Shannon's Theory of Secrecy System

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Shannon's Information Theory Paper

- "Mathematical Theory of Communication", published in 1948
- Main claim:
  - All sources of data have a **rate**
  - All channels have a **capacity**
  - If the **capacity** is greater than the **rate**, transmission with **no errors** is possible
- Introduced concept of **entropy** of a random variable/process

Definition of a Cryptosystem:

- A cryptosystem can be viewed as a distribution of plaintexts P, a set of ciphertexts C, a distribution of possible keys (K) and an encoding transformation, with its inverse (D).

Definition of Cryptosystem Modern Variations

Confidentiality

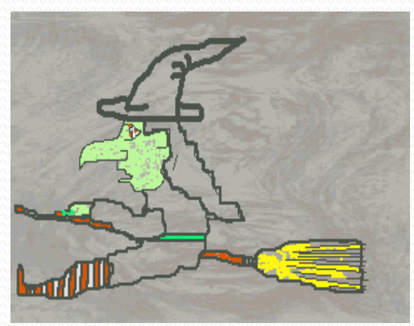
Dear Alice,

Data Integrity

Bob

Alice

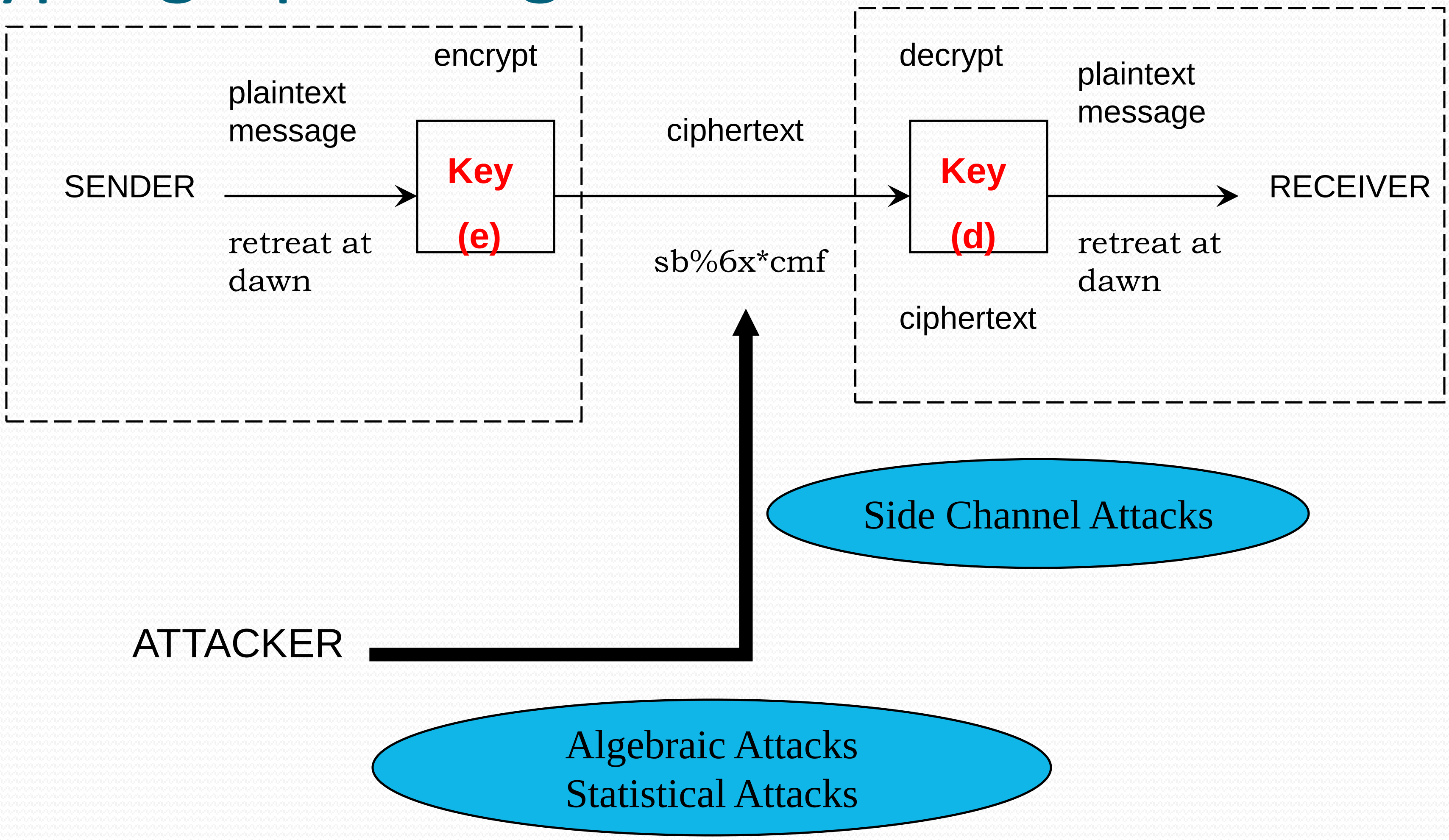
Authentication



Non Repudiation

Eve

Cryptographic Algorithms



Shannon’s 1948 Paper

- Published one year after his monumental “information theory” paper
- “transformed cryptography from art to science”

Main Contributions

- Notions of **theoretical security** and **practical security**
- Observation that the secret is all in the key, not in the algorithm
- **Product ciphers** and **mixing transformations** – inspiration for **DES, AES and …………**
- Proof that **Vernam's cipher** (one-time pad) was **theoretically secure**

Theoretical and Practical Security

Theoretical and Practical Security

- **Theoretically secure** cryptosystems cannot be broken – even by an all-powerful adversary
- **Practically secure** cryptosystems "require a large amount of work to solve"
- Bad news:
  - The only **theoretically secure** cryptosystem is the **one-time pad**
  - The only **practically secure** cryptosystem is… the **one-time pad**

Shannon's theory

- 1949, "Communication theory of Secrecy Systems" in Bell Systems Tech. Journal.
- Two issues:
  - What is the concept of perfect secrecy? Does there any cryptosystem provide perfect secrecy?
    - It is possible when a key is used for only one encryption
  - How to evaluate a cryptosystem when many plaintexts are encrypted using the same key?

Shannon’s 1949 Paper

- Approaches to evaluate the security of Cryptosystem
  - Computational Security
  - Provable Security
  - Unconditional Security

Computational Security

- Concerns the computational effort required to break a cryptosystem

  Definition

  A Cryptosystem is said to be computationally Secure if the best algorithm for breaking it requires atleast N operations where N is some specified, very large number.

  Problem - No known cryptosystem can be proved to be secure.

  - Specific attack like Exhaustive Key Search

Provable Security

Definition

A Cryptosystem is said to be provably Secure if the security of the system can be reduced to some well-studied problem that is considered to be difficult

Example "A given cryptosystem is secure if a given integer n cannot be factored"

- relative not an absolute proof

Unconditional Security

Definition

A cryptosystem is said to be unconditionally secure if it cannot be broken, even with infinite computational resources.

- it cannot be studied from the point of view of computational complexity as we allow computation time is infinite
- can be studied with Probability Theory

One-Time Pad

- Unconditional security !!!
- Described by Gilbert Vernam in 1917
- Use a random key that was truly as long as the message, no repetitions

$$P = C = K = (\mathbb{Z}_2)^n \quad x = (x_1, \ldots, x_n) \qquad K = (K_1, \ldots, K_n)$$

$$e_K(x) = (x_1 + K_1, \ldots, x_n + K_n) \mod 2$$

For ciphertext $y = (y_1, \ldots, y_n)$

$$d_K(y) = (y_1 + K_1, \ldots, y_n + K_n) \mod 2$$

Example: one-time pad

- Given ciphertext with Vigenère Cipher:
  ANKYODKYUREPFJBYOJDSPLREYIUNOFDOIUERFPLUYTS

Decrypt by hacker 1:

```
CT: ANKYODKYUREPFJBYOJDSPLREYIUNOFDOIUERFPLUYTS
Key: pxlmvmsydofuyrvzwc tnlebnecvgdupahfzzlmnyih
PT:  mr mustard with the candlestick in the hall
```

Decrypt by hacker 2:

```
CT: ANKYODKYUREPFJBYOJDSPLREYIUNOFDOIUERFPLUYTS
Key:pftgpmiydgaxgoufhklllmhsqdqogtewbqfgyovuhwt
PT:miss scarlet with the knife in the library
```

Which one?

Problem with one-time pad

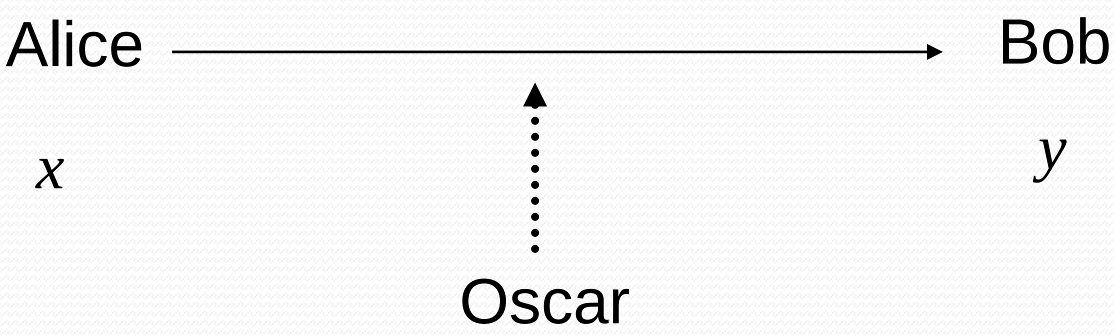
- Truly random key with arbitrary length?
- Distribution and protection of long keys
  - The key has the same length as the plaintext!
- One-time pad was thought to be unbreakable, but there was no mathematical proof until Shannon developed the concept of perfect secrecy 30 years later.

Perfect secrecy

- When we discuss the security of a cryptosystem, we should specify the type of attack that is being considered
  - Ciphertext-only attack
- Unconditional security assumes infinite computational time
  - Theory of computational complexity
  - Probability theory

Perfect secrecy

- **Definition:** A cryptosystem has perfect secrecy if $\Pr[x|y] = \Pr[x]$ for all $x \in P$, $y \in C$
- Idea: Oscar can obtain no information about the plaintext by observing the ciphertext



Elementary Probability Theory

Discrete random variable

- **Def:** A *discrete random variable*, say **X**, consists of a finite set X and a probability distribution defined on X.
- The probability that the random variable **X** takes on the value x is denoted $\Pr[\mathbf{X}=x]$ or $\Pr[x]$
- $0 \le \Pr[x]$ for all $x \in X$, $\sum_{x \in X} \Pr[x] = 1$

Discrete random variable

- Ex. Consider a coin toss to be a random variable defined on {head, tails} , the associated probabilities Pr[head]=Pr[tail]=1/2
- Ex. Throw a pair of dice. It is modeled by Z={(1,1), (1,2), …, (2,1), (2,2), …, (6,6)}

  where Pr[(i,j)]=1/36 for all i, j.

  sum=4 corresponds to {(1,3), (2,2), (3,1)} with probability 3/36

Joint and conditional probability

- **X** and **Y** are random variables defined on finite sets X and Y, respectively.
- **Def:** the joint probability $\Pr[x, y]$ is the probability that **X**$=x$ and **Y**$=y$
- **Def:** the conditional probability $\Pr[x|y]$ is the probability that **X**$=x$ given **Y**$=y$

$$\Pr[x, y] = \Pr[x|y]\Pr[y] = \Pr[y|x]\Pr[x]$$

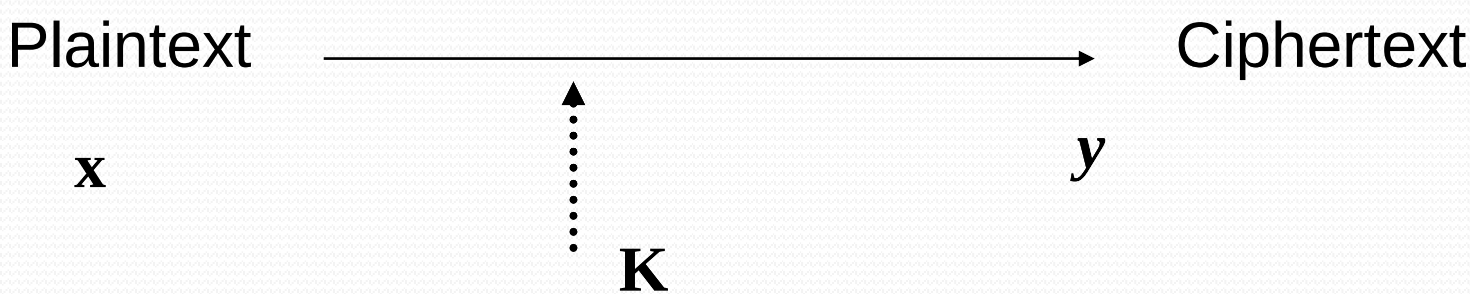
Bayes' theorem

- If Pr[y] > 0, then
$$\Pr[x \mid y] = \frac{\Pr[x]\Pr[y \mid x]}{\Pr[y]}$$
- Ex. Let **X** denote the sum of two dice.
  Y is a random variable on {D, N}, **Y**=D if the two dice are the same. (double)

$$\Pr[D \mid 4] = \frac{\Pr[4 \mid D]\Pr[D]}{\Pr[4]} = \frac{(1/6)(1/6)}{3/36} = \frac{1}{3}$$

Definitions

- Assume a cryptosystem (P,C,K,E,D) is specified, and a key is used for one encryption
- Plaintext is denoted by random variable **x**
- Key is denoted by random variable **K**
- Ciphertext is denoted by random variable **y**



Perfect secrecy

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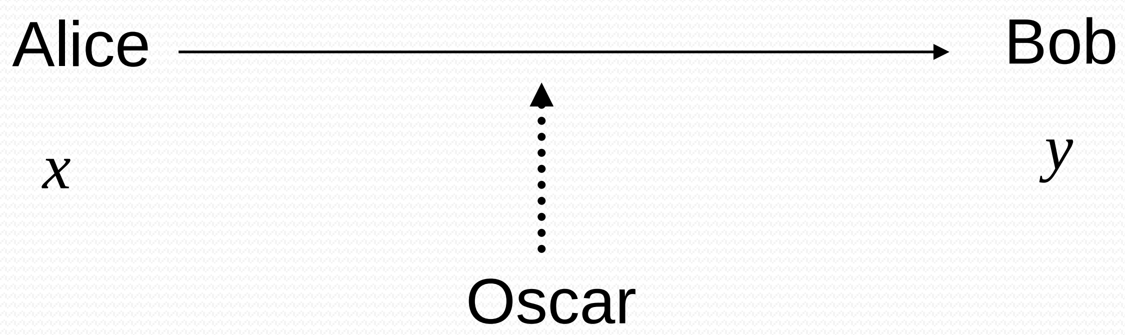

Relations among **x, K, y**

- Ciphertext is a function of **x** and **K**

$$\Pr[\mathbf{y} = y] = \sum_{\{K : y \in C(K)\}} \Pr[\mathbf{K} = K] \Pr[\mathbf{x} = d_K(y)]$$

- **y** is the ciphertext, given that **x** is the plaintext

$$\Pr[\mathbf{y} = y \mid \mathbf{x} = x] = \sum_{\{K : x = d_K(y)\}} \Pr[\mathbf{K} = K]$$

Relations among **x, K, y**

- **x** is the plaintext, given that **y** is the ciphertext

$$\Pr[\mathbf{x} = x \mid \mathbf{y} = y] = \frac{\Pr[x]\Pr[y \mid x]}{\Pr[y]}$$

$$= \frac{\Pr[\mathbf{x} = x] \times \sum_{\{K : x = d_K(y)\}} \Pr[\mathbf{K} = K]}{\sum_{\{K : y \in C(K)\}} \Pr[\mathbf{K} = K] \Pr[\mathbf{x} = d_K(y)]}$$

Ex. Shift cipher has perfect secrecy

- Shift cipher: P=C=K=Z_{26} , encryption is defined as

- Ciphertext:

$$e_K(x) = (x + K) \mod 26$$

$$\begin{aligned}\Pr[\mathbf{y} = y] &= \sum_{K \in Z_{26}} \Pr[\mathbf{K} = K] \Pr[\mathbf{x} = d_K(y)] \\ &= \sum_{K \in Z_{26}} \tfrac{1}{26} \Pr[x = y - K] \\ &= \tfrac{1}{26} \sum_{K \in Z_{26}} \Pr[x = y - K] = \tfrac{1}{26}\end{aligned}$$

Ex. Shift cipher has perfect secrecy

- $\Pr[y|x]$

$$= \Pr[\mathbf{K} = (y - x) \mod 26] = \tfrac{1}{26}$$

- Apply Bayes' theorem

$$\Pr[x \mid y] = \frac{\Pr[x]\Pr[y \mid x]}{\Pr[y]}$$

$$= \frac{\Pr[x]\,\tfrac{1}{26}}{\tfrac{1}{26}} = \Pr[x]$$

Perfect secrecy

Perfect secrecy when |K|=|C|=|P|

- (P,C,K,E,D) is a cryptosystem where |K|=|C|=|P|.
- The cryptosystem provides perfect secrecy iff
  - every keys is used with equal probability 1/|K|
  - For every $x \in P$, $y \in C$, there is a unique key K such that

Ex. One-time pad in Z_2

$$e_K(x) = y$$

Shannon's Product Ciphers and Modern Encryption Algorithms

Product Cryptosystems

- Different cryptosystems can be combined to create a new cryptosystem.

- Given two cryptosystems with the same message space, consider a probabilistic combination of the two systems: with probability p use system A, otherwise use system B.

Product Cryptosystems

- Another way to use two cryptosystems is to encrypt and decrypt messages consecutively. We call this a **product cipher**.
- He believes that a combination of an initial transposition (Permutation) with alternating substitutions and linear operations may do the trick.
- Both DES and AES use Shannon's ideas of Product System and of type Substitution Permutation Network (SPN).

Conventional Encryption Principles

- Basic ingredients of the scheme:
  a) Plaintext (P)
     - Message to be encrypted
  b) Secret Key (K)
     - Shared among the two parties
  c) Ciphertext (C)
     - Message after encryption
  d) Encryption algorithm
     - Uses P and K
  e) Decryption algorithm
     - Uses C and K

Types of algorithms

- Private Key : The encryption key and decryption key are easily derivable from each other
  - Block Cipher : Fixed blocks of data
  - Stream Cipher : Block Size = 1
- Public Key : Infeasible to determine the decryption key, d from the encryption key, e.

- Security of the scheme
  - Depends on the secrecy of the key
  - Does not depend on the secrecy of the algorithm
- Assumptions that we make:
  - Algorithms for encryption/decryption are known to the public
  - Keys used are kept secret

Simplified Model of Encryption/Decryption