Camera Geometry (Week 03 – Lectures 11 – 15)

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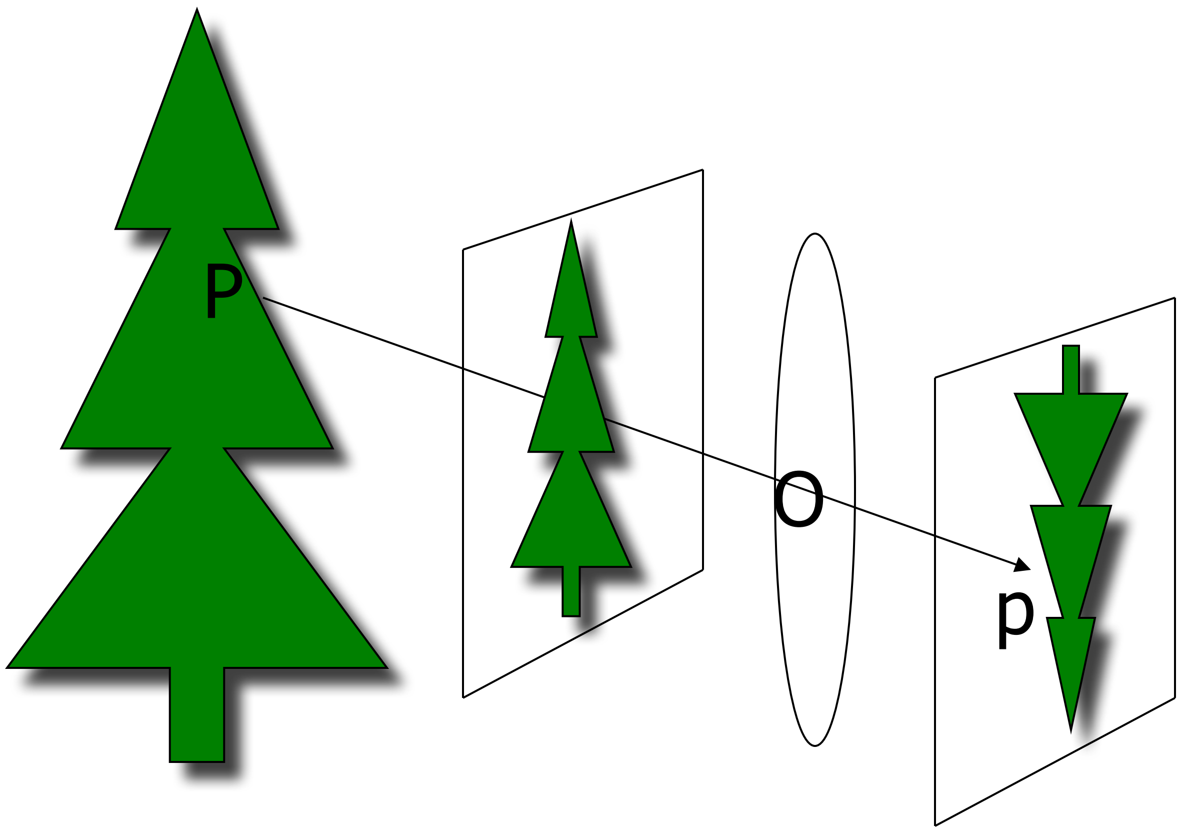
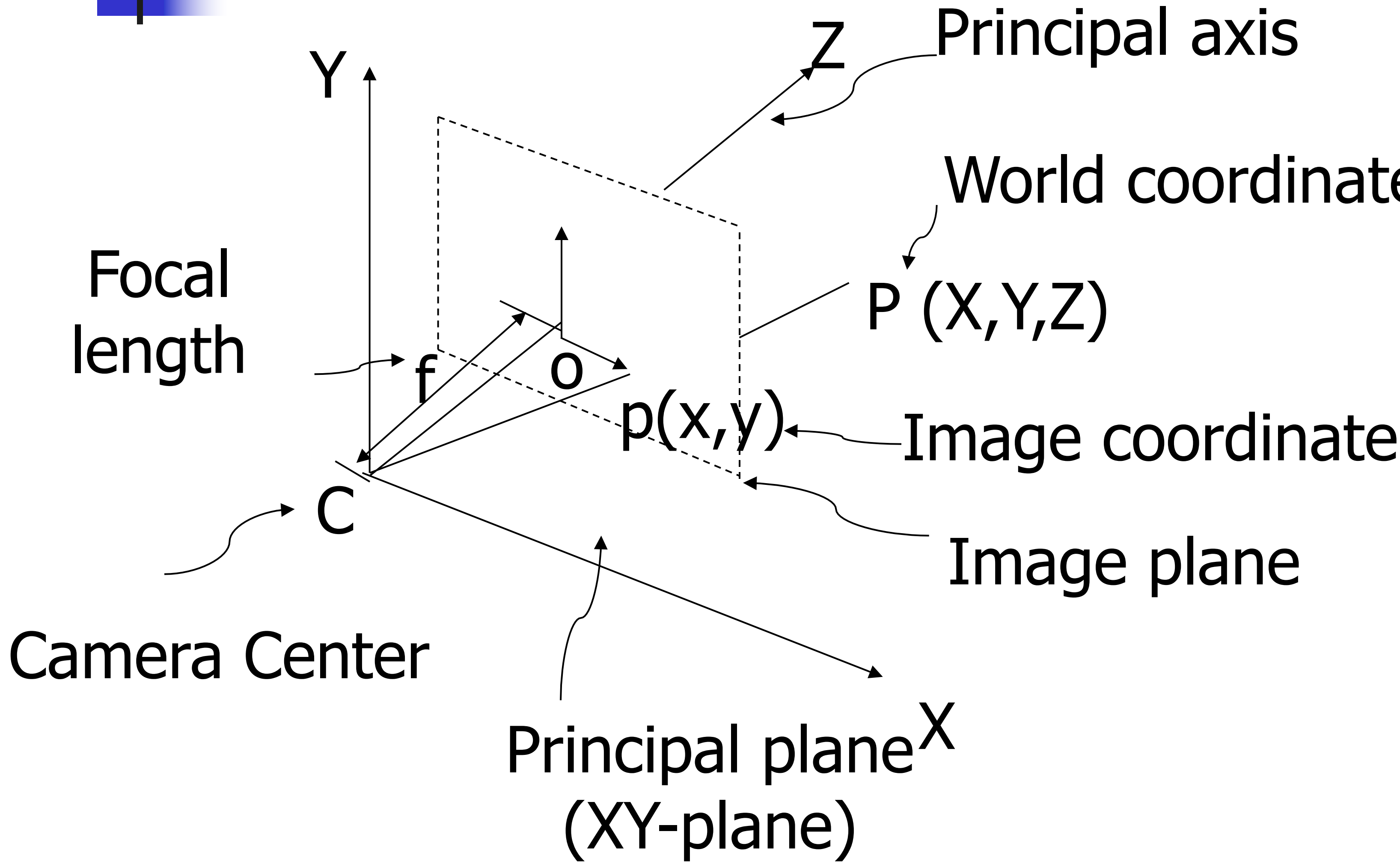
Image formation in optical camera



Pinhole camera

$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$



Pinhole Camera: Mapping from P3→P2

$$x = \frac{fX}{Z}$$
$$y = \frac{fY}{Z}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} fX \\ fY \\ Z \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

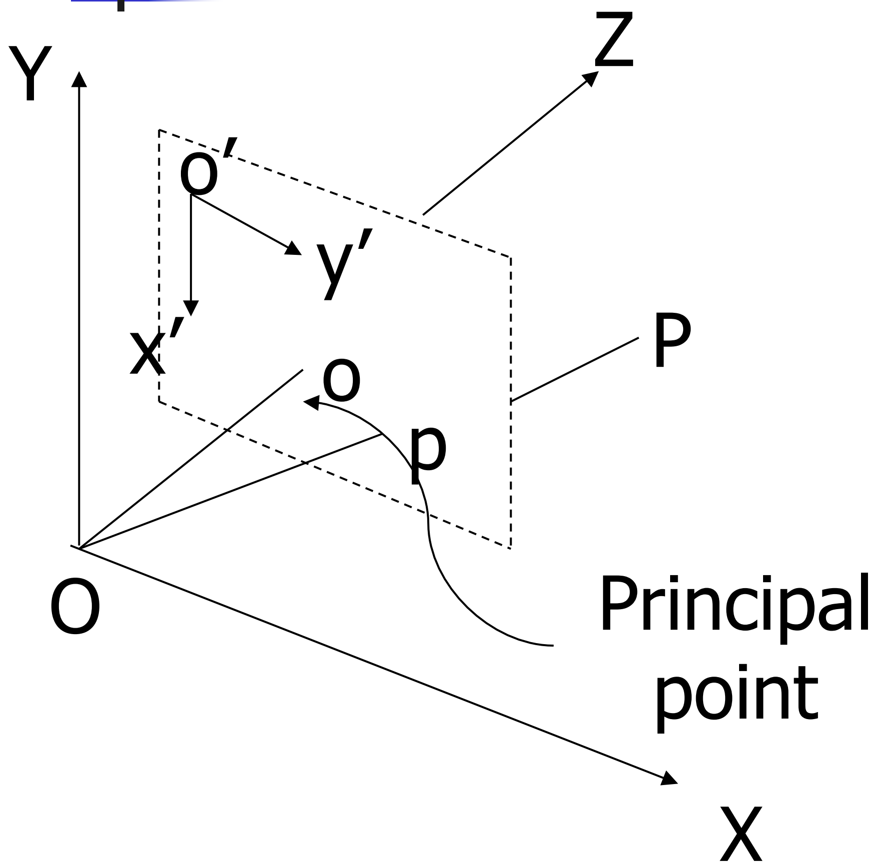
Projection Matrix (P)

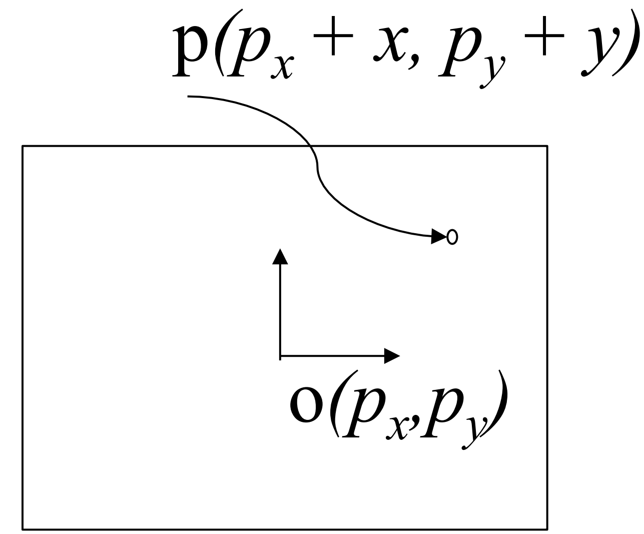
$$P = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = diag(f, f, 1)[I \quad | \quad 0]$$

Offset of principal point

$$x = \frac{fX}{Z}$$

$$y = \frac{fY}{Z}$$





$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Projection Matrix under the offset

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} \equiv \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = K[I \quad | \quad 0]$$

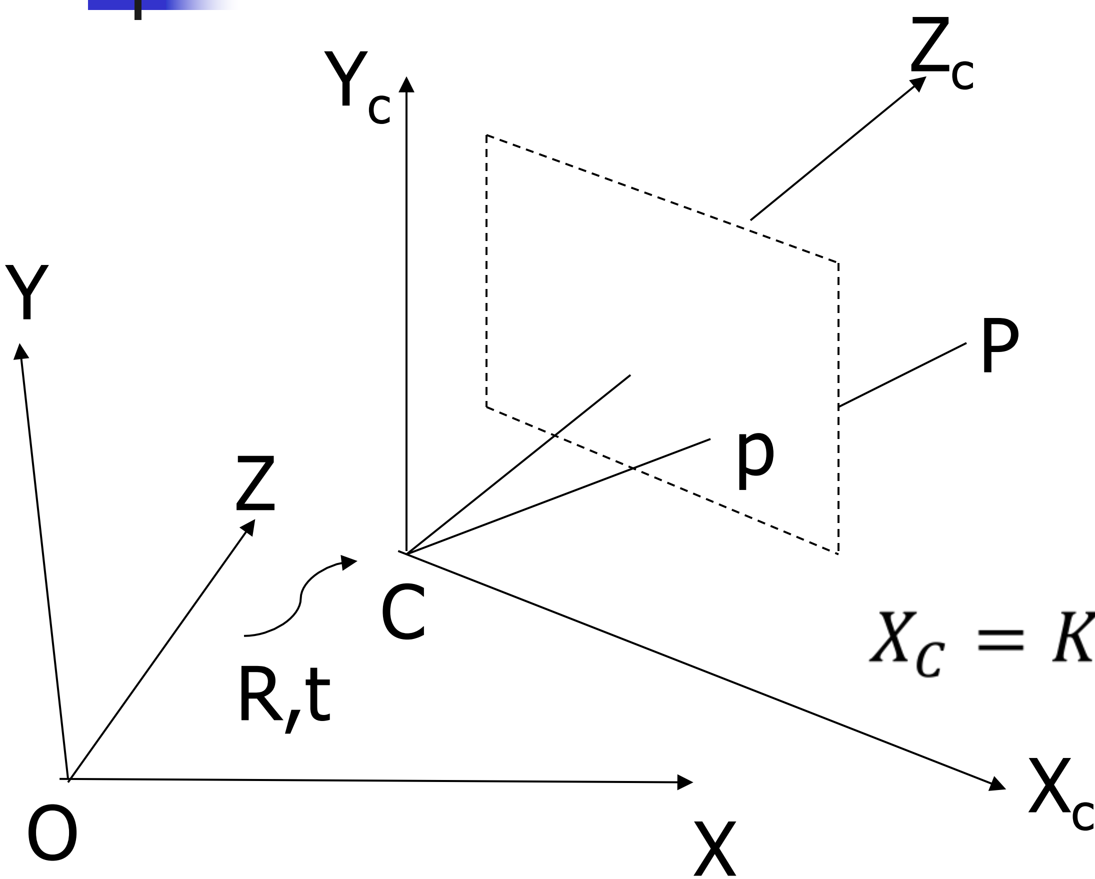
$$\boldsymbol{x} = K[I \quad | \quad 0]\boldsymbol{X}$$

K (Camera Calibration Matrix)

$$\tilde{X} \equiv Inhomogeneous\ Coordinate$$

$$X = \begin{bmatrix} \tilde{X} \\ 1 \end{bmatrix} \equiv Homogeneous\ Coordinate$$

Shifting of world coordinate



$$\widetilde{X_c} = R\left(\tilde{X} - \tilde{C}\right)$$

$$X_C = \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = K[I \quad | \quad 0]\boldsymbol{X_c}$$

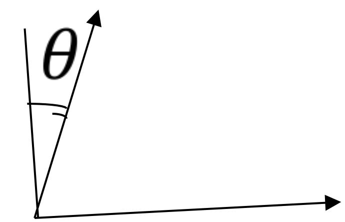
$$X_C = K[I \quad | \quad 0] \begin{bmatrix} R & -R\tilde{C} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$P = KR[I \quad | \quad -\tilde{C}] = K[R \quad | \quad t]$$

CCD Camera model

$$P = KR[I \quad | \quad -\tilde{C}] = K[R \quad | \quad t]$$

$$where \;\; K = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix}$$



$$Let \; \alpha_x = f.m_x,$$
$$\alpha_y = f.m_y$$

No. of pixels per unit length

$$s = \tan\theta$$

$$K = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

General Projective Camera

$$P = KR[I \quad | \quad -\tilde{C}] = K[\underbrace{R}_{3} \quad | \quad \underbrace{t}_{3}]$$

$$where\ K = \underbrace{\begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix}}_{5}$$

11 d.o.f

Extrinsic parameters: R, t

Intrinsic parameters: K

$|K| = \alpha_x\ \alpha_y > 0$

$$P = [M \quad | \quad p_4] = M[I \quad | \quad M^{-1}p_4] = KR[I \quad | \quad -\tilde{C}]$$

$$where\ M = KR\ and\ p_4\ is\ the\ last\ column\ of\ P.$$

K^{-1}

$$K^{-1} = \begin{bmatrix} \frac{1}{\alpha_x} & -\frac{s}{\alpha_x \alpha_y} & \frac{s p_y - \alpha_y p_x}{\alpha_x \alpha_y} \\ 0 & \frac{1}{\alpha_y} & -\frac{p_y}{\alpha_y} \\ 0 & 0 & 1 \end{bmatrix}$$

Upper triangular matrix.

$$\boldsymbol{x} = K[I \quad | \quad 0]\boldsymbol{X}_c$$

$K^{-1}\boldsymbol{x}$ provides you the image coordinate in canonical form for the above.

Properties of projective camera matrix $P=[M \mid p_4]$

Rank of P: 3;
Size: 3x4;
d.o.f.=11;

# of extrinsic params: 6
# of intrinsic params: 5

$$\boldsymbol{x} = P\boldsymbol{X} \longleftarrow$$ Two independent equations

Minimum # of point correspondences between world and image coordinates required to estimate P: 6

Estimation of the camera matrix (P)

$$P = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i} = (x_i \quad y_i \quad w_i)^T \ for\ i = 1,2,\dots,n \geq 6$$

$$P\boldsymbol{X_i} \equiv \boldsymbol{x_i}$$

$$\Longrightarrow P\boldsymbol{X_i} \times \boldsymbol{x_i} = 0$$

$$\Longrightarrow \begin{bmatrix} 0^T & -w_i\boldsymbol{X_i^T} & y_i\boldsymbol{X_i^T} \\ w_i\boldsymbol{X_i^T} & 0^T & -x_i\boldsymbol{X_i^T} \\ -y_i\boldsymbol{X_i^T} & x_i\boldsymbol{X_i^T} & 0^T \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$$

Redundant, as x_i x (1) + y_i x (2)= w_i x (3)

Estimation of the camera matrix (P)

$$P = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i} = (x_i \quad y_i \quad w_i)^T \ for\ i = 1,2,\ldots,n \geq 6$$

$$\begin{bmatrix} 0^T & -w_i\boldsymbol{X_i^T} & y_i\boldsymbol{X_i^T} \\ w_i\boldsymbol{X_i^T} & 0^T & -x_i\boldsymbol{X_i^T} \\ \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$$

For n correspondences $\quad A_{2n\times 12} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = 0$

Minimize $||A\boldsymbol{p}||$ subject to $||\boldsymbol{p}|| = 1$

Use similar techniques, such as DLT.

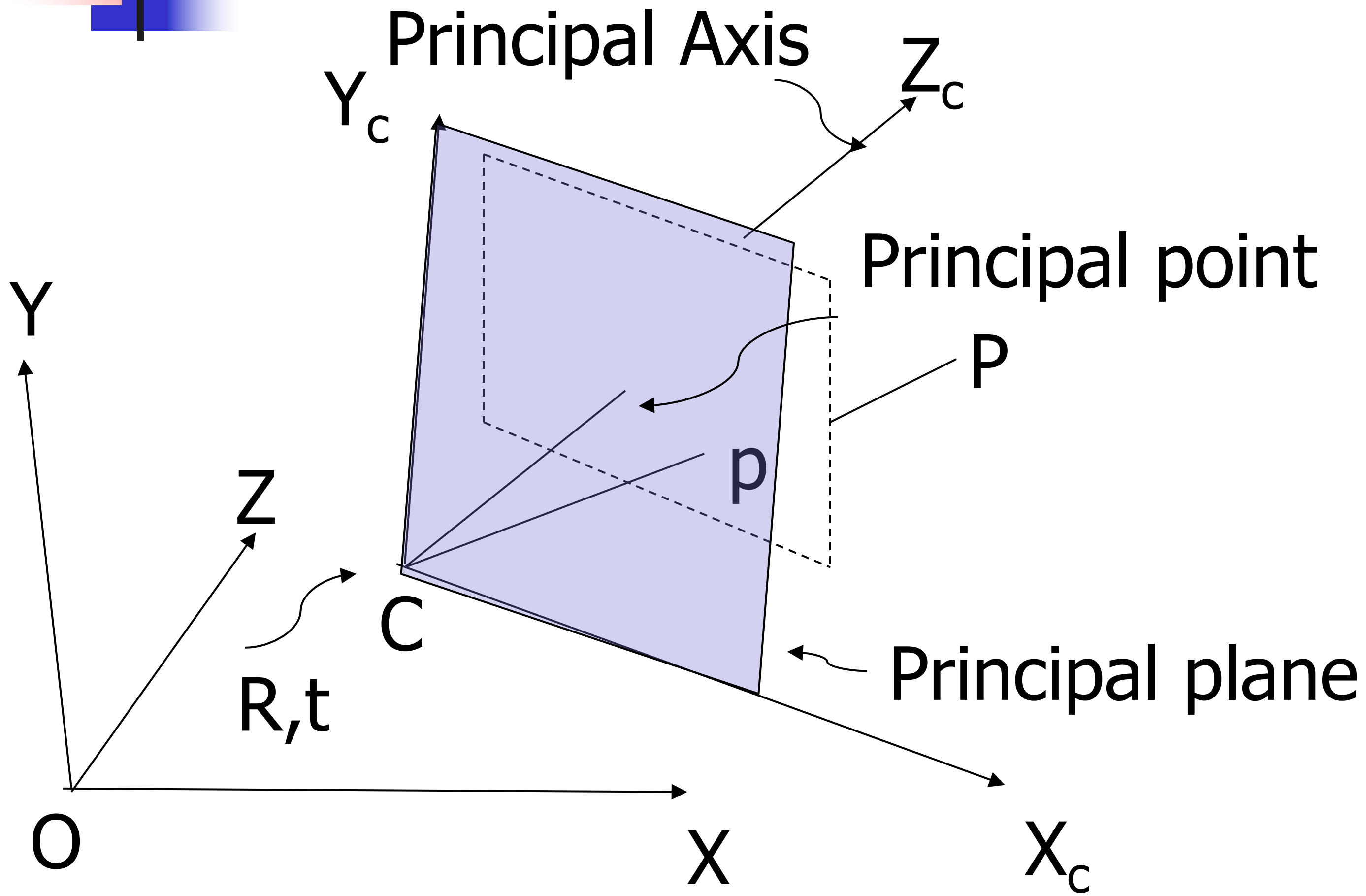
Properties of projective camera matrix $P=[M \mid p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

1. Camera Center (C): 1-D right null space of P, i.e. $PC=0$.
   1. Finite camera: M non-singular.
   2. Camera at infinity: M singular C= $\begin{bmatrix} d \\ 0 \end{bmatrix}$
2. Column points: p_1, p_2, and p_3 are vanishing points of X, Y and Z axes. p_4 is the image of coordinate origin.

$$p_1 = [p_1 \quad p_2 \quad p_3 \quad p_4] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad p_4 = [p_1 \quad p_2 \quad p_3 \quad p_4] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Principal plane, axis, and point



Properties of projective camera matrix $P=[M \mid p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

3. Principal plane: Plane parallel to image plane: r_3; As any point belonging to this plane should be imaged at $[x\ y\ 0]^{\mathrm{T}}$, $r_3^T \boldsymbol{X}=0$.
4. Axes plane: $r_1^T \boldsymbol{X}=0$ → Imaged at y-axis of the image coordinate, i.e. plane containing camera center ($r_1^T \boldsymbol{C}=0$) and y-axis of image plane.
5. Similarly, $r_2^T \boldsymbol{X}=0$ → Plane defined by camera center ($r_2^T \boldsymbol{C}=0$) and x-axis of image plane.
6. Principal point: $M.\ \boldsymbol{mr}_3$; $\boldsymbol{mr}_3$ is third row of M.

Properties of projective camera matrix $P=[M\,|\,p_4]$

$$P \equiv [p_1 \quad p_2 \quad p_3 \quad p_4] \equiv \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

6. Principal point: $M.\, \boldsymbol{mr_3}$; $\boldsymbol{mr_3}$ is third row of M. A point at infinity along the normal of $r_3^T X=0$ plane is projected to the principal point ($\boldsymbol{x_0}$).

$$\boldsymbol{x_0} = P \begin{bmatrix} p_{31} \\ p_{32} \\ p_{33} \\ 0 \end{bmatrix} = M.\, \boldsymbol{mr_3}$$

7. Principal Ray: $\boldsymbol{mr_3}$; $\boldsymbol{mr_3}$ is the third row of M. A point at infinity along the normal of $r_3^T X=0$ plane is projected to the principal point ($\boldsymbol{x_0}$). det(M). $\boldsymbol{mr_3}$ directed towards front of camera.

Projective camera on points

Forward projection: Mapping of vanishing points $(\boldsymbol{d}, 0)^T$ on the plane at infinity (π_∞):

$$\boldsymbol{x} = [M \mid p_4]\begin{bmatrix}\boldsymbol{d}\\0\end{bmatrix} = M\boldsymbol{d}$$

Only affected by M.

Back Projection:

$$[M \mid p_4]\begin{bmatrix}M^{-1}\boldsymbol{x}\\0\end{bmatrix} = \boldsymbol{x}$$

$$D=\begin{bmatrix}M^{-1}\boldsymbol{x}\\0\end{bmatrix}$$

$$X(\mu) = \mu D + C$$

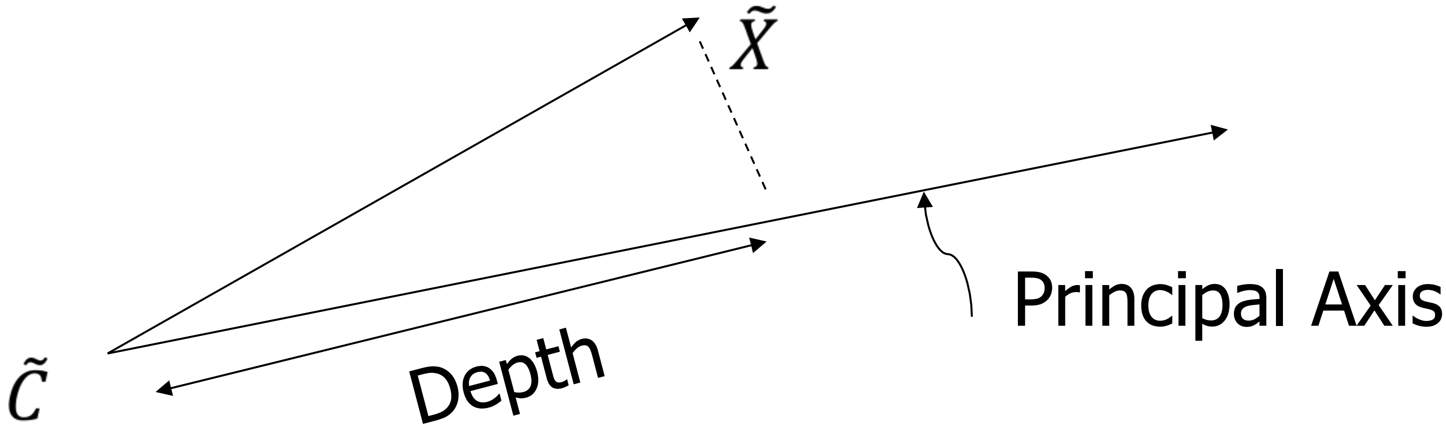
$$=\mu\begin{bmatrix}M^{-1}\boldsymbol{x}\\0\end{bmatrix}+\begin{bmatrix}\tilde{C}\\1\end{bmatrix}$$

$$=\mu\begin{bmatrix}M^{-1}\boldsymbol{x}\\0\end{bmatrix}+\begin{bmatrix}-M^{-1}p_4\\1\end{bmatrix}$$

$$C=\begin{bmatrix}\tilde{C}\\1\end{bmatrix}$$

$$X(\mu) = \begin{bmatrix}M^{-1}(\mu\boldsymbol{x}-p_4)\\1\end{bmatrix}$$

Depth of points



Unit vector

$$Depth = \widehat{mr_3^T}.(\tilde{X} - \tilde{C})$$

Computing camera center for $P = [M \mid p_4]$

$$M = [p_1 \quad p_2 \quad p_3] \qquad \tilde{C} = [X_c \quad Y_c \quad Z_c]^{\mathsf{T}}$$

$$PC = 0 \implies [M \quad | \quad p_4]\begin{bmatrix}\tilde{C}\\ 1\end{bmatrix} = 0$$

$$\implies M\tilde{C} = -p_4$$

$$X_c = \frac{|-p_4 \quad p_2 \quad p_3|}{|p_1 \quad p_2 \quad p_3|} \qquad Y_c = \frac{|p_1 \quad -p_4 \quad p_3|}{|p_1 \quad p_2 \quad p_3|}$$

$$Z_c = \frac{|p_1 \quad p_2 \quad -p_4|}{|p_1 \quad p_2 \quad p_3|}$$

Camera parameters from P

$$
\begin{aligned}
P &= [M \quad | \quad p_4] \\
&= [M \quad | \quad -M\tilde{C}] \\
&= K[R \quad | \quad -R\tilde{C}]
\end{aligned}
$$

1. RQ-decomposition of M s.t. $M=KR$, where K is an upper-triangular matrix and R is an orthogonal matrix.
2. Obtain camera center using $M\tilde{C} = -p_4$.
3. From R get the orientation of camera.
4. From K get elements of calibration matrix.

Exercise -1

Consider the following projection matrix.

$$P = \begin{bmatrix} -9 & 2 & 3 & 1 \\ 3 & -9 & 6 & 1 \\ 2 & 6 & -10 & 1 \end{bmatrix}$$

Compute the following:

(i) Camera center
(ii) Vanishing point of X-axis.
(iii) Image point of origin.
(iv) Vanishing point of the line with the direction cosines 2:3:4.

Solution

$$P = \begin{bmatrix} -9 & 2 & 3 & 1 \\ 3 & -9 & 6 & 1 \\ 2 & 6 & -10 & 1 \end{bmatrix}$$

M

p_4

$$\tilde{C} = -M^{-1} p_4$$

$$Cofactor(M) = \begin{bmatrix} 54 & 42 & 36 \\ 38 & 84 & 58 \\ 39 & 63 & 75 \end{bmatrix} \qquad M^{-1} = -\frac{1}{294}\begin{bmatrix} 54 & 38 & 39 \\ 42 & 84 & 63 \\ 36 & 58 & 75 \end{bmatrix}$$

$$\det(M) = -9(90-36) + 2(12+30) + 3(18+18) = -294$$

$$\tilde{C} = \frac{1}{294}\begin{bmatrix} 131 \\ 189 \\ 169 \end{bmatrix}$$

Solution (Contd.)

$$P = \begin{bmatrix} -9 & 2 & 3 & 1 \\ 3 & -9 & 6 & 1 \\ 2 & 6 & -10 & 1 \end{bmatrix}$$

p_1 p_4

Vanishing point of X-axis: $P\,[\,1\ 0\ 0\ 0]^T = p_1$

Image point of origin: $P\,[\,0\ 0\ 0\ 1]^T = p_4$

Vanishing point of the line with the direction cosines 2:3:4

$P\,[\,2\ 3\ 4\ 0]^T$

$= [\,0\ 3\ -18]^T$

Exercise-2

- Consider the following projection matrix of an optical camera based imaging system.

$$P = \begin{bmatrix} 8 & 5 & 4 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -5 & 8 & 1 \end{bmatrix}$$

Answer the following with respect to P.

(a) Given an image point (2,7) in R^2, compute its corresponding scene point if it is known that the point is at a distance of 40 units from the center of camera.

Ans. 2(a)

$$P = \left[\underbrace{\begin{matrix}8 & 5 & 4\\ 7 & 8 & 9\\ 1 & -5 & 8\end{matrix}}_{M}\ \ \underbrace{\begin{matrix}0\\ 0\\ 1\end{matrix}}_{p_4}\right]$$

$$M^{-1} = \frac{1}{465}\begin{bmatrix}109 & -60 & 13\\ -47 & 60 & -44\\ -43 & 45 & 29\end{bmatrix}$$

Camera center:

$$\tilde{C} = -M^{-1}p_4 = \frac{1}{465}\begin{bmatrix}-13\\ 44\\ -29\end{bmatrix}$$

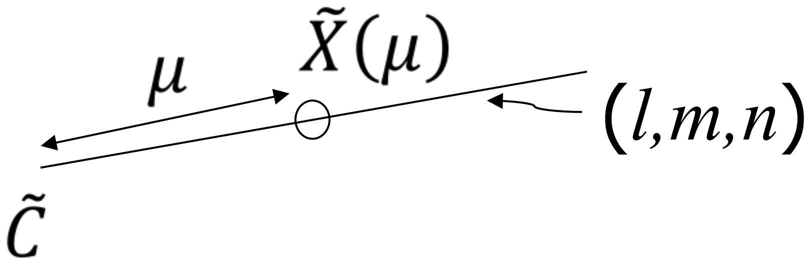
Direction ratio (l,m,n):

$$\begin{bmatrix}l\\ m\\ n\end{bmatrix} = M^{-1}\begin{bmatrix}2\\ 7\\ 1\end{bmatrix} = \frac{1}{155}\begin{bmatrix}-63\\ 94\\ 86\end{bmatrix}$$

$$\tilde{X}(\mu) = \tilde{C} + \frac{\mu}{\sqrt{l^2+m^2+n^2}}\begin{bmatrix}l\\ m\\ n\end{bmatrix}$$

Where μ is the distance from $\tilde{C}$.

=40



$$P = \begin{bmatrix} 8 & 5 & 4 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -5 & 8 & 1 \end{bmatrix}$$

Q. 2(b)

- Compute the principal plane of the imaging system.

Image point of a point in a principal plane: $(x, y, 0)$

$r_3^T X = 0$ $\Rightarrow$ The last row of P.

$\Rightarrow$ (1,-5, 8, 1)

Exercise-3

- Consider the following camera matrix.

$$P = \begin{bmatrix} 7 & 4 & 9 & 0 \\ 2 & 3 & 6 & 0 \\ 1 & 5 & 8 & 0 \end{bmatrix}$$

Consider four image points $x_1=(2,5)$, $x_2=(7,9)$, $x_3=(-1,3)$ and $x_4=(4,-1)$. Let the camera center be denoted as O. Compute the dihedral angle between planes of Ox_1x_2 and Ox_3x_4. A dihedral angle of two planes is the angle between their normals.

Ans.

$x_1=(2,5)$,
$x_2=(7,9)$,
$x_3=(-1,3)$
and
$x_4=(4,-1)$

$$P = \begin{bmatrix} 7 & 4 & 9 & 0 \\ 2 & 3 & 6 & 0 \\ 1 & 5 & 8 & 0 \end{bmatrix}$$

$$\Pi_1 = P^T(x_1 \times x_2) = \begin{bmatrix} 35 \\ 86 \\ 142 \\ 0 \end{bmatrix} \Rightarrow \hat{n}_1 = \frac{1}{\sqrt{35^2 + 86^2 + 142^2}} \begin{bmatrix} 35 \\ 86 \\ 142 \end{bmatrix}$$

$$\Pi_2 = P^T(x_3 \times x_4) = \begin{bmatrix} -27 \\ 24 \\ 22 \\ 0 \end{bmatrix} \Rightarrow \hat{n}_2 = \frac{1}{\sqrt{27^2 + 24^2 + 22^2}} \begin{bmatrix} -27 \\ 24 \\ 22 \end{bmatrix}$$

$$\theta = \cos^{-1}(\hat{n}_1 . \hat{n}_2) = 53.752^\circ$$

Cameras at ∞

$P = [M \quad | \quad p_4]$ where M is singular.

Affine: Last row of P $[0 \quad 0 \quad 0 \quad 1]$

Non-affine: Otherwise

Affine Camera:

1. Principal plane → Plane at ∞ (π_∞).
2. Camera center lies on π_∞.
3. Points at ∞ are mapped to points at ∞.
4. Parallel lines remain parallel after projection.

$$P\begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Affine projection

$$\begin{bmatrix} \widetilde{\boldsymbol{x}} \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & t_1 \\ m_{21} & m_{22} & m_{23} & t_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{X}} \\ 1 \end{bmatrix}$$

$$[\widetilde{\boldsymbol{x}}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} [\widetilde{\boldsymbol{X}}] + \boldsymbol{t}$$

$$\widetilde{\boldsymbol{x}} = M_{2\times 3}\widetilde{\boldsymbol{X}} + \boldsymbol{t}$$

- Affine projection matrix: 8 d.o.f.
- For estimating the matrix, it requires four point correspondences.

$$[\tilde{\boldsymbol{x}}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix} [\tilde{\boldsymbol{X}}] + \boldsymbol{t}$$

Affine Camera

$$\tilde{\boldsymbol{x}} = M_{2\times 3}\tilde{\boldsymbol{X}} + \boldsymbol{t}$$

Camera Center➔ Direction of parallel rays ($\boldsymbol{d}$)

$$M_{2\times 3}\boldsymbol{d} = 0$$

- Image of the world origin: $\boldsymbol{t}$
- Principal plane for projection matrix P_A is the plane at ∞.
- Parallel world lines remain parallel in image.
- $M_{2\times 3}$ should be of rank 2, to ensure P_A to be of rank 3.

$$\begin{bmatrix}\widetilde{\boldsymbol{x}}\\1\end{bmatrix}=\begin{bmatrix}m_{11} & m_{12} & m_{13} & t_1\\ m_{21} & m_{22} & m_{23} & t_2\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{\boldsymbol{X}}\\1\end{bmatrix}$$

Estimation of an affine camera

$$\boldsymbol{X_i} \leftrightarrow \boldsymbol{x_i}=(x_i, y_i, 1),\ for\ i=1,2,3,\ldots,n$$

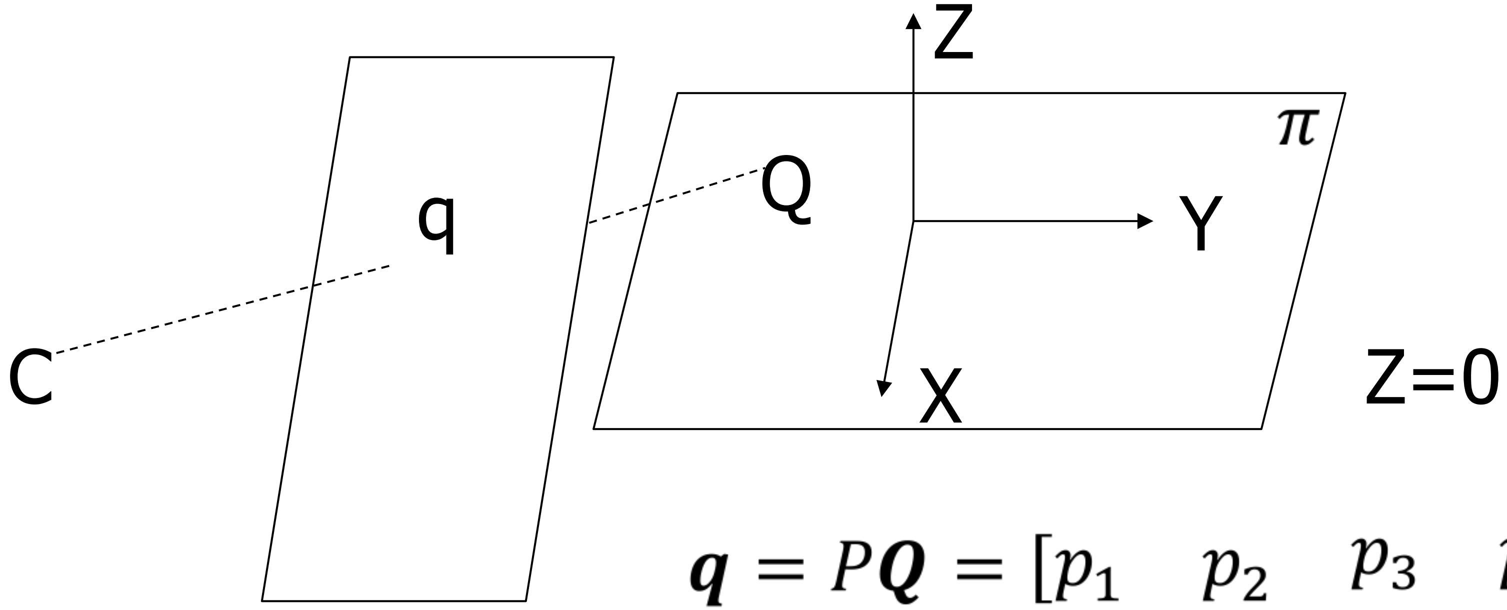
$$\boldsymbol{r_3}^T=[0 \quad 0 \quad 0 \quad 1]$$

$$\begin{bmatrix}\boldsymbol{X_i} & 0^T\\ 0^T & \boldsymbol{X_i}\end{bmatrix}\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=\begin{bmatrix}x_i\\ y_i\end{bmatrix}$$

For n points $$A_{2n\times 8}\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=b_{2n\times 1}$$

$$\begin{bmatrix}\boldsymbol{r_1}\\ \boldsymbol{r_2}\end{bmatrix}=[A^TA]^{-1}A^Tb$$

Projective Camera on plane

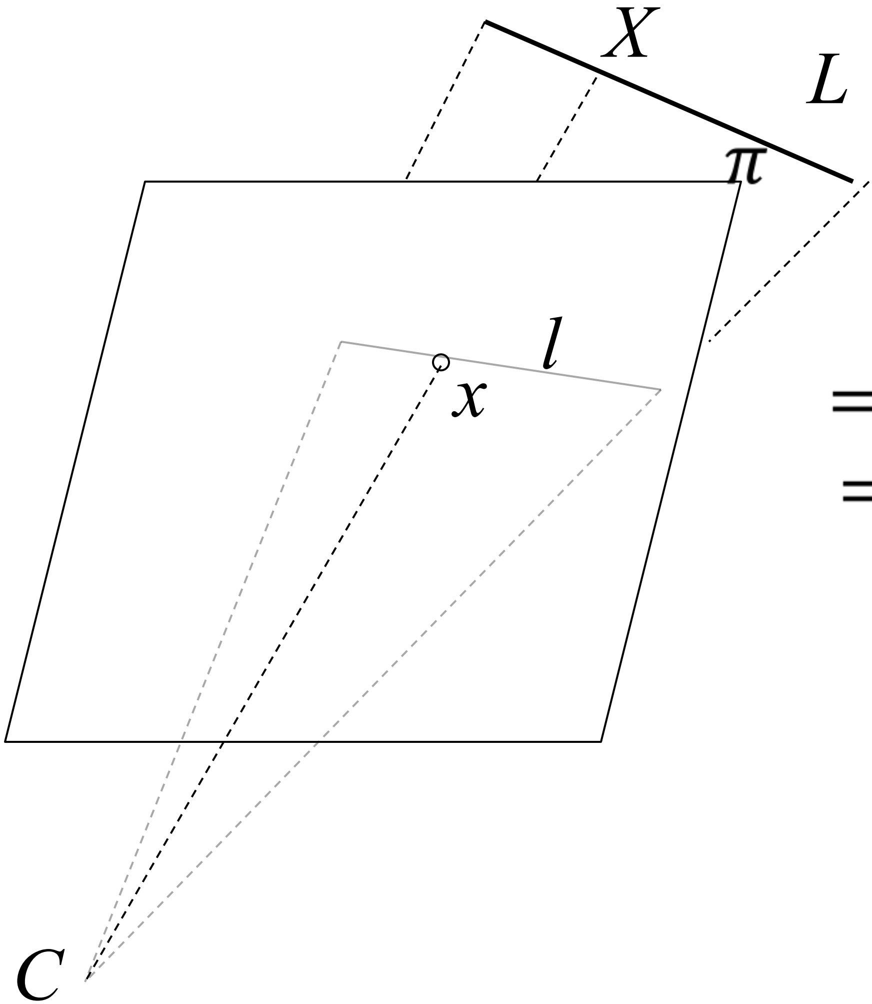


$$\boldsymbol{q} = P\boldsymbol{Q} = [p_1 \quad p_2 \quad p_3 \quad p_4]\begin{bmatrix} X \\ Y \\ 0 \\ 1 \end{bmatrix}$$

$$= [p_1 \quad p_2 \quad p_4]\begin{bmatrix} X \\ Y \\ 1 \end{bmatrix} = H\,Q_{\pi}$$

Perspective projection of scene plane is a projective transformation.

Projective camera on a line



$$x^T l = 0$$

$$\Longrightarrow (PX)^T l = 0$$

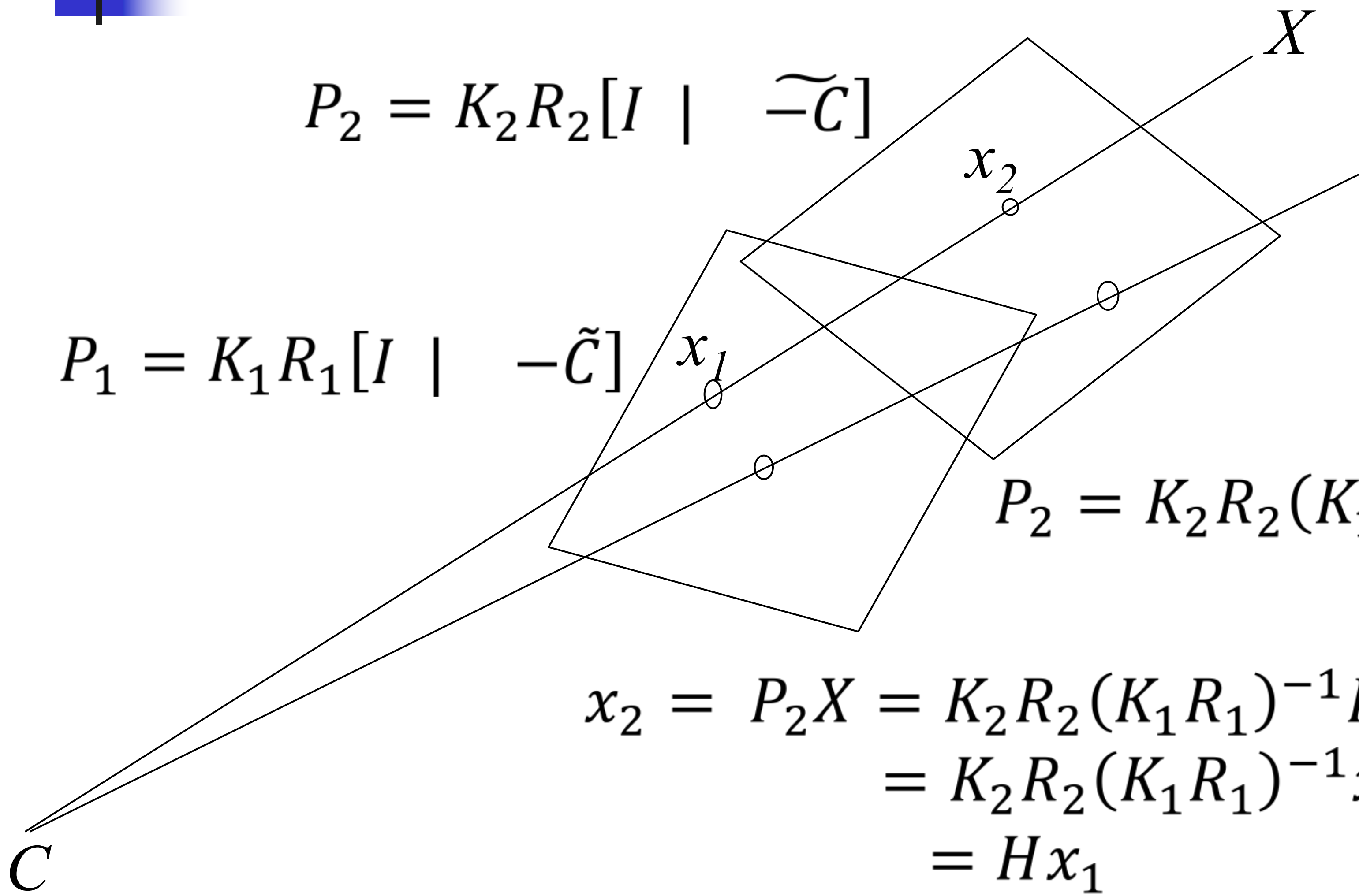
$$\Longrightarrow X^T \underbrace{P^T l}_{\pi} = 0$$

$$\pi \equiv P^T l$$

Fixed camera center and moving image plane

$$P_2 = K_2 R_2 [I \mid -\tilde{C}]$$

$$P_1 = K_1 R_1 [I \mid -\tilde{C}]$$

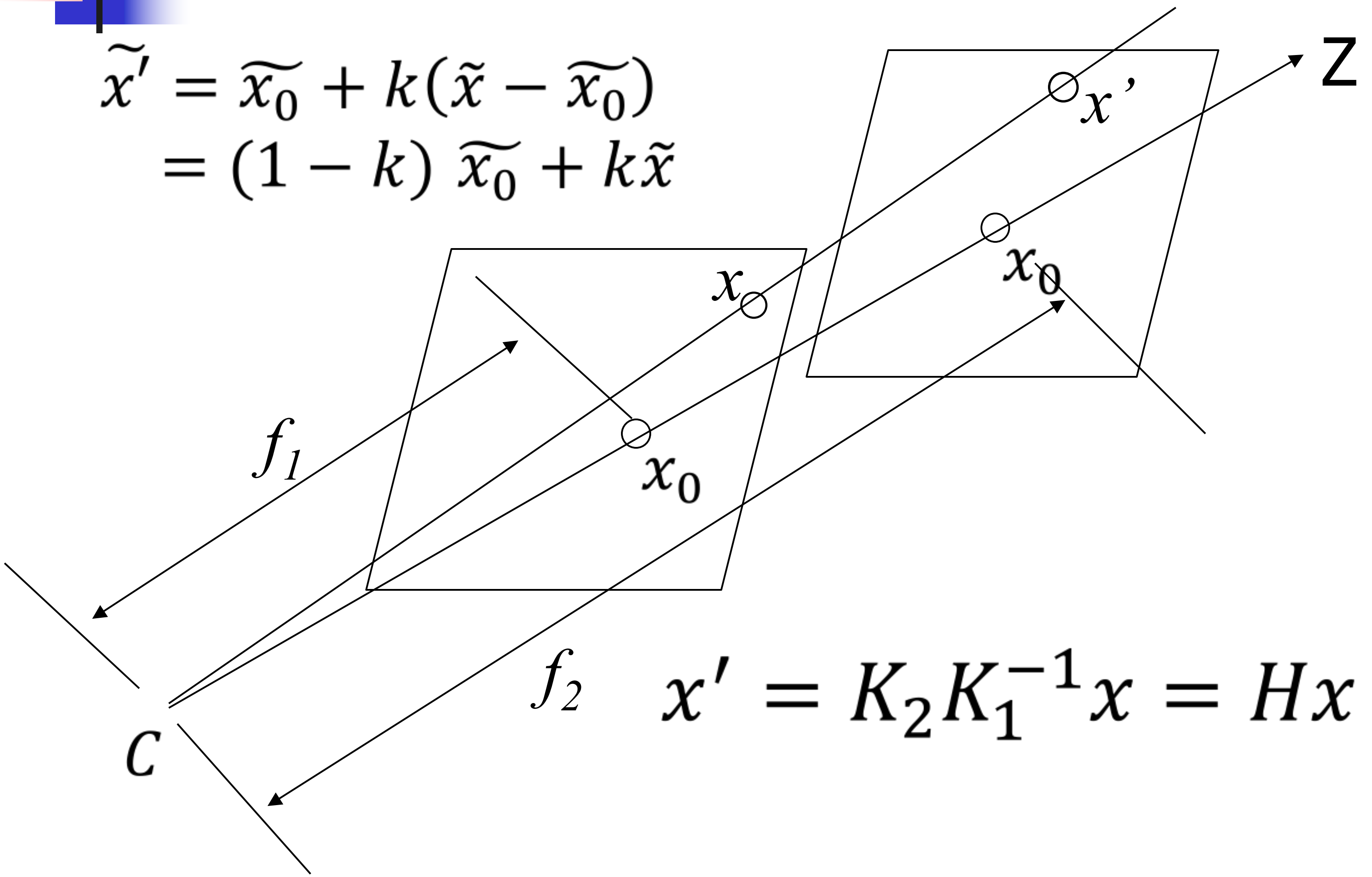


$$P_2 = K_2 R_2 (K_1 R_1)^{-1} P_1$$

$$\begin{aligned} x_2 = P_2 X &= K_2 R_2 (K_1 R_1)^{-1} P_1 X \\ &= K_2 R_2 (K_1 R_1)^{-1} x_1 \\ &= H x_1 \end{aligned}$$

Simple zooming ($k=f_2/f_1$, $R=I$)

$$\widetilde{x'} = \widetilde{x_0} + k(\tilde{x} - \widetilde{x_0})$$
$$= (1-k)\,\widetilde{x_0} + k\tilde{x}$$



$$x' = K_2 K_1^{-1} x = Hx$$

Simple Zooming

$$x' = K_2 K_1^{-1} x = Hx$$

$$\Rightarrow H = \begin{bmatrix} kI & (1-k)\,\widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} = K_2 K_1^{-1}$$

$$\Rightarrow K_2 = \begin{bmatrix} kI & (1-k)\,\widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} K_1$$

$$= \begin{bmatrix} kI & (1-k)\,\widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} A & \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} kA & k\widetilde{x_0} + (1-k)\,\widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

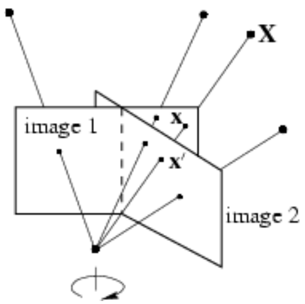
$$= \begin{bmatrix} kA & \widetilde{x_0} \\ \mathbf{0} & 1 \end{bmatrix}$$

$$= K_1 \begin{bmatrix} kI & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} = K_1 \, . \, diag(k,k,1)$$

The effect of zooming by a factor k is to multiply the calibration matrix K on the right by $diag(k,k,1)$.

Rotation about an axis passing through the camera center (assuming at origin)

$$x = K[I \mid 0]X$$

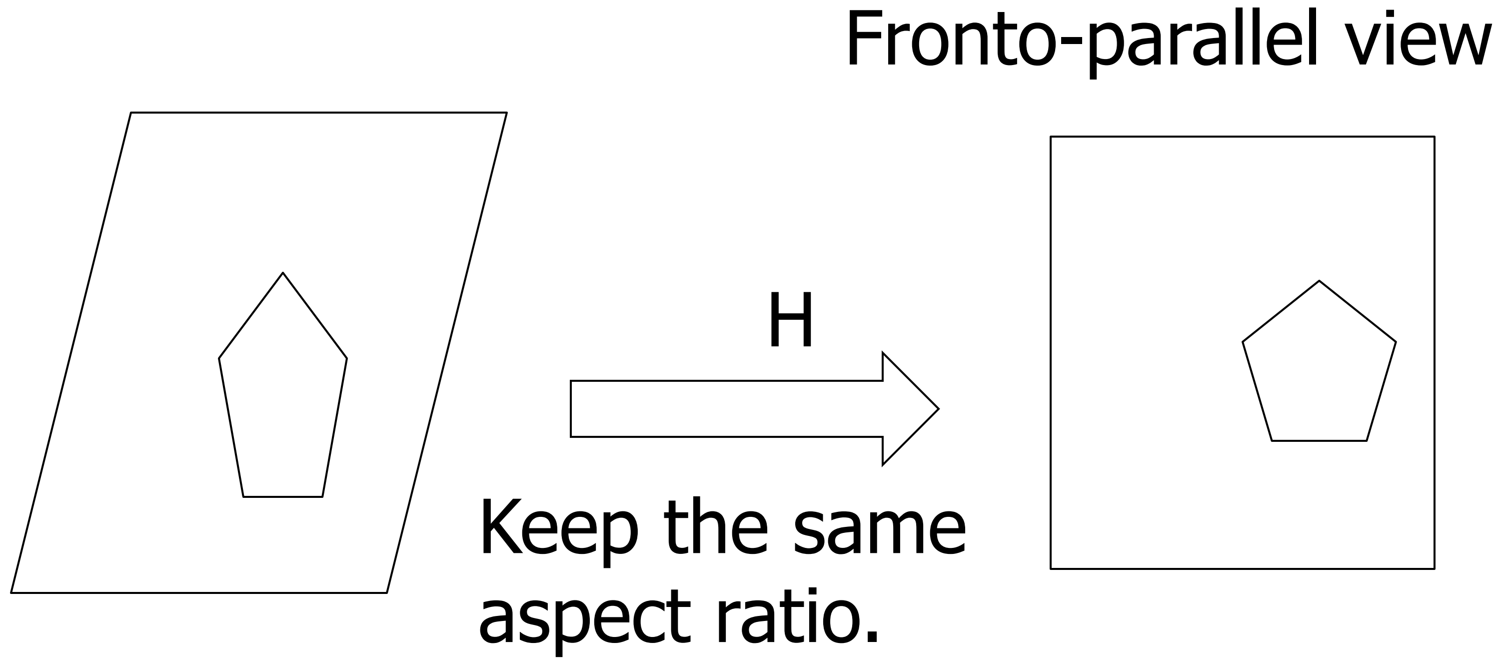


$$\begin{aligned} x' &= K[R \mid 0]X \\ &= KRK^{-1}K[I \mid 0]X \\ &= KRK^{-1}x \\ \Longrightarrow H &= KRK^{-1} \end{aligned}$$

- H has the same eigen values (upto scale) as R, namely $\mu, \mu e^{i\theta}$, and $\mu e^{-i\theta}$, where μ is the scale factor.
- H is also known as *conjugate rotation* homography and can be used to measure the angle of rotation of two views.
- The eigen vector corresponding to the real eigen value (i.e. μ) is the vanishing point of the rotation axis.

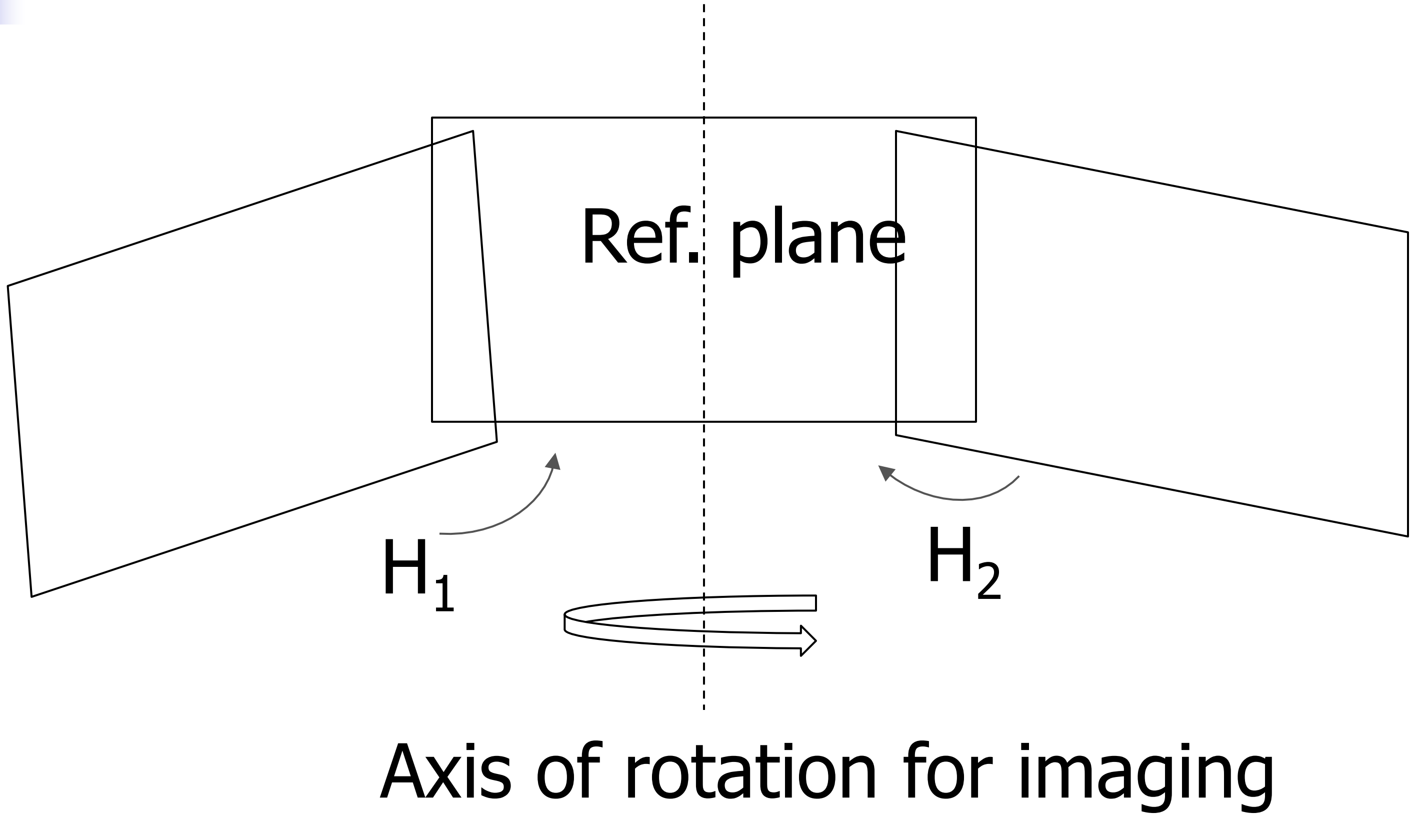
From Hartley and Zisserman, "Multiple view geometry in computer vision", Cambridge Univ. Press (2000)

Application-I: Generation of synthetic view



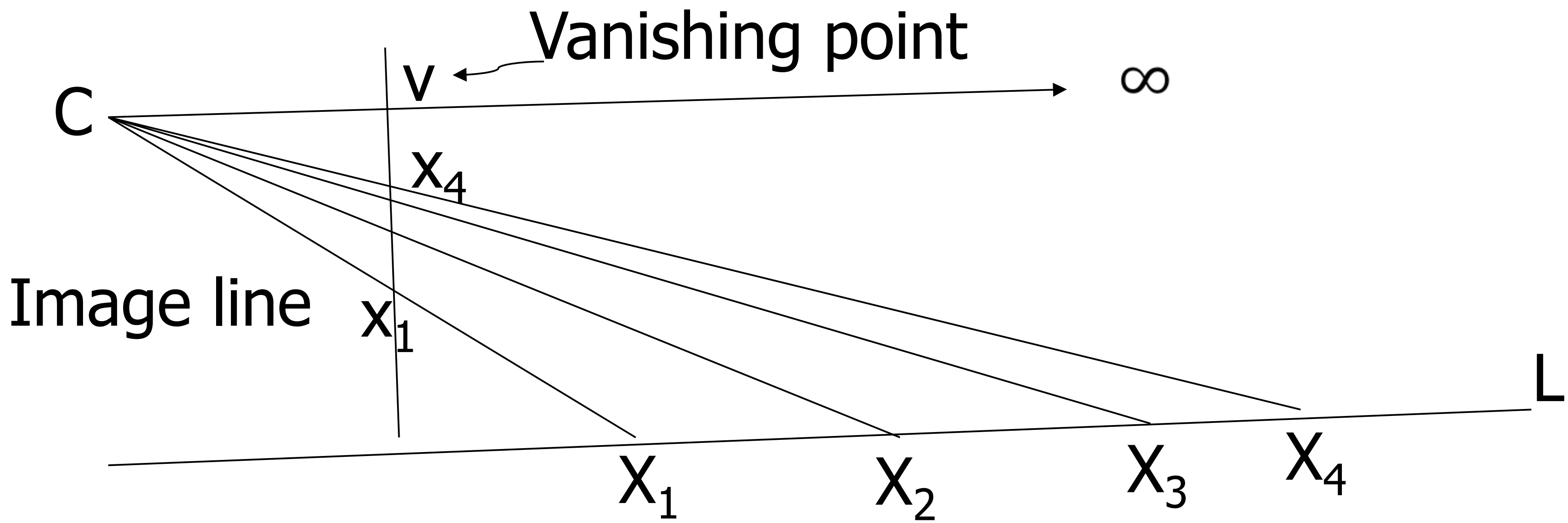
1. Compute H.
2. Warp the source image with H.

Planar panoramic mosaicing



Vanishing points

Vanishing points: images at points at ∞.

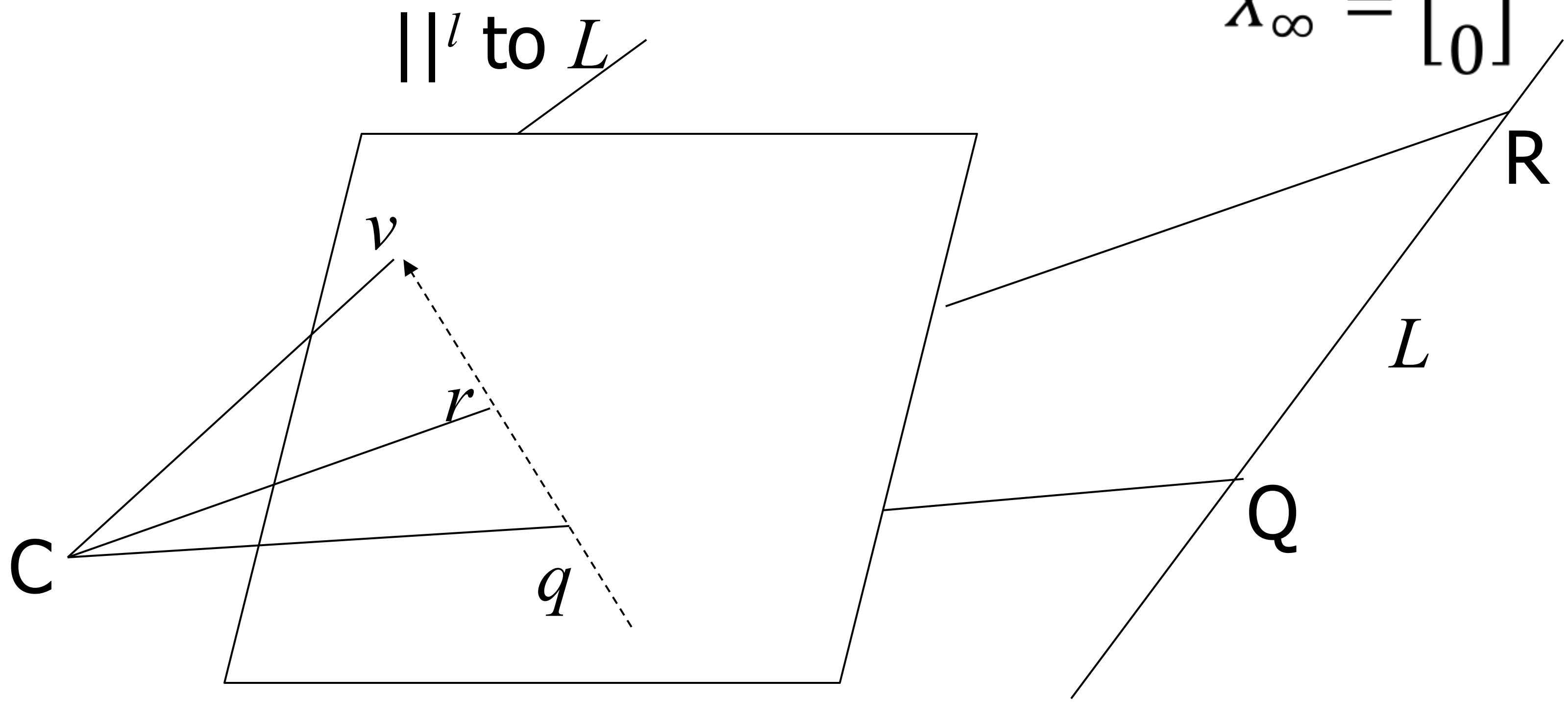


Vanishing point of a line L is the intersecting point in the image plane parallel to L and passing through the camera center C.

$$v = PX_\infty = K[I \mid 0]\begin{bmatrix}\mathbf{d}\\0\end{bmatrix} = K\mathbf{d}$$

Vanishing points

$$X_\infty = \begin{bmatrix}\mathbf{d}\\0\end{bmatrix}$$



Vanishing points are independent of camera position, if it is not rotated.

$$v = PX_\infty = K[I \quad | \quad 0] \begin{bmatrix} \boldsymbol{d} \\ 0 \end{bmatrix} = K\boldsymbol{d}$$

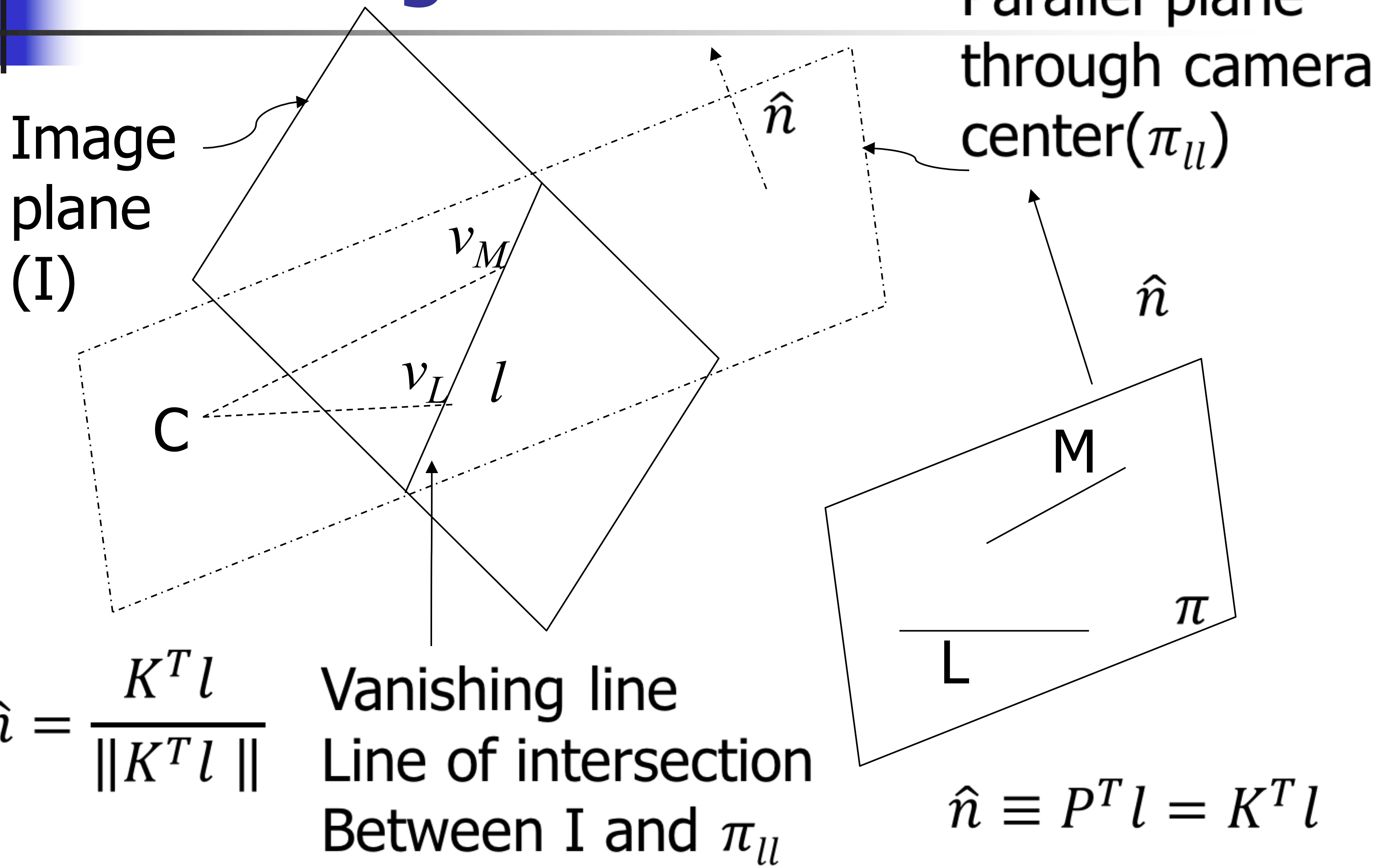
Vanishing points

Vanishing points are independent of camera position, if it is not rotated.
With rotation R it becomes $v' = KR\boldsymbol{d}$.
If we know v, v', and K, we can compute R.

$$\hat{\boldsymbol{d}} = \frac{K^{-1}v}{\|K^{-1}v\|} \qquad \hat{\boldsymbol{d}}' = \frac{K^{-1}v'}{\|K^{-1}v'\|} \qquad \hat{\boldsymbol{d}}' = R\boldsymbol{d}$$

Two independent constraints on R and it can be computed.

Vanishing lines



$$\hat{n} = \frac{K^T l}{\|K^T l\|}$$

$$\hat{n} \equiv P^T l = K^T l$$

Exercise

Suppose a camera has the following projection matrix P.

$$P = \begin{bmatrix} 8 & 5 & 4 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -5 & 8 & 1 \end{bmatrix}$$

Given a line l in the image coordinate space by the following equation:

$$3x+4y=5$$

Compute the normal of the plane for which the line appears as a horizon (vanishing line).

Ans.

$$P = \begin{bmatrix} 8 & 5 & 4 & 0 \\ 7 & 8 & 9 & 0 \\ 1 & -5 & 8 & 1 \end{bmatrix}$$

$l = [\ 3\ 4\ {-5}]^T$

Plane formed by the camera center and the line l: $P^T l$

$$\begin{bmatrix} 8 & 7 & 1 \\ 5 & 8 & -5 \\ 4 & 9 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = \begin{bmatrix} 47 \\ 72 \\ 8 \\ -5 \end{bmatrix}$$

All planes parallel to this plane have the vanishing line l.

$$\hat{n} = \frac{1}{\sqrt{47^2 + 72^2 + 8^2}} \begin{bmatrix} 47 \\ 72 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} .54 \\ .83 \\ .09 \end{bmatrix}$$

Computing vanishing line

- Identify groups of sets of parallel lines in a plane at different directions.
- Obtain their vanishing points.
- Get the line among them.

Summary

- Pinhole camera model provides the projection matrix which maps a 3D point to an image point.
- Projection matrix:
  - 3x4
  - Dof: 11
  - 5 intrinsic parameters and 6 extrinsic parameters.
  - Minimum 6 point correspondences required for estimation
- Affine projection matrix
  - Last row $[0\ 0\ 0\ 1]^T$
  - Dof:8
  - Minimum 4 point correspondences required to estimate.

Summary (contd.)

- Geometry encoded in a projection matrix
  - $P=[\ M \mid p_4]$ or $P=[p_1\ p_2\ p_3\ p_4]$ or $P=[r_1^T\ ;\ r_2^T\ ;\ r_3^T]$
  - Camera Center: $-M^{-1}p4$
    - For affine projection matrix: Right zero of M (A direction).
  - Vanishing points
    - X-axis: p_1
    - Y-axis: p_2
    - Z-axis: p_3
  - Image of world origin: p_4
  - Special planes passing through the camera center
    - Principal plane: $r_3^TX=0$
      - Direction of optical axis: $<r_{31}, r_{32}, r_{33}>$
      - Principal point: $M\ [r_{31}\ \ r_{32}\ r_{33}]^T$
    - Plane formed with x-axis of image coordinate system: $r_1^TX=0$
    - Plane formed with y-axis of image coordinate system: $r_2^TX=0$

Summary (contd.)

- Geometric derivatives from Projection Matrix: $P = [M|p_4]$
  - Projection ray formed at image point **x**.
    - Direction ratio: $M^{-1}\mathbf{x}$
    - A point on the ray:
      - Camera center $(-M^{-1}p_4)$
  - Plane formed with a line l in the image plane with the camera center: $P^T l$
  - Vanishing point of a line with direction **d**
    - $M\mathbf{d}$