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# On approximating Euclidean metrics by digital distances in 2D and 3D

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**Abstract**

In this paper a geometric approach is suggested to find the closest approximation to Euclidean metric based on geometric measures of the digital circles in 2D and the digital spheres in 3D for the generalized octagonal distances. First we show that the vertices of the digital circles (spheres) for octagonal distances can be suitably approximated as a function of the number of neighborhood types used in the sequence. Then we use these approximate vertex formulae to compute the geometric features in an approximate way. Finally we minimize the errors of these measurements with respect to respective Euclidean discs to identify the best distances. We have verified our results by experimenting with analytical error measures suggested earlier. We have also compared the performances of the good octagonal distances with good weighted distances. It has been found that the best octagonal distance in 2D ($\{1,1,2\}$) performs equally good with respect to the best one for the weighted distances ($\langle 3,4\rangle$). In fact in 3D, the octagonal distance $\{1,1,3\}$ has an edge over the other good weighted distances. © 2000 Elsevier Science B.V. All rights reserved.

*Keywords:* Digital geometry; Distance function; Neighborhood sequence; Octagonal distances; Weighted distances; Digital circle; Digital sphere

## 1. Introduction

Distance transforms are widely used in digital shape analysis. In a distance transform each object point in the image has a value measuring the shortest distance to the background points. There are different applications of distance transforms such as in morphological operations, skeletonization, computing Voronoi diagrams, template matching, geometric transformation (Aswatha Kumar et al., 1996), data compression (Aswatha Kumar et al., 1995), computation of cross-sections (Mukherjee et al., 1998), rendering of 3D objects (Mukherjee et al., 1999) etc. An overview of some of these applications is given in (Borgefors, 1994). The distance transform was first proposed by Blum (1964) for a continuous binary picture on a 2-dimensional plane. Then Rosenfeld and Pfaltz (1968) incorporated the idea in digital images. The basic idea behind the computation is to determine the global minimum distances from the background by propagating the local distances between neighboring pixels starting from the boundaries. Rosenfeld and Pfaltz used two types of neighbors,

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4-neighbor (horizontal and vertical neighbors), called cityblock distance and 8-neighbor (horizontal, vertical and diagonal neighbors), also called as chessboard distance. One of the prime motivations of using different digital distance functions is to approximate the Euclidean metric. But the 4-neighbor or 8-neighbor distance functions are far from the Euclidean metric. Rosenfeld and Pfaltz (1968) introduced the concept of octagonal distance in 2D for digital pictures which is also a metric. They proved that an alternate sequence of cityblock and chessboard motions defines a new integer valued metric which can approximate the true Euclidean metric better, than the conventional cityblock and chessboard distances. They have also introduced the concept of generalized distances and concluded an optimal distance should be such that for a distance with $m$ cityblock motions followed by $n$ chessboard motions, $2n/m$ should be as close as possible to $\sqrt{2}$. However a detailed careful investigation has been carried out by Das and Chatterji (1990). Toriwaki et al. (1981) discussed the sequential algorithm for the octagonal distance by calculating the sequence of pictures using the four raster scanning modes. Das and Chatterji (1990) have extended the definition to allow for arbitrarily long cyclic sequences of cityblock and chessboard motions called neighborhood sequences. This general definition has been shown to be octagonal still, since it always corresponds to discs of constant radii which are digital octagons. Detailed analysis of such octagons with respect to the area and perimeter errors for an Euclidean circle shows that in every such neighborhood sequence the actual order in which the two motions are arranged is of little consequence in an asymptotic sense (of the distance value) so long as the length of the sequence and the numbers of cityblock and chessboard motions remain constant. Thus for all practical purposes and for the ease of analysis it is sufficient to restrict the attention to the neighborhood sequences which are sorted (increasing). The sorted order in the neighborhood also guarantees metricity which is a basic necessity in any analysis.

In this paper we have considered generalized octagonal distances both in 2D and 3D digital spaces. We have evaluated their performances in approximating the Euclidean metric in respective dimensions. Earlier Das (1992) presented a detailed analysis for obtaining the best approximations to Euclidean metric over a set of octagonal distances in 2D. In our work, we have taken a different approach based on the geometry of the digital discs, for the comparative assessments of their performances. Following the similar approach we are also able to get relative performance measures for 3D octagonal distances. It may be noted here that Danielsson (1993) also adopted a similar approach for evaluating octagonal distances in 2D and 3D. But his treatment in 3D was incomplete, as he had considered only those 3D octagonal distances whose digital spheres are of the shape of a convex polyhedron with 26 faces. But there are other octagonal distances having digital spheres in the shape of polyhedra with 6, 12, 14, 18 and 20 faces also. That is why Danielsson (1993) could not explain why some of the 3D octagonal distances performed better than others. In our work, we have considered all possible shapes of the digital spheres for the set of 3D octagonal distances and presented our analysis for obtaining good octagonal distances. It may be noted here that an initial version of this work was briefly reported in (Aswatha Kumar et al., 1995).

It is interesting to note that there are other distance tranforms as well for approximating the Euclidean norms. Borgefors (1984, 1986, 1993, 1996) considered many such distance transforms. One such interesting distance transform is the *weighted distance transform* (WDT) both in 2D and 3D. In 2D the weighted distance functions are positive linear combinations of the cityblock and chessboard distance functions. Borgefors (1984, 1986) has shown that $\langle 3,4 \rangle$, $\langle 2,3 \rangle$, and $\langle 8,11 \rangle$ (for notations and explanations please refer Borgefors, 1984) are some of the good choices in 2D for approximating Euclidean metrics. Similar ideas have been extended to 3D and it has been shown (Borgefors, 1996) that $\langle 3,4,5 \rangle$, $\langle 8,11,13 \rangle$ and $\langle 13,17,23 \rangle$ (for notations and explanations please refer Borgefors, 1996) are reasonably good approximations to the Euclidean metric. In our work we have also presented the relative performances between the good octagonal distances and the good weighted distances.

In Section 2 we present the definitions and notations for generalized octagonal distances both in 2D and 3D. In this section we also present the properties of their digital discs. In Section 3 the computation of geometric features related to the shape of these discs are described. Subsequently, the error analysis is presented for obtaining good octagonal distances. Finally we have also presented the comparative assessments of the performances of good octagonal distances and good weighted distances based on two performance measures.

## 2. Generalized octagonal distances

In 2D Euclidean space $\mathbb{R}^2$, where $\mathbb{R}$ is the set of real numbers, a circle $C(q,b)$, centered at $q \in \mathbb{R}^2$ and having radius $b \in \mathbb{R}$ is defined as the set of points in $\mathbb{R}^2$ whose Euclidean distance from the center $q$ is less than or equal to $b$, i.e.

$$C(q,b) = \{p \mid (p \in \mathbb{R}^2) \text{ and } (E(p,q) \leqslant b)\},$$

where $E(p,q)$ is the Euclidean metric. In 3D Euclidean space $\mathbb{R}^3$, a sphere $S(q,B)$ is defined similarly.

In 2D digital space $Z^2$ or 3D digital space $Z^3$ these definitions are immediately extended where digital, instead of Euclidean distances are used. In 2D digital plane, two types of motion are natural. They are cityblock motion and chess board motion. The distance $d(B)$ between two points is defined as the length of a shortest path between these two points which is restricted by a particular type of motion $B$. The cityblock movement is marked here as 1-neighbor or $B = \{1\}$ as it allows a unit change in at most one of the coordinates whereas the chessboard movement allows a unit change in both the coordinates and hence it is termed as 2-neighbor or $B = \{2\}$. The distance between two points $(i_1, j_1)$ and $(i_2, j_2)$ is given by $d(\{1\}) = |i_1 - i_2| + |j_1 - j_2|$ for the cityblock movement and $d(\{2\}) = \max(|i_1 - i_2|, |j_1 - j_2|)$ for the chessboard movement whereas the Euclidean distance is given by

$$d_e = \sqrt{(i_1 - i_2)^2 + (j_1 - j_2)^2}.$$

In a generalization to cityblock or chessboard distances, the generalized octagonal distances $d(B)$ have been defined by Das and Chatterji (1990) where the type of motion at each step from point $(i_1, j_1)$ to $(i_2, j_2)$ is determined by a sequence of neighborhoods $B = \{b(1), b(2), \ldots, b(n)\}$ where $\forall i,\ b(i) = 1$ or 2. As it is impractical to work with an infinite sequence, one usually is concerned with the neighborhood sequences which are cyclic in nature with cycle length $p = |B|$ as $B = \{b(1), b(2), \ldots, b(p), b(1), b(2), \ldots, b(p), \ldots\}$. It may be noted that a special case of $d(B)$ with $B = \{1, 2\}$ was defined by Rosenfeld and Pfaltz (1968) under the name $d_{\text{oct}}$. It has been proved that if $B$ is sorted the distance function defined by neighborhood sequence $B$ is a metric. In this paper we restrict our attention to such distances only. The different distance functions of neighborhood length upto 4 in 2D are $\{1\}, \{1,1,1,2\}, \{1,1,2\}, \{1,2\}, \{1,1,2,2\}, \{1,2,2\}, \{1,2,2,2\}$ and $\{2\}$. As the digital circles are of convex polygons, the vertices of these polygons can be determined from the following lemma (Das and Chatterji, 1990).

**Lemma 1.** *For any $B$ and $r \geqslant 0$, a vertex of the digital circle of $d(B)$ of radius $r$ in all the positive quadrants is*

$$\underline{x}^{(2)}(r) = (x_1(r), x_2(r))$$

*such that*

(a) $x_1(r) = r$ *and*
(b) $x_2(r) = \lfloor r/p \rfloor (f(p) - p) + f(r \bmod p) - (r \bmod \mathrm{p})$, *where* $f(i) = \sum_{j=1}^{i} b(j),\ 1 \leqslant i \leqslant n$ *and*
(c) *all other vertices of the digital circles are reflections and permutations of the coordinates of this vertex.*

Similarly in 3D digital space $Z^3$, three types of motions have been identified. They are face neighborhood (type 1), edge neighborhood (type 2) and vertex neighborhood (type 3). The notion of the neighborhood sequences is also naturally extended to 3D. The sorted neighborhood sequences of length upto 4 are $\{1\}, \{1,1,2\}, \{1,1,3\}, \{1,2\}, \{1,2,2\}, \{1,2,3\}, \{1,3\}, \{1,3,3\}, \{2\}, \{2,2,3\}, \{2,3\}, \{2,3,3\}, \{3\}, \{1,1,1,2\}, \{1,1,1,3\}, \{1,1,2,2,\}, \{1,1,2,3\}, \{1,1,3,3\}, \{1,2,2,2\}, \{1,2,2,3\}, \{1,2,3,3\}, \{1,2,$

3,3},{1,3,3,3},{2,2,2,3},{2,2,3,3},{2,3,3,3}. The spheres of these distances are convex digital polyhedra. The vertices of these spheres are given by the following lemma.

**Lemma 2.** *A vertex of the octagonal digital sphere of radius r in all the positive octant with* $x_1(r) \geqslant x_2(r) \geqslant x_3(r)$ *is*

$$\underline{x}^{(3)}(r) = (x_1(r), x_2(r), x_3(r))$$

*such that*

(a) $x_i(r) = \lfloor \frac{r}{p} \rfloor (f_i(p) - f_{i-1}(p)) \quad + f_i(r \bmod p) - f_{i-1}(r \bmod p)$, $i = 1, 2, 3$ *where* $f_i(j) = \sum_{1 \leqslant k \leqslant j} b_i(k)$ *and*

$$b_i(j) = \begin{cases} b(j) & \text{if } b(j) \leqslant i, \\ i & \text{otherwise,} \end{cases}$$

(b) *all other vertices of the sphere are reflections and permutations of the coordinates of this vertex.*

It may be noted here for $r < p$ some of the vertices will merge to form degenerate circles and spheres in both 2D and 3D.

## 3. Vertex approximation

In this section we present an approximation for the digital circles and digital spheres of generalized octagonal distances. This approximation will be used in the subsequent sections to compute various properties of the digital circles (spheres), and further to identify the best digital approximant to the Euclidean metric. We present these approximate expressions for the coordinates of the vertices in Lemmas 3 and 4. The proof of Lemma 4 is presented in (Mukherjee et al., 1999). Following the same approach one can prove Lemma 3. Hence the proofs are not presented here.

### 3.1. Digital circles (2D)

$B$ being sorted we can write $B$ as a doublet $(\alpha_1, \alpha_2)$ where $\alpha_1$ 1s are followed by $\alpha_2$ 2s. That is $B = \{1^{\alpha_1} 2^{\alpha_2}\}$ and $\alpha_1 + \alpha_2 = p$. If we approximate $\underline{x}^{(2)}(r)$ by $\underline{x}'^{(2)}(r)$ where $\underline{x}'^{(2)} = (r, r\alpha_2/p)$, the difference between $\underline{x}^{(2)}(r)$ and $\underline{x}'^{(2)}(r)$ is bounded by the following lemma.

**Lemma 3.** *For any* $r > 0$ *and for any* $B$, *we have*

(i) $r - x_1(r) = 0$;

(ii)

$$0 \leqslant \left\{ r\alpha_2/p - x_2(r) \leqslant \alpha_1 - (\alpha_1^2/p) \right\} \leqslant \begin{cases} p/4 & p \text{ even}, \\ \dfrac{(p^2-1)}{4p} & p \text{ odd}. \end{cases}$$

### 3.2. Digital sphere (3D)

To represent a digital sphere as a convex polyhedron in 3D, we need to compute the quantities $x_1(r)$, $x_2(r)$ and $x_3(r)$ for a given value of $r$. We provide approximate formulae below to compute $x_2(r)$ and $x_3(r)$. We have assumed here that the neighborhood elements in $B$ are sorted. As $B$ is sorted we can use an alternative representation for it as triplet $(\alpha_1, \alpha_2, \alpha_3)$, $B = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3}$. Clearly $\alpha_1 + \alpha_2 + \alpha_3 = p$ and $\alpha_1 + \alpha_2 + \alpha_3 \geqslant 0$. We shall approximate $\underline{x}^{(3)}(r)$ by $\underline{x}'^{(3)}(r)$ where $\underline{x}'^{(3)}(r) = (r, r(\alpha_2 + \alpha_3)/p, r\alpha_3/p)$. It can be proved that the difference between $\underline{x}^{(3)}(r)$ and $\underline{x}'^{(3)}(r)$ is bounded (Mukherjee et al., 1999) by the following lemma.

**Lemma 4.** *For any* $r > 0$ *and for any* $B$, *we have*

(i) $r - x_1(r) = 0$;

(ii)

$$0 \leqslant \left\{ \begin{matrix} r(\alpha_2 + \alpha_3)/p - x_2(r) \leqslant \alpha_1 - (\alpha_1^2/p) \\ r\alpha_3/p - x_3(r) \leqslant \alpha_3 - (\alpha_3^2/p) \end{matrix} \right\} \leqslant \begin{cases} p/4 & p \text{ even}, \\ \dfrac{(p^2-1)}{4p} & p \text{ odd}, \end{cases}$$

*where* $x_i(r)$*s are as defined in Lemma* 2.

## 4. Error analysis based on the geometric features

Different features of digital circles (spheres) are computed so that they could be compared to those of an Euclidean circle (sphere). These computations are described in the following sections. Throughout we use the approximation presented in the last section to carry out the computations.

### 4.1. Perimeter, area, volume and shape parameters

Here we present few lemmas related to the computation of the geometric features of the digital discs. The proofs are given in (Aswatha Kumar et al., 1995).

**Lemma 5.** *The perimeter and area of digital circle of radius r for an octagonal distance in 2D are*

$$P_{dc} = 4rP(m) \quad \textit{and} \quad A_{dc} = r^2F(m),$$

*where* $P(m) = (2-\sqrt{2})m + \sqrt{2}$, $F(m) = 2 + 4m - 2m^2$ *and* $m = \alpha_2/(\alpha_1+\alpha_2) = \alpha_2/p$.

**Lemma 6.** *Volume V and the surface area A of the polyhedron of radius r for a 3D octagonal metric is given by*

$$V = (4/3)r^3T(m,n) \quad \textit{and} \quad A = 4r^2G(m,n),$$

*where*

$$\begin{aligned} T(m,n) = \{&1 + 3m + 3n + 6mn + 3m^2 - 6n^2 \\ &+ 3mn^2 - 6m^2n - 2m^3 + n^3\}, \end{aligned}$$

$$\begin{aligned} G(m,n) = \{&m^2(3-2\sqrt{3}) + n^2(\sqrt{3}-3) + mn(2\sqrt{3} \\ &- 6\sqrt{2} + 6) + m(2\sqrt{3}) + n(6\sqrt{2} - 4\sqrt{3}) \\ &+ \sqrt{3}\}, \end{aligned}$$

$$m = (\alpha_2+\alpha_3)/(\alpha_1+\alpha_2+\alpha_3)$$

*and*

$$n = (\alpha_3)/(\alpha_1+\alpha_2+\alpha_3).$$

It should be noted that for $m = 0, n = 0$, an octagonal face in the digital sphere degenerates to a point, a rectangular one degenerates to a straight line segment and a hexagon degenerates to a triangle. Similarly, for $m \neq 0, n = 0$, octagon degenerates to a square, rectangle degenerates to a straight line and hexagon degenerates to a triangle, and for $m = 1, n = 1$, the octagon degenerates to a square, rectangle degenerates to a straight line and the hexagon degenerates to a point. But in all such cases the expression given in Lemma 6 holds.

Next we define shape feature based on the above measurements. In 2D, shape feature is defined as

$$\begin{aligned} S^{(2)}_{dc} &= \frac{(\text{perimeter})^2}{(\text{area})} \\ &= \frac{(4rP(m))^2}{r^2F(m)} \\ &= \frac{16(m(2-\sqrt{2})+\sqrt{2})^2}{2+4m-2m^2} \end{aligned}$$

and in 3D

$$\begin{aligned} S^{(3)}_{dc} &= \frac{\text{area}^3}{\text{volume}^2} \\ &= \frac{(4r^2G(m,n))^3}{(\frac{4}{3}r^3T(m,n))^2} \\ &= \frac{6(G(m,n))^3}{(T(m,n))^2}. \end{aligned}$$

### 4.2. Different error measures

For evaluating the performances of different octagonal distances we have used simple error measures based on different features like area, volume and shape parameters of digital circles and spheres. The respective error measures are defined below.

*Error measures in 2D:*

$$\begin{aligned} \text{Perimeter error} = E^{(2)}_p &= \left| \frac{2\pi r - 4rP(m)}{r} \right| \\ &= 2|\pi - 2P(m)|, \end{aligned}$$

$$\begin{aligned} \text{Area error} = E^{(2)}_a &= \left| \frac{\pi r^2 - r^2F(m)}{r^2} \right| \\ &= |\pi - F(m)| \end{aligned}$$

and

$$\text{Shape feature error} = E_s^{(2)} = \left| \pi - \frac{(m(2-\sqrt{2})+\sqrt{2})^2}{2+4m-2m^2} \right| = |\pi - S(m)|,$$

where

$$S(m) = \frac{(m(2-\sqrt{2})+\sqrt{2})^2}{2+4m-2m^2}.$$

*Error measures in* 3*D*:

$$\text{Volumetric error} = E_v^{(3)} = |\pi - T(m,n)|,$$

$$\text{Surface area error} = E_a^{(3)} = |\pi - (G(m,n))|,$$

$$\text{Shape feature error} = E_s^{(3)} = \left| \pi - \frac{G(m,n)^3}{T(m,n)^2} \right|.$$

With the above definitions, our objective is to minimize the errors by varying the values of $m$ and $n$. In 2D, the error measures $E_p^{(2)}$ and $E_a^{(2)}$ are minimized by solving the following equations separately.

1. For perimeter error:

$$P(m) = \pi/2, \quad 0 \leqslant m \leqslant 1. \tag{1}$$

2. For area error:

$$F(m) = \pi, \quad 0 \leqslant m \leqslant 1. \tag{2}$$

3. For shape feature error:

$$S(m) = \pi, \quad 0 \leqslant m \leqslant 1. \tag{3}$$

By solving Eq. (1) one finds that $E_p^{(2)}$ is zero at $m = 0.284$, and the solution of Eq. (2) shows that $E_a^{(2)}$ is zero at $m = 0.3449$. It is possible that over a large neighborhood sequence of length $p$ (say $p \geqslant 1000$) value of $m$ could be set as close as possible to one of these theoretical values (either $m \simeq 0.284$ or $m = 0.3449$). But in that case the shape of the digital circle will be valid only after the radius is greater than the large value $p$ and for a practical image it has little significance. Hence smaller is the value of $p$, better is our approximation to an Euclidean circle for a larger range of radii. That is why we have considered the neighborhood sequence of length not more than 4 in 2D.

To observe a good choice over this set of metrics, we have enumerated the error value at each case (Table 1). In the table the minimum errors are highlighted (given in boldface). One may observe that the closest practical distance (for $p \leqslant 4$) is with the neighborhood sequence $B = \{1,1,1,2\}$ for minimum $E_p^{(2)}$ and the closest practical distance for getting minimum $E_a^{(2)}$ and $E_s^{(2)}$ is that with $B = \{1,1,2\}$. The plots of $E_a^{(2)}$, $E_p^{(2)}$ and $E_s^{(2)}$ vs. $m$ are shown in Figs. 1–3.

In 3D, different combinations of $m$ and $n$ will give minimum error ($= 0$) for each error measure. Hence we have to solve the following equations separately.

1. For surface area error:

$$G(m,n) = \pi, \quad 0 \leqslant m \leqslant 1, \ 0 \leqslant n \leqslant m. \tag{4}$$

Table 1
Performances of different 2D octagonal distances based on different error measures

| $B$ | $E_a^{(2)}$ | $E_p^{(2)}$ | $E_s^{(2)}$ |
|---|---|---|---|
| {1} | 1.142 | 0.313 | 3.142 |
| {1112} | 0.266 | **0.020** | 1.114 |
| {112} | **0.031** | 0.077 | **0.189** |
| {12} | 0.358 | 0.272 | 3.5194 |
| {122} | 0.636 | 0.468 | 7.830 |
| {1222} | 0.733 | 0.566 | 10.396 |
| {2} | 0.858 | 0.858 | 20.172 |

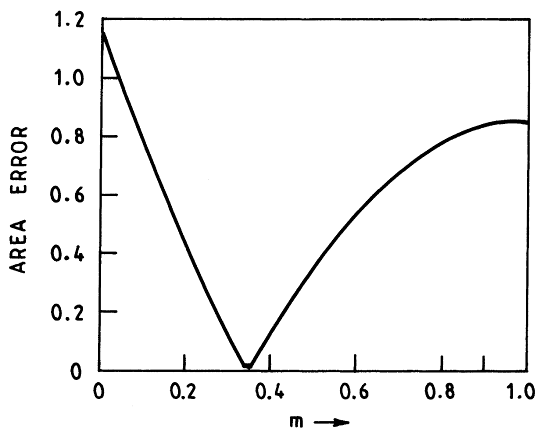


Fig. 1. Plot of area error vs. $m$ for the digital circle (area error $\simeq 0$ for $m = 0.35$).

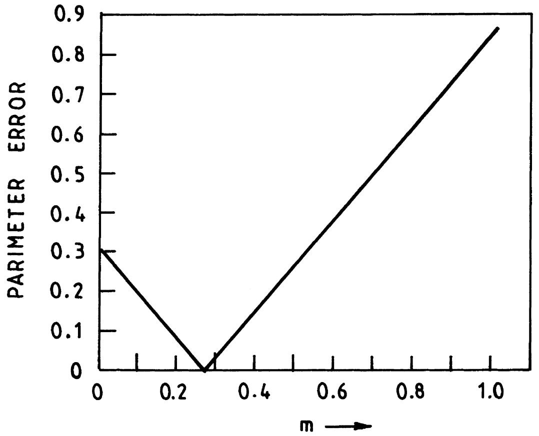


Fig. 2. Plot of perimeter error vs. $m$ for the digital circle (perimeter error $= 0$ for $m = 0.284$).

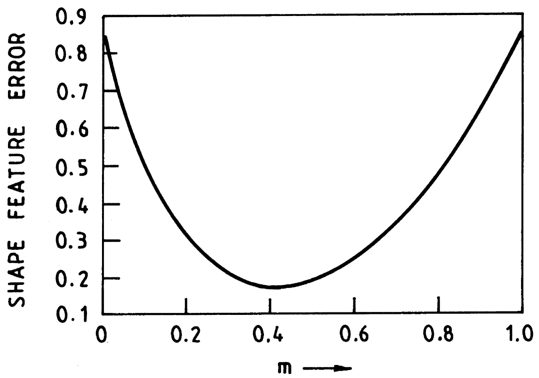


Fig. 3. Plot of shape feature error vs. $m$ for the digital circle (shape feature error is minimum for $m = 0.41$).

2. For volume error:

$$T(m,n) = \pi, \quad 0 \leqslant m \leqslant 1, \ 0 \leqslant n \leqslant m. \tag{5}$$

3. Shape feature error:

$$\frac{(G(m,n))^3}{(T(m,n))^2} = \pi, \quad 0 \leqslant m \leqslant 1, \ 0 \leqslant n \leqslant m. \tag{6}$$

To get a solution of (4), we have solved Eq. (4) for a constant $m$, $0 \leqslant m \leqslant 1$. It is observed that the domain of solution lies in $0.2940 \leqslant m \leqslant 0.4310$ and $0.0014 \leqslant n \leqslant 0.2925$. In Fig. 4, we have plotted values of $n$ for given $m$, where the surface area error $E_a^{(3)}$ is minimum.

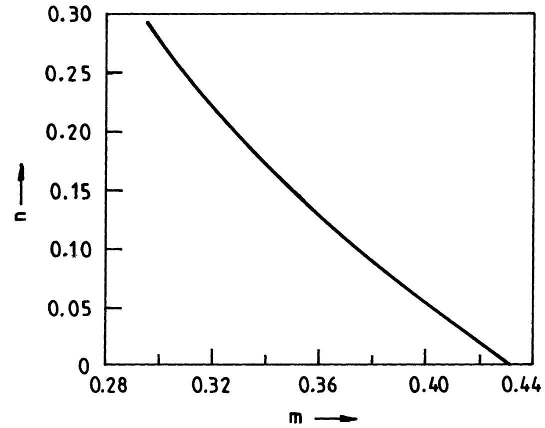


Fig. 4. Plot of $m$ vs. $n$ for surface area error $= 0$ for the digital sphere.

Similar analysis has been carried out for $E_v^{(3)}$. In this case for the ease of computation, we have approximated Eq. (5) as follows (without considering the $n^3$ term):

$$n^2(3m - 6) + n(-6m^2 + 6m + 3) + (-2m^3 + 3m^2 + 3m + 1) = \pi. \tag{7}$$

It is observed that the domain of solution lies in $0.3330 \leqslant m \leqslant 0.5310$ and $0.0005 \leqslant n \leqslant 0.3286$. In Fig. 5, the plot of $n$ vs. $m$ (where $E_v^{(3)}$ becomes zero) is shown.

It is interesting to note that the distance with $B = \{1, 1, 3\}$ (with $m = 0.33$ and $n = 0.33$) gives

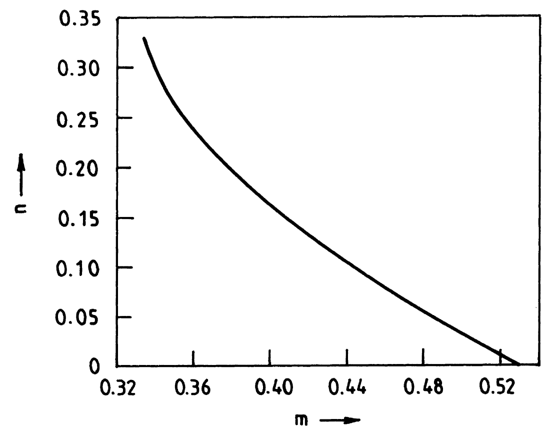


Fig. 5. Plot of $m$ vs. $n$ for volume error $= 0$ for the digital sphere.

Table 2
Performances of different 3D octagonal distances based on different error measures

| $B$ | $E_{\mathrm{v}}^{(3)}$ | $E_{\mathrm{a}}^{(3)}$ | $E_{\mathrm{s}}^{(3)}$ |
|---|---|---|---|
| {1} | 2.142 | 1.410 | 2.050 |
| {11123} | 0.1144 | 0.241 | 0.510 |
| {1111223} | 0.083 | 0.246 | 0.598 |
| {112} | 0.882 | 0.3064 | 1.323 |
| {113} | **0.044** | **0.1804** | **0.470** |
| {12} | 0.142 | 0.2064 | 1.028 |
| {122} | 0.599 | 0.694 | 0.890 |
| {123} | 1.636 | 1.289 | 0.670 |
| {13} | 1.108 | 0.913 | 0.548 |
| {133} | 2.007 | 1.603 | 0.889 |
| {2} | 1.854 | 1.590 | 1.090 |
| {223} | 2.562 | 2.295 | 1.797 |
| {23} | 2.733 | 2.541 | 2.176 |
| {233} | 2.821 | 2.718 | 2.515 |
| {3} | 2.858 | 2.858 | 2.858 |

the closest realization of a point in the solution space of Eqs. (7) and (4). The distance $B = \{1, 2\}$ gives also somewhat nearer value of $(m, n)$ to the solution region of Eq. (7). In Table 2, we have computed the error values for some of the octagonal distances. The minimum error values are highlighted (given in boldface) in the table. It may be observed that in all the three cases errors against the distance with $B = \{1, 1, 3\}$ are minimum. We also observe that the good octagonal distances as reported by Danielsson (1993) with $B = \{1, 1, 1, 2, 3\}$ ($m = 0.40$ and $n = 0.20$) and with $B = \{1, 1, 1, 1, 2, 2, 3\}$ ($m \simeq 0.42$ and $n \simeq 0.14$) also satisfy the bounds of $m$ and $n$ in the solution space for the Eqs. (4) and (7). The errors are also less for those distances (refer Table 2).

## 5. Relative performances between good octagonal distances and good weighted distances

For evaluating the relative performances between good octagonal distances and good weighted distances (Borgefors, 1984, 1986, 1996), we have considered the analytical error measures used earlier. One such measure was used by Das (1992) for studying the performances of the octagonal distances in 2D. He had used the following normalized average difference between Euclidean metric and the digital metric for the octagonal distances.

$$E_{\mathrm{av}}^{(2)} = \left\{ \sum_{i=0}^{M} \sum_{j=0}^{i} |E(i,j) - d((i,j);B)| \Big/ \sum_{i=0}^{M} \sum_{j=0}^{i} 1 \right\} \Big/ M,$$

where $E(i,j)$ and $d(i,j;B)$ denote respectively the Euclidean distance and the octagonal distance given a neighborhood sequence $B$ in 2D of a point $(i,j)$ from the origin, i.e. $(0,0)$.

We have also experimentally computed the average error between Euclidean and octagonal metrices in one octant of a finite image space in 3D. The functional form of the normalized average error $E_{\mathrm{av}}^{(3)}$ is

$$E_{\mathrm{av}}^{(3)} = \left\{ \sum_{i=0}^{M} \sum_{j=0}^{i} \sum_{k=0}^{j} |E(i,j,k) - d((i,j,k);B)| \Big/ \sum_{i=0}^{M} \sum_{j=0}^{i} \sum_{k=0}^{j} 1 \right\} \Big/ M,$$

where $E(i,j,k)$ and $d(i,j,k;B)$ denote respectively the Euclidean distance and the octagonal distance given a neighborhood sequence $B$ in 3D of a point $(i,j,k)$ from the origin, i.e. $(0,0,0)$.

In her work, Borgefors (1984, 1986, 1996) used a different kind of analytical error function. She had used normalized maximum difference between the Euclidean and the weighted distances both in 2D and 3D. In our experimentations we have also used these analytical error measures in 2D and 3D as given below.

For 2D:

$$E_{\max\_\mathrm{diff}}^{(2)} = \max_{0 \leqslant j \leqslant i \leqslant M} \{|E(i,j) - d((i,j);B)|\}/M.$$

For 3D:

$$E_{\max\_\mathrm{diff}}^{(3)} = \max_{0 \leqslant k \leqslant j \leqslant i \leqslant M} \{|E(i,j,k) - d((i,j,k);B)|\}/M.$$

In our experimentation we have taken the value of $M$ as 512. This value is chosen because in many applications sizes of the digital images in 2D or 3D are well within these ranges (i.e. $512 \times 512$ in 2D and $512 \times 512 \times 512$ in 3D). From Tables 1 and 2 we have considered the octagonal distances $\{1,1,2\},\{1,1,1,2\}$ and $\{1,2\}$ in 2D and $\{1,1,3\}$, $\{1,1,1,2,3\},\{1,1,1,1,2,2,3\},\{1,2\}$ and $\{1,2,2\}$ in 3D as representative octagonal distances. For representative good weighted distances, we have chosen the good distances found by Borgefors in both 2D (Borgefors, 1984, 1986) and 3D (Borgefors, 1996). They are $\langle 2,3\rangle$, $\langle 3,4\rangle$, $\langle 5,7\rangle$ and $\langle 8,11\rangle$ in 2D and $\langle 3,4,5\rangle$, $\langle 8,11,13\rangle$, $\langle 13,17,22\rangle$, $\langle 13,17,23\rangle$, $\langle 16,21,27\rangle$ and $\langle 16,21,28\rangle$ in 3D. For the explanations of their notational representations and functional forms please refer Borgefors (1984, 1996). We have presented their relative performances in Tables 3 and 4, respectively. One may observe that in 2D (refer Table 3), $\{1,1,2\}$ again outperforms other octagonal distances, while for the weighted distances $\langle 3,4\rangle$ is found to be equally good. Interestingly in 3D (refer Table 4) the octagonal distance $\{1,1,3\}$ performs better than those octagonal distances found earlier by Danielsson (1993). In fact considering both the analytical error measures $\{1,1,3\}$ is found to be the best among all the distances (including the weighted distances) considered here.

Table 3
Relative performances of good 2D octagonal distances and good weighted distances

| Distance functions | $E^{(2)}_{av}$ | $E^{(2)}_{max\_diff}$ |
|---|---|---|
| *Octagonal distances* | | |
| $\{112\}$ | 0.026 | 0.087 |
| $\{1112\}$ | 0.056 | 0.187 |
| $\{12\}$ | 0.043 | 0.118 |
| *Weighted distances* | | |
| $\langle 2,3\rangle$ | 0.068 | 0.134 |
| $\langle 3,4\rangle$ | 0.025 | 0.081 |
| $\langle 5,7\rangle$ | 0.035 | 0.083 |
| $\langle 8,11\rangle$ | 0.030 | 0.073 |

Table 4
Relative performances of good 3D octagonal distances and good weighted distances in 3D

| Distance functions | $E^{(3)}_{av}$ | $E^{(3)}_{max\_diff}$ |
|---|---|---|
| *Octagonal distances* | | |
| $\{113\}$ | 0.027 | 0.105 |
| $\{11123\}$ | 0.034 | 0.146 |
| $\{1111223\}$ | 0.044 | 0.180 |
| $\{12\}$ | 0.070 | 0.269 |
| $\{122\}$ | 0.049 | 0.214 |
| *Weighted distances* | | |
| $\langle 3,4,5\rangle$ | 0.043 | 0.118 |
| $\langle 8,11,13\rangle$ | 0.043 | 0.107 |
| $\langle 13,17,22\rangle$ | 0.043 | 0.107 |
| $\langle 13,17,23\rangle$ | 0.034 | 0.107 |
| $\langle 16,21,27\rangle$ | 0.036 | 0.103 |
| $\langle 16,21,28\rangle$ | 0.043 | 0.103 |

## 6. Conclusion

By analyzing the geometry of digital circles in 2D and digital spheres in 3D, we have identified the best octagonal distances which are closer approximation to Euclidean norm. A generalization of this approach in $n$-dimension seems to be interesting. In the present paper our motivation is to identify good approximate digital distances for the Euclidean distance in 2D and 3D. We follow an approach based on the geometry of the circles (spheres) of these distances. First we show that under the condition of sorted order the vertices of the circles (spheres) can be suitably approximated as a function of the number of neighborhood types used in the sequence. Then we used these approximate vertex formulae to compute the perimeter, surface area and volume in an approximate way. Finally we minimize the errors of area, volume and a shape feature (defined to capture the circularity and sphericity) to identify the best distances. We have verified our results by experimenting with analytical error measures suggested earlier. We have also compared the performances of the good octagonal distances with good weighted distances. It has been found that the best octagonal distance in 2D ( $\{1,1,2\}$) performs equally good with respect to the best one for the weighted distances ($\langle 3,4\rangle$). In fact in 3D, the octagonal distance $\{1,1,3\}$ has an edge over the other good weighted distances.

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## References


Aswatha Kumar, M., Mukherjee, J., Chatterji, B.N., Das, P.P., 1996. Representation of 2D and 3D binary images using medial circles and spheres. Int. J. Pattern Recognition Artificial Intell. 10, 365–387.

Aswatha Kumar, M., Mukherjee, J., Chatterji, B.N., Das, P.P., 1995. Image compression using medial circles and medial spheres. In: Pattern Recognition, Image Processing and Computer Vision: Recent Advances, Proc. ICPIC, pp. 74–79.

Aswatha Kumar, M., Mukherjee, J., Chatterji, B.N., Das, P.P., 1995. A geometric approach to obtain best octagonal distances. In: Ninth Scandinavian Conf. Image Process., pp. 491–498.

Borgefors, G., 1984. Distance transformations in arbitrary dimensions. Comput. Vision Graphics Image Process. 27, 321–345.

Borgefors, G., 1986. Distance transformations in digital images. Comput. Vision Graphics Image Process. 34, 344–371.

Borgefors, G., 1993. Centres of maximal discs in the 5-7-11 distance transforms. In: Proc. Eighth Scandinavian Conf., pp. 1-5–111.

Borgefors, G., 1996. On digital transforms in three dimensions. Comput. Vision Image Understanding 64, 368–376.

Borgefors, G., 1994. Applications of distance transforms. In: Arcelli, C. et al. (Eds.), Aspects of Visual Form Processing, pp. 83–108.

Blum, H., 1964. A transformation for extracting new descriptors of shape. In: Walthen Dunn (Ed.), Perception of Speech and Visual Form, pp. 362–380.

Das, P.P., Chatterji, B.N., 1990. Octagonal distances for digital pictures. Inform. Sci. 50, 123–150.

Das, P.P., 1992. Best simple octagonal distance in digital geometry. J. Approx. Theory 68 (2), 155–174.

Danielsson, P.E., 1993. 3D octagonal metrics. In: Proc. Eighth Scandinavian Conf., pp. 727–736.

Mukherjee, J., Das, P.P., Aswatha Kumar, M., Chatterji, B.N., Das, P.P., 1998. Discrete shading algroithms for 3D objects from medial sphere representation. In: Computer Vision, Graphics and Image Processing: Recent Advances, Proc. ICVGIP, pp. 185–190.

Mukherjee, J., Aswatha Kumar, M., Chatterji, B.N., Das, P.P., 1999. Discrete shading of three dimensional objects from medial axis transform. Pattern Recognition Lett. 20, 1533–1544.

Rosenfeld, A., Pfaltz, J.L., 1968. Distance functions in digital pictures. Pattern Recognition 1, 33–61.

Toriwaki, J.I., Yokoi, S., Fukumara, T., 1981. On generalised distance transformation of digital pictures. IEEE Trans. Pattern Anal. Machine Intell. 3 (4), 424–443.