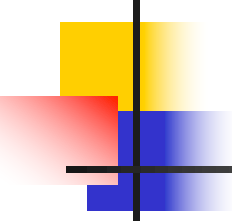
# Computation of structure



$$P = \begin{bmatrix} 7 & 4 & -6 & 3 \\ 8 & -1 & 2 & -5 \\ 9 & -10 & 4 & 1 \end{bmatrix} \quad P' = \begin{bmatrix} 6 & 4 & -6 & 10 \\ 8 & -5 & 2 & -7 \\ 9 & -10 & 6 & 2 \end{bmatrix}$$

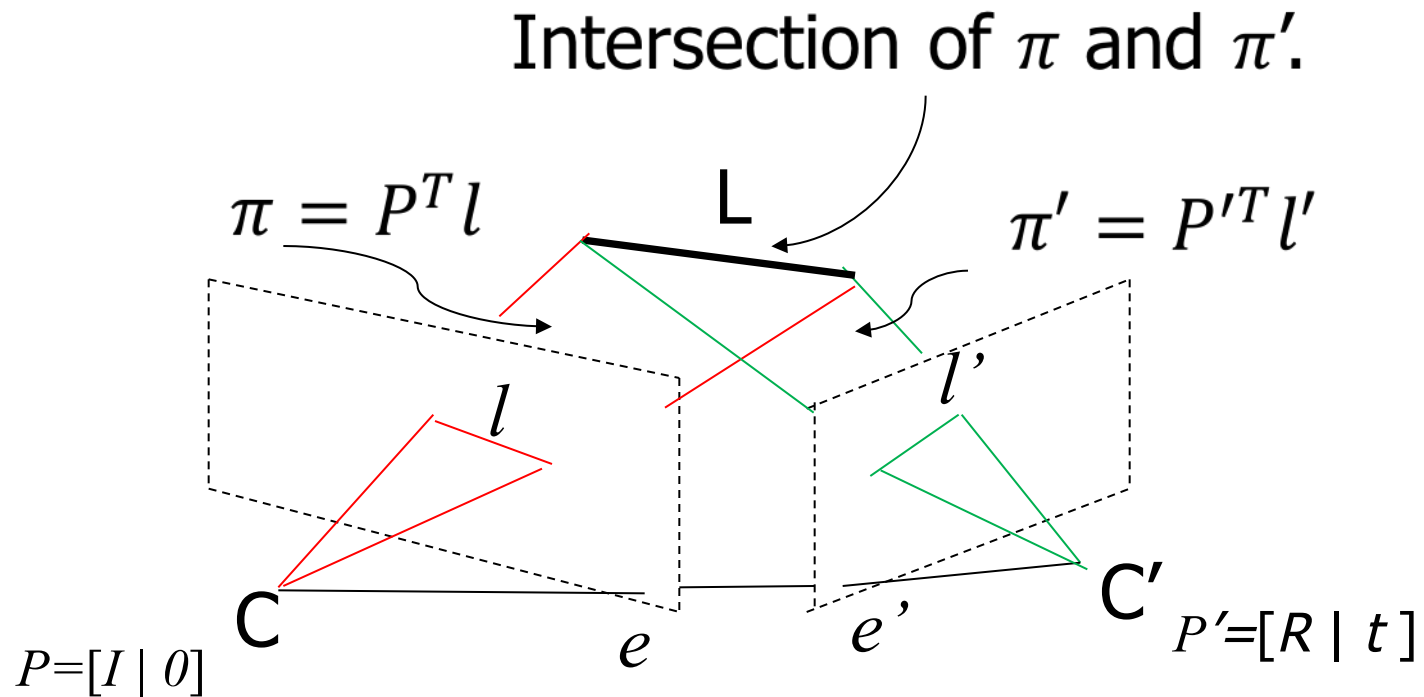
4 Ans.

- $P=[M|m] \quad P'=[M'|m']$
- $C=-M^{-1}m = [0.35 \ 1 \ 1.62]^T$
- $C'=-M'^{-1}m' = [13 \ 35 \ 38]^T$
- $x=[0 \ 3.5 \ 1]^T \quad x'=[2/3 \ -1/3 \ 1]^T$
- $d_x = M^{-1} x = [0.32 \ 0.47 \ 0.69]$
- $d_{x'} = M'^{-1} x' = [1.36 \ 3.67 \ 3.91]$
- $d_x \times d_{x'} = [0.7 \ 0.33 \ -0.55]$

- 
- 
- $(C+s d_x) - (C'+t d_{x'}) = [-12.85 \ -33.93 \ -36.58] + s[0.32 \ 0.47 \ 0.69] - t[1.36 \ 3.67 \ 3.91]$
  - $0.32(-12.85+0.3s-1.36t)+0.47(-33.93+0.47s-3.67t)+0.69(-36.58+0.69s-3.91t)=0 \quad -(1)$
  - $1.36(-12.85+0.3s-1.36t)+3.67(-33.93+0.47s-3.67t)+3.91(-36.58+0.69s-3.91t)=0 \quad -(2)$
  - Solve (1) and (2) to get s and t, and the point.



# Line reconstruction



$$L = \begin{bmatrix} \pi \\ \pi' \end{bmatrix}$$

A convenient way of representing 3D line.

# Plane Induced Homography

*Proof:*

$$x' = P'X = [A|a]X$$

$$\text{Now, } x = PX = [I|0]X$$

$$\text{So any point in } \overrightarrow{CX} \text{ is } X = \begin{bmatrix} x \\ \rho \end{bmatrix}$$

$$\text{When it intersects } \pi, \pi^T \begin{bmatrix} x \\ \rho \end{bmatrix} = 0.$$

$$\Rightarrow v^T x + \rho = 0$$

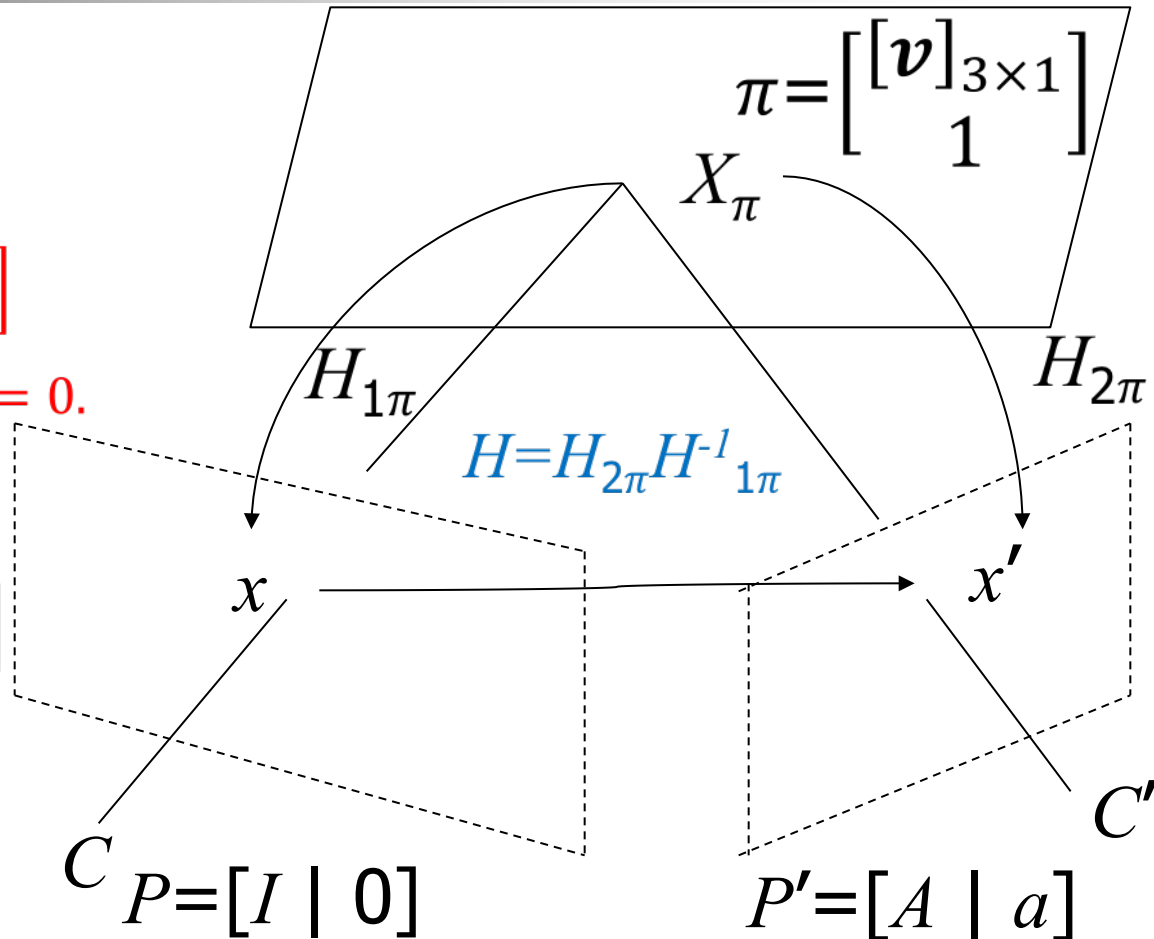
$$\Rightarrow \rho = -v^T x$$

$$\text{So, } x' = P'X = [A|a] \begin{bmatrix} x \\ -v^T x \end{bmatrix}$$

$$= Ax - a v^T x$$

$$= (A - a v^T)x$$

$H$

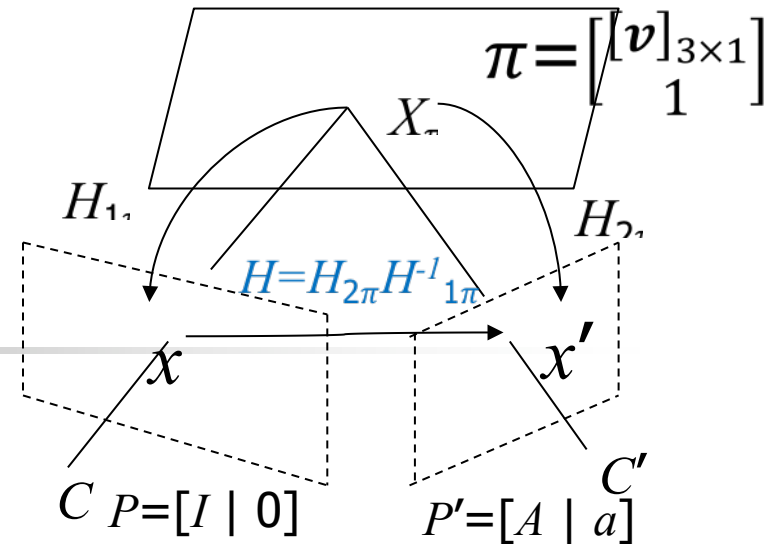


# Plane induced homography

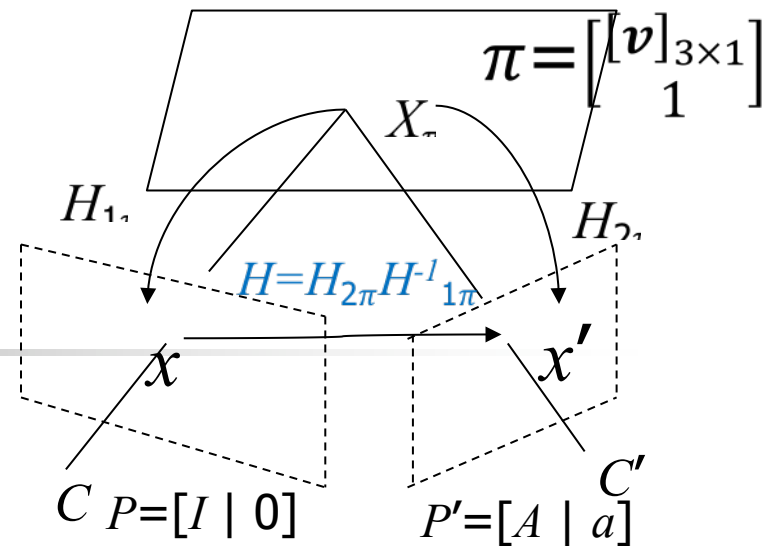
A transformation  $H$  between two stereo images is plane induced homography if  $F$  is decomposed into  $[e']_x H$ .

Hence,  $P=[I|0]$  &  $P'=[H|e']$ .

Given  $P=[I|0]$ ,  $P'=[A|a]$ , & a plane induced homography  $H$ , the plane can be recovered by solving  $kH=A-av^T$ , (linear equations for unknowns  $k$  and  $v$ ).



# Plane induced homography



A transformation  $H$  between two stereo images is plane induced homography if  $F$  is decomposed into  $[e']_x H$ . Hence,  $P = [I | 0]$  &  $P' = [H | e']$ .

Plane at infinity

$H$  is the transformation w.r.t. plane  $[0 \ 0 \ 0 \ 1]^T$  in the camera coordinate.

Given  $P = [I | 0]$ ,  $P' = [A | a]$ , & a plane induced homography  $H$ , the plane can be recovered by solving  $kH = A - a\mathbf{v}^T$ , (linear equations for unknowns  $k$  and  $\mathbf{v}$ ).



# Homography compatible stereo geometry

$H$  is compatible iff  $H^T F$  is skew symmetric, i.e.

$$H^T F + F^T H = 0$$

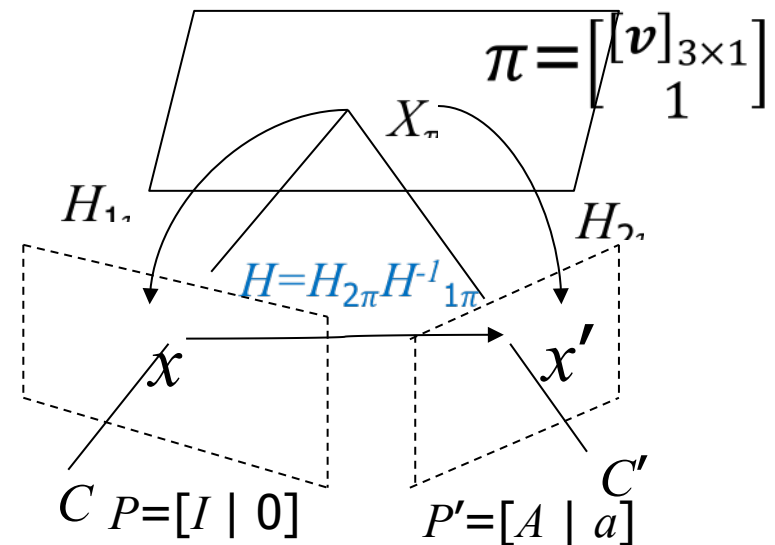
$$x'^T F x = 0$$

$$\text{And, } x' = Hx$$

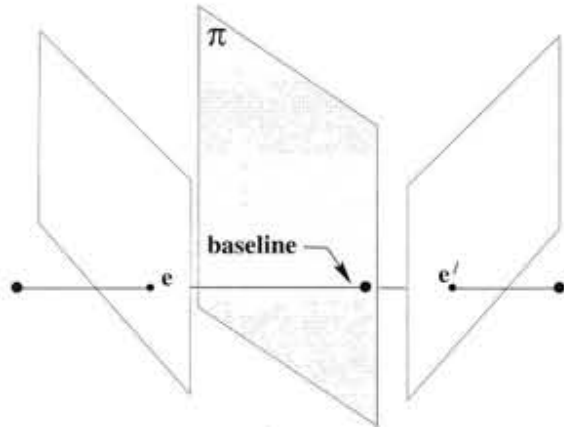
$$\Rightarrow (Hx)^T F x = 0$$

$$\Rightarrow x^T H^T F x = 0$$

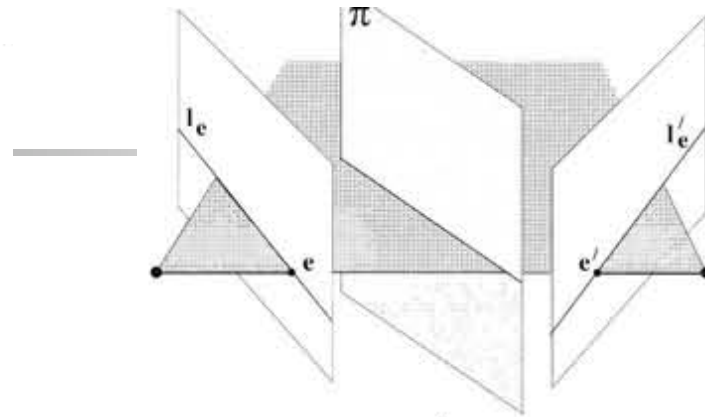
As this is true for all  $x$ ,  
 $H^T F$  is skew symmetric.



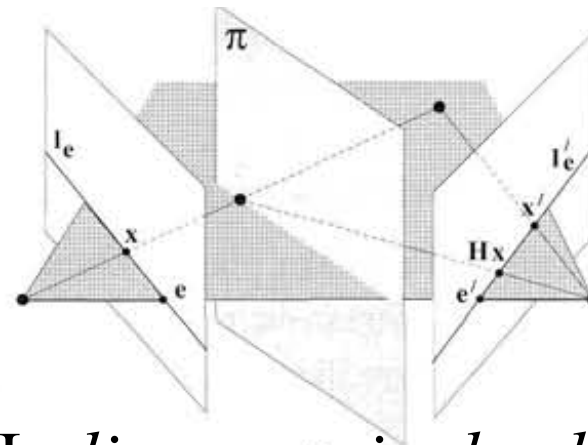
# Plane induced $H$ and epipolar constraints



*Epipoles mapped by  $H$ , as  $e' = He$ , since they are images of the point on the plane where the baseline intersects it.*

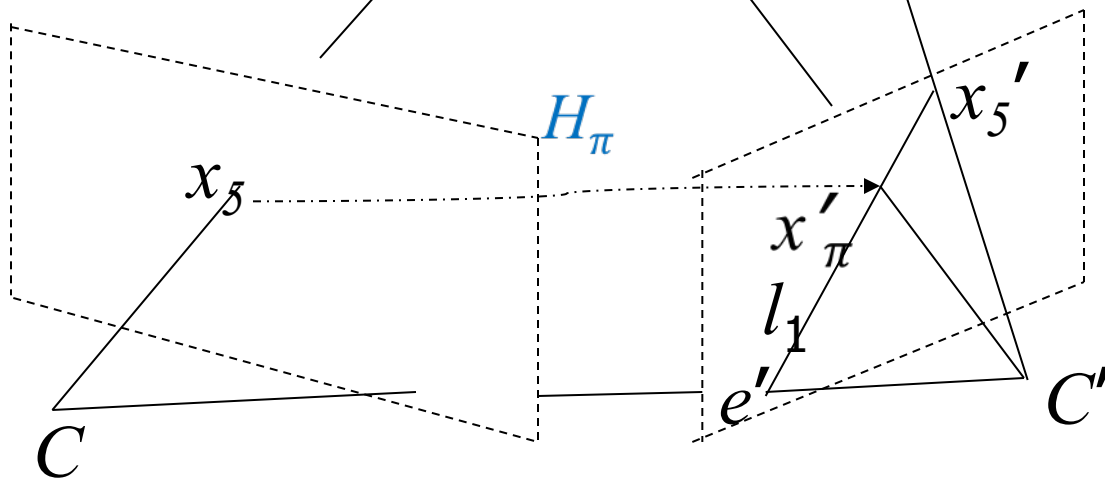
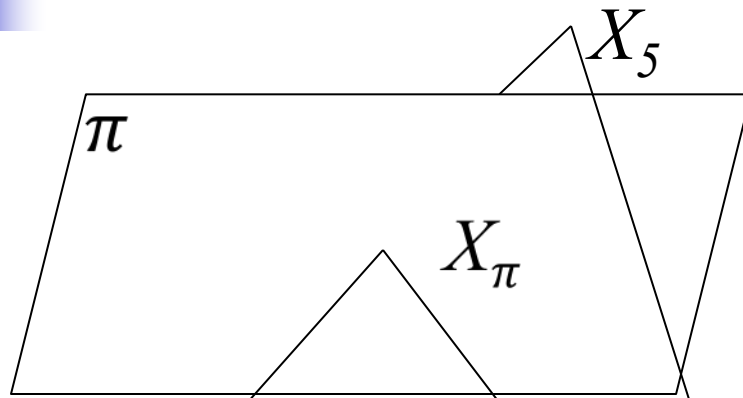


*Epipolar lines are mapped by the homography as  $H^{-T}l'_e = l_e$ .*



*$Hx$  lies on epipolar line  $l'_e$   
so  $l'_e = Fx = x' x (Hx)$ .*

# Computing F from 6 points out of which 4 are coplanar



1. Use 4 coplanar points to compute  $H_\pi$ .
2.  $l_1 = H_\pi(x_5) \times x_5'$
3.  $l_2 = H_\pi(x_6) \times x_6'$
4.  $e' = l_1 \times l_2$
5.  $F = [e']_\times H_\pi$ .



# Given $F$ and 3 point correspondences, compute $H$ .

---

## First Method

1. Obtain  $(P=[I|0], P'=[A|a])$  from  $F$  and construct 3 scene points,  $X_1, X_2$ , &  $X_3$ .
2. Obtain plane  $(v^T, 1)^T$ .
3. Compute  $H=A-av^T$

## Second Method

1. Obtain  $(e, e')$  from  $F$ .
2. Use 3 correspondences +  $(e, e')$ , to obtain  $H$ .

Any 3 points can bipartition the image space, w.r.t. the plane formed by them.

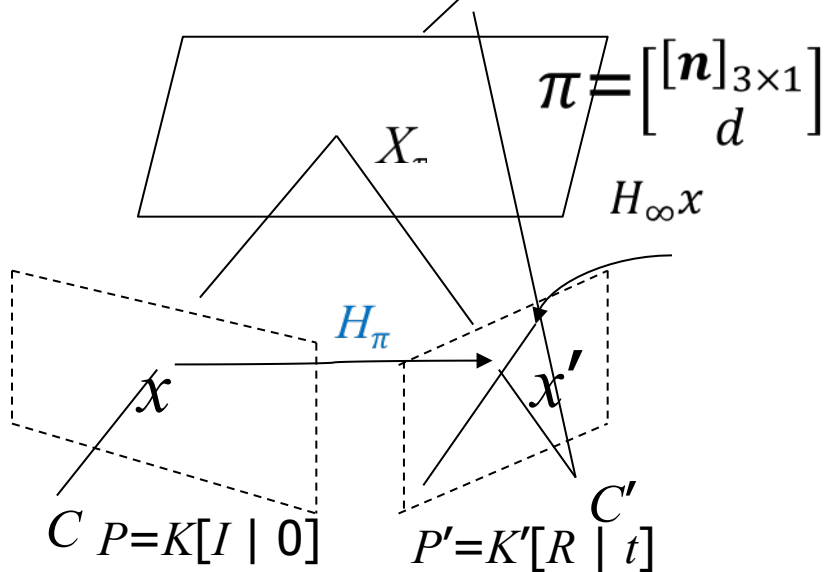
# Infinite Homography

$$H = (A - a v^T)$$

$$\text{For } P=[I | 0], P'=[A | a]$$

$$\text{and plane}=[v^T \ 1]^T$$

Homography induced by the plane at  $\alpha$ .



$$H_\pi = (K'R - e'v^T)K^{-1}$$

where  $v = \frac{n}{d}$  and  $e' = K't$

$$H_\pi = K' \left( R - \frac{tn^T}{d} \right) K^{-1}$$

$$= K'RK^{-1} - K' \frac{tn^T}{d} K^{-1}$$

$$\text{As } d \rightarrow \infty, H_\infty = K'RK^{-1}$$

$$x' = K'RK^{-1}x + \frac{K't}{Z}$$

Vanishing point  $\rightarrow$   $= H_\infty x + \frac{K't}{Z}$

As  $Z \rightarrow \infty$ ,  
 $x'$  is the image of point on  $\pi_\infty$ .



# $H_\alpha$ and Vanishing points

---

- $H_\alpha$  maps vanishing points between two images.
- $H_\alpha$  can be computed by identifying three non-collinear vanishing points given  $F$  or from 4 vanishing points.
- Let  $P=[M | m]$ ,  $P'=[M' | m']$ ,  $X=[x_\alpha^T 0]^T$  (a point at infinity).

$$x=PX=M x_\alpha$$

$$x'=P'X=M' x_\alpha$$

$$x'=M'M^{-1}x \rightarrow H_\alpha=M'M^{-1}$$

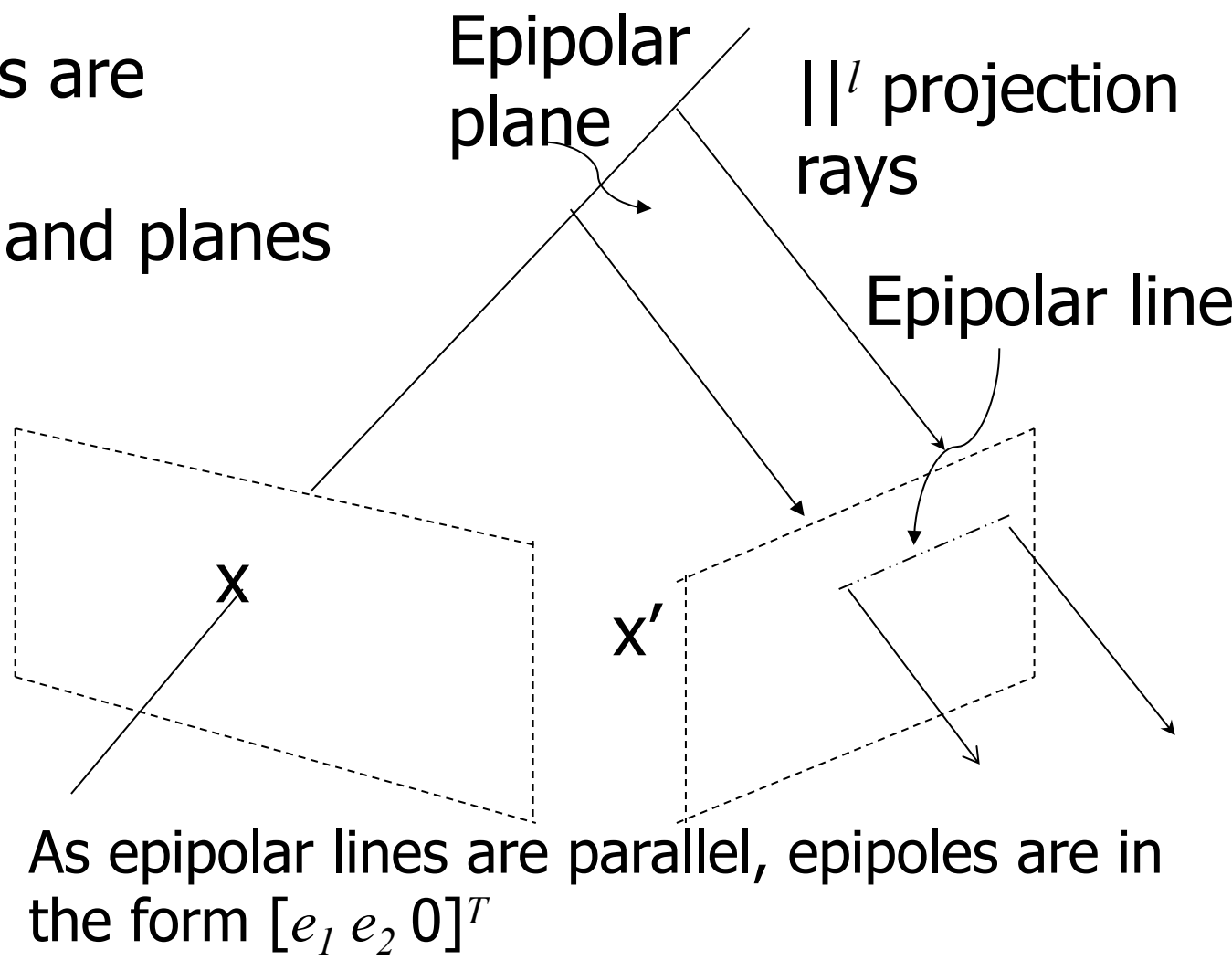
# Affine epipolar geometry

- Projection rays are parallel.
- Epipolar lines and planes are parallel.

Form of  $F$ :

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ e & d & c \end{bmatrix}$$

$a, b, c, d, e$  all non-zero.

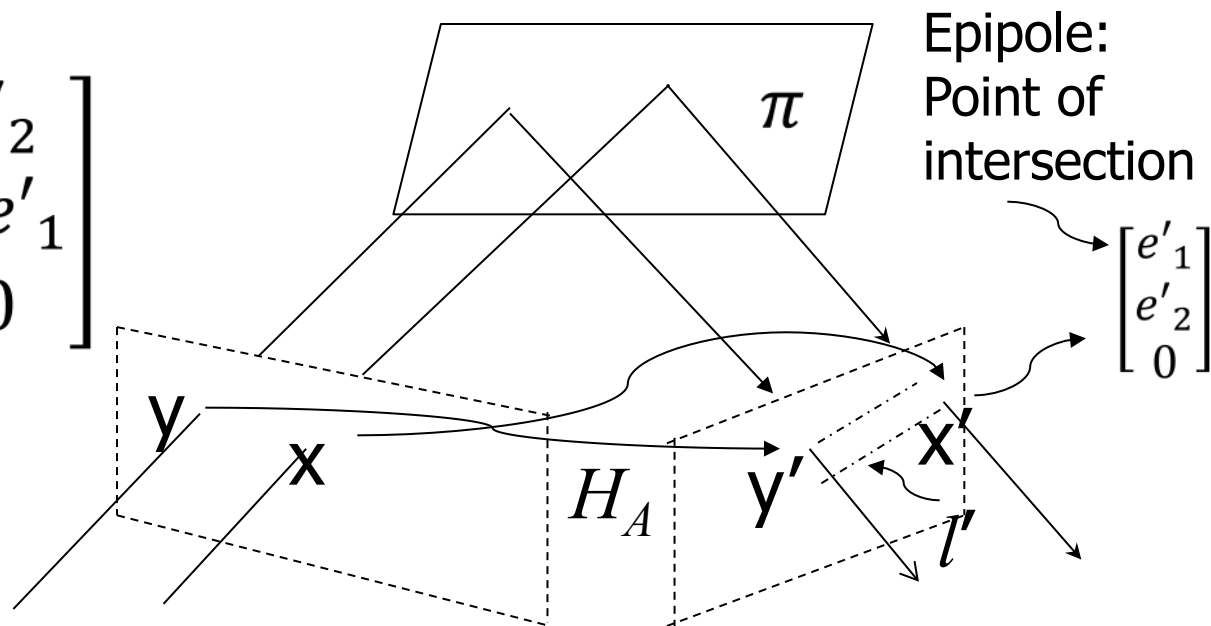


# Affine stereo

$$\begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ e & d & c \end{bmatrix} \begin{matrix} l' = e' \times H_A x \\ = [e']_{\times} H_A x \\ \Rightarrow F_A = [e']_{\times} H_A \end{matrix}$$

$$[e']_{\times} = \begin{bmatrix} 0 & 0 & e'_2 \\ 0 & 0 & -e'_1 \\ -e'_2 & e'_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{b} \\ -\mathbf{b}^T & 0 \end{bmatrix}$$

$$H_A = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$



$F_A$ : d.o.f.: 4

$$F_A = \begin{bmatrix} 0 & \mathbf{b} \\ -\mathbf{b}^T A & -\mathbf{b}^T \mathbf{t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & e'_2 \\ 0 & 0 & -e'_1 \\ e & d & c \end{bmatrix}$$

left epipole:  $[-d \ e \ 0]^T$   
 right epipole:  $[-b \ a \ 0]^T$





## Estimating $F_A$

$$F_A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ e & d & c \end{bmatrix}$$

Epipolar lines:

$$l' = F_A \mathbf{x} = [a \quad b \quad ex + dy + c]^T$$

$$l = F_A^T \mathbf{x}' = [e \quad d \quad ax' + by' + c]^T$$

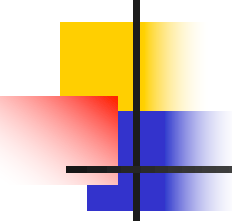
Point correspondence: *Reduced to a single linear equation.*

$$\mathbf{x}'^T F_A \mathbf{x} = 0 \Rightarrow ax' + by' + ex + dy + c = 0$$

$$[A]_{N \times 5} \mathbf{f}_{5 \times 1} = 0 \quad \text{Solve using DLT.}$$

Minimum 4 point correspondences required to get  $F_A$ .

Singularity constraint is satisfied by the structure of  $F_A$ .



## Estimating $F_A$ (another approach)

$$F_A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ e & d & c \end{bmatrix}$$

1. Compute  $H_A$  using 3 point-correspondences.
2.  $l' = H_A x_4' \times x_4'$  (say,  $[l_1 \ l_2 \ l_3]^T$ )
3. Get  $e'$  from  $l'$  as  $[l_2 \ -l_1 \ 0]$
4.  $F_A = [e']_x H_A$



# Summary

---

- Epipolar geometry in a stereo imaging system.
  - epipoles, scene point, corresponding image points, and camera centers lie on the same plane.
- Fundamental matrix (F) characteristics and unique to the stereo set-up.
  - Transformation invariant.
  - 3x3 singular matrix with d.o.f. 7 and rank 2.
  - Given an image point  $x$ , its epipolar line  $l = Fx$ .
  - For any pair of correspondence point  $(x, x')$ ,  $x'^T F x = 0$ .
  - Given epipoles  $(e, e')$ ,  $Fe = e'^T F = 0$
  - Given camera matrices  $(P, P')$ , F is unique.
    - $P = [I|0]$ ,  $P' = [M|m] \rightarrow F = [m]_x M$ .
    - $P = [M|m]$ ,  $P' = [M'|m'] \rightarrow F = [m' - M'M^{-1}m]_x M'M^{-1}$



# Summary

---

- Given a homography  $H$  ( $4 \times 4$  non-singular matrix) in  $P^3$ , if  $(P, P') \rightarrow F$ , then  $(PH, P'H) \rightarrow F$ .
- Given a fundamental matrix  $F$ , there exist a family of stereo setups (pairs of camera matrices).
  - $([I \mid 0], [[e']_x F + e' v^T \mid k e'])$ ,  $v$ : any arbitrary 3 vector,  $k$ : a scalar constant.
- Given camera matrices  $(P, P')$  and a corresponding pair of image points  $(x, x')$ , it is possible to reconstruct the respective 3-D scene point  $X$ .



# Summary

---

- Fundamental matrix of calibrated cameras is called Essential Matrix  $E$ .
- $E = [t]_{\times} R$ , given  $([I|0], [R|t])$ .
- $E$  is an essential matrix iff two of its singular values are equal and the third one is zero.
- $[t]_{\times}$  and  $R$  can be computed through decomposition of  $E$  s.t.  $E = SR$ , where  $S$  is a skew symmetric matrix and  $R$  is orthogonal.



# Summary

---

- SVD of  $E=U \text{ diag}(1,1,0) V^T$
- Two possible decomposition of  $E=SR$
- $S=UZU^T$  and  $R=UWV^T$  or  $UW^TV^T$

$$z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad P' = \begin{bmatrix} UWV^T & | & +u_3 \\ UW^TV^T & | & +u_3 \end{bmatrix} \text{ or } \begin{bmatrix} UWV^T & | & -u_3 \\ UW^TV^T & | & -u_3 \end{bmatrix} \quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One of the above is valid in viewing a point from both the cameras.



# Summary

---

- Given a set of pairs of corresponding points, it is possible to estimate  $F$ .
  - Minimum 7 pairs required.
- Parametric representation of  $F$ .

$$F = \begin{bmatrix} a & b & \alpha a + \beta b \\ c & d & \alpha c + \beta d \\ \alpha' a + \beta' c & \alpha' b + \beta' d & \alpha \alpha' a + \alpha \beta' c + \alpha' \beta b + \beta \beta' d \end{bmatrix}$$

$$\text{Epipoles: } e' = [\alpha \quad \beta \quad -1]^T \quad e = [\alpha' \quad \beta' \quad -1]^T$$

$$e = [\alpha \quad \beta \quad -1]^T \quad e' = [\alpha' \quad \beta' \quad -1]^T$$



# Summary

---

- Given a set of pairs of corresponding points, it is possible to estimate camera matrices and scene points up to projective (4x4 homography matrix) ambiguity.
  - $(PH)(H^{-1}X) \leftrightarrow (P'H)(H^{-1}X)$
- Given a pair of corresponding lines  $l$  and  $l'$ , and camera matrices  $(P, P')$  possible to reconstruct respective 3D line  $L$ .
  - Intersection of planes  $P^T l$  and  $P'^T l'$





# Summary

---

- A plane induces homography between corresponding image points in a stereo set-up.
  - Given a plane  $(\mathbf{v}^T, 1)$ , and camera matrices  $([I|0], [A|a])$ ,  
 $H = (A - a\mathbf{v}^T)$ 
    - Homography at infinity: Plane at infinity  $(\mathbf{0}^T, 1)$  induces  $H = A$ .
- Affine epipolar geometry simplifies the structure of fundamental matrix.
  - **Left** epipole:  $[-d \ e \ 0]^T$
  - **Right** epipole:  $[-b \ a \ 0]^T$

$$F_A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ e & d & c \end{bmatrix}$$