PROJECTIVE COORDINATES AND COMPACTIFICATION IN ELLIPTIC, PARABOLIC AND HYPERBOLIC 2-D GEOMETRY

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Abstract. A result that the upper half plane is not preserved in the hyperbolic case, has implications in physics, geometry and analysis. We discuss in details the introduction of projective coordinates for the EPH cases. We also introduce appropriate compactification for all the three EPH cases, which results in a sphere in the elliptic case, a cylinder in the parabolic case and a crosscap in the hyperbolic case.
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1. Introduction

Geometry is an essential branch of Mathematics. It deals with the properties of figures in a plane or in space [7]. Perhaps, the most influencial book of all time, is 'Euclids Elements', written in around 3000 BC [14]. Since Euclid, geometry usually meant the geometry of Euclidean space of two-dimensions (2-D, Plane geometry) and three-dimensions (3-D, Solid geometry). A close scrutiny of the basis for the traditional Euclidean geometry, had revealed the independence of the parallel axiom from the others and consequently, non-Euclidean geometry was born and in projective geometry, new "points" (i.e., points at infinity and points with complex number coordinates) were introduced [1, 3, 9, 26]. In the eighteenth century, under the influence of Steiner, Von Staudt, Chasles and others, projective geometry became one of the chief subjects of mathematical research [7]. Its popularity was partly due to its great aesthetic charm and partly due to its intimate connection with non-Euclidean geometry and algebra [7].

Also, conic sections are the curves obtained by intersecting a circular cone by a plane: ellipse (including circles), parabolas and hyperbolas [26]. Beginning from the early age of mathematics as a science, we repeatedly face the division of different mathematical objects into three main classes. In very different areas, these classes preserve the names obtained by the very first example- the classification of conic sections: elliptic, parabolic and hyperbolic which will be abbreviated as EPH- classification. Many authors have put forward their ideas, in this regard from different considerations [4, 8, 10, 12, 16, 18, 19]. In our work, we describe geometry of two-dimensional spaces in the spirit of the Erlangen Program of Felix Klein, influenced by the works of M. Sophus Lie. A point transformation of the plane onto itself that carries "circle" into "circle", is called a circular or a Möbius transformation [9]. In our case, we study objects in a plane and their properties which are invariant under Möbius transformations of the group $SL_2(\mathbb{R})$. We also introduce the projective coordinates and compactification for the three EPH cases.

2. $SL_2(\mathbb{R})$ group and Clifford Algebras in EPH cases

The group $SL_2(\mathbb{R})$ is considered as the simplest of the noncompact semisimple Lie groups. We use representations of the group $SL_2(\mathbb{R})$ in Clifford algebras with two generators. In forming the foundations for elliptic, parabolic and hyperbolic cases, we consider Clifford algebras to be a very suitable candidate. There will be three different Clifford algebras $\mathcal{C}\ell(e)$, $\mathcal{C}\ell(p)$ and $\mathcal{C}\ell(h)$ corresponding to *elliptic*, *parabolic*, and *hyperbolic* cases respectively. The notation $\mathcal{C}\ell(a)$ refers to *any* of these three algebras.

A Clifford algebra $\mathcal{C}\ell(a)$ as a 4-dimensional linear space is spanned by 1, e_1, e_2, e_1e_2 with *non-commutative* multiplication defined by the identities, in Lounesto [21]:

$$e_1^2 = -1, \qquad e_2^2 = \begin{cases} -1, & \text{for } \mathcal{C}\ell(e)\text{—elliptic case} \\ 0, & \text{for } \mathcal{C}\ell(p)\text{—parabolic case} \\ 1, & \text{for } \mathcal{C}\ell(h)\text{—hyperbolic case} \end{cases}, \qquad e_1e_2 = -e_2e_1. \tag{2.1}$$

It contains both the plane $\mathbb{E}$ and the vector plane $\mathbb{R}^a$ so that

$$\mathbb{R}^a \quad \text{is spanned by } e_1 \text{ and } e_2, \quad \mathbb{E} \quad \text{is spanned by 1 and } e_1e_2 (= e_{12}).$$

The only common point of the two planes is the zero 0. The two planes are both parts of the same algebra $\mathcal{C}\ell(a)$. The vector plane $\mathbb{R}^a$ and the plane $\mathbb{E}$ are incorporated as separate substructures in the Clifford algebra $\mathcal{C}\ell(a) = \mathcal{C}\ell(a)^+ \oplus \mathcal{C}\ell(a)^-$ so that the plane $\mathbb{E}$ is the *even part* $\mathcal{C}\ell(a)^+$ and the vector plane $\mathbb{R}^a$ is the *odd part* $\mathcal{C}\ell(a)^-$. The names even and odd mean that the elements are products of an even or odd numbers of vectors. We then have the following inclusions

$$\begin{aligned} \mathcal{C}\ell(a)^+\, \mathcal{C}\ell(a)^+ &\subset \mathcal{C}\ell(a)^+, \\ \mathcal{C}\ell(a)^-\, \mathcal{C}\ell(a)^+ &\subset \mathcal{C}\ell(a)^-, \\ \mathcal{C}\ell(a)^+\, \mathcal{C}\ell(a)^- &\subset \mathcal{C}\ell(a)^-, \\ \mathcal{C}\ell(a)^-\, \mathcal{C}\ell(a)^- &\subset \mathcal{C}\ell(a)^+. \end{aligned}$$

These observations are expressed by saying that the Clifford algebra $\mathcal{C}\ell(a)$ has an *even-odd grading*. We take u, v as coordinates on our axes, see Figure 1. Next we define projection on axes as $E_1(ue_1+ve_2) = u$ and $E_2(ue_1+ve_2) = v$.

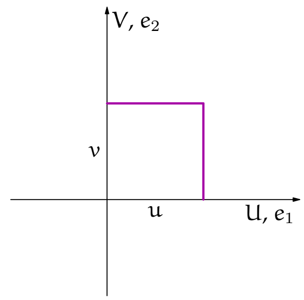

FIGURE 1. Correspondence between notations on $\mathbb{R}^a$.

The group $SL_2(\mathbb{R})$ (Lang [20]) consists of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{with } a, b, c, d \in \mathbb{R} \text{ and the determinant } ad - bc = 1.$$

The transformation T defined by $w = T(z) = \frac{az+b}{cz+d}$, where a, b, c, d are complex constants, $ad - bc \neq 0$ and w, z are complex variables is called a *bilinear transformation* or a *Möbius transformation* or a *linear fractional transformation*. The constant $ad - bc$ is called the *determinant* of the transformation. This transformation is said to be *normalized* if $ad - bc = 1$. The Möbius transformation above, is undefined when the denominator is a divisor of zero.

The two-dimensional subalgebra of $\mathcal{C}\ell(e)$ spanned by 1 and $\mathrm{i} = e_2e_1 = -e_1e_2$ is *isomorphic* to (and can replace in all calculations!) the field of complex numbers $\mathbb{C}$. For any $\mathcal{C}\ell(a)$, we identify $\mathbb{R}^2$ with the set of vectors $w = ue_1+ve_2$, where $(u, v) \in \mathbb{R}^2$.

(e) In the elliptic case of $\mathcal{C}\ell(e)$, this maps

$$(u, v) \longmapsto e_1(u + \mathrm{i}v) = e_1 z, \text{ with } z = u + \mathrm{i}v \text{ a standard complex number.} \tag{2.2}$$

Similarly, as seen in Yaglom [28] and in Kisil [19],

(p) in the parabolic case $\epsilon = e_2e_1$ (such that $\epsilon^2 = 0$) is known as *dual unit* and all expressions $u + \epsilon v$, where $u, v \in \mathbb{R}$ form *dual numbers*, and

(h) in the hyperbolic case $e = e_2e_1$ (such that $e^2 = 1$) is known as *double unit* and all expressions $u + ev$, where $u, v \in \mathbb{R}$ constitute *double numbers*.

We denote $\mathbb{R}^2$ by $\mathbb{R}^e$, $\mathbb{R}^p$ or $\mathbb{R}^h$ to highlight which of the Clifford algebras is used in the present context. The notation $\mathbb{R}^a$ assumes $\mathcal{C}\ell(a)$.

An isomorphic realisation of $SL_2(\mathbb{R})$ [15, 20, 27] with the same multiplication is obtained if we replace a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix}$ within any $\mathcal{C}\ell(a)$. The advantage of the latter form is that we can define the *Möbius transformation* of $\mathbb{R}^a \to \mathbb{R}^a$ for *all* three algebras $\mathcal{C}\ell(a)$ by the same expression:

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : \ ue_1 + ve_2 \ \longmapsto \ \frac{a(ue_1+ve_2) - be_1}{ce_1(ue_1+ve_2)+d}, \tag{2.3}$$

where the expression $\frac{a}{b}$ in a non-commutative algebra is always understood as ab^{-1}, (Cnops [5, 6]). Therefore $\frac{ac}{bc} = \frac{a}{b}$ but $\frac{ca}{cb} \neq \frac{a}{b}$ in general.

Again in the elliptic case the transformation (2.3) is equivalent to

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : \ e_1 z \ \longmapsto \ \frac{e_1(a(u+e_2e_1v)-b)}{-c(u+e_2e_1v)+d} = e_1\frac{az-b}{-cz+d}, \text{ where } z = u+\mathrm{i}v,$$

which is the standard form of a Möbius transformation. One can straightforwardly verify that the map (2.3) is a left action of $SL_2(\mathbb{R})$ on $\mathbb{R}^a$, i.e. $g_1(g_2w) = (g_1g_2)w$.

To study the finer structure of Möbius transformations, it is useful to decompose an element g of $SL_2(\mathbb{R})$ into the product $g = g_a g_n g_k$:

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & \chi e_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\phi & e_1\sin\phi \\ e_1\sin\phi & \cos\phi \end{pmatrix}, \tag{2.4}$$

where the values of parameters are as follows:

$$\alpha = \sqrt{c^2+d^2}, \qquad \chi = \frac{d - a(c^2+d^2)}{c} = \frac{-b(c^2+d^2)-c}{d}, \qquad \phi = \tan^{-1}\frac{c}{d}. \tag{2.5}$$

Consequently $\cos\phi = \frac{d}{\sqrt{c^2+d^2}}$ and $\sin\phi = \frac{c}{\sqrt{c^2+d^2}}$. The product (2.4) gives a realisation of the *Iwasawa decomposition* for semisimple Lie group $SL_2(\mathbb{R}) = ANK$, where A is diagonal, N is nilpotent and K is maximal compact and A normalizes N (Lang [20, § III.1]).

In all three EPH cases, the subgroups A and N act through Möbius transformations uniformly: For any type of the Clifford algebra $\mathcal{C}\ell(a)$, the subgroup N defines shifts $ue_1 + ve_2 \mapsto (u+\chi)e_1 + ve_2$ along the "real" axis U by χ. The subgroup A defines dilations $ue_1 + ve_2 \mapsto \alpha^{-2}(ue_1+ve_2)$ by the factor α^{-2} which fixes origin $(0,0)$. By contrast the actions of the subgroup K is significantly different between the EPH cases and correlates with the names chosen for $\mathcal{C}\ell(e)$, $\mathcal{C}\ell(p)$, $\mathcal{C}\ell(h)$.

3. (Non)-Invariance of the upper half plane

The important difference between the hyperbolic case and the two others (elliptic and parabolic), is that

Theorem 3.1. *In the elliptic and parabolic cases, the upper half plane in* $\mathbb{R}^a$ *is preserved by Möbius transformations from* $SL_2(\mathbb{R})$*. However in the hyperbolic case, any point* (u,v) *with* $v>0$ *can be mapped to an arbitrary point* (u',v') *with* $v'\neq 0$*.*

Proof. We know by equation (2.3) that the Möbius mapping is given by

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : ue_1+ve_2 \longrightarrow \frac{a(ue_1+ve_2)-be_1}{ce_1(ue_1+ve_2)+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}). \tag{3.1}$$

As defined in earlier $e_1^2=-1$ in all the three EPH cases but e_2^2 may be $-1,0$ or 1 respectively.
Now the inverse of the denominator of the Möbius transformation exists when the denominator is not a divisor of zero i.e. $\Big(ce_1(ue_1+ve_2)+d\Big)$ is not a divisor of zero.

Therefore $\Big((d-cu)+cve_{12}\Big)^{-1} = \Big((d-cu)-cve_{12}\Big)\Big((d-cu)^2-c^2v^2e_2^2\Big)^{-1}$.
This is equivalent to $v\neq 0$ and $u\neq d/c$ in the elliptic case.

We now show that the upper half plane is preserved in the elliptic and parabolic cases but not in the hyperbolic case. To this end we decompose the Möbius transformation into coordinates:

$$\frac{aue_1+ave_2-be_1}{d-cu+cve_{12}} = u'e_1+v'e_2, \tag{3.2}$$

$$\text{where} \quad u' = \frac{(au-b)(d-cu)+acv^2e_2^2}{(d-cu)^2-c^2v^2e_2^2} \quad \text{and} \quad v' = \frac{v}{(d-cu)^2-c^2v^2e_2^2}.$$

So, we now know the coordinate-wise form of the Möbius transformation in $\mathbb{R}^a$.

Now we want to know whether $v>0$ implies $E_2\left(\frac{aw-be_1}{ce_1w+d}\right)=v'>0$, where E_2 as is defined in the notation (see Section 2) and $w=ue_1+ve_2$. There are three cases:

(1) **Elliptic Case**: Here $e_2^2=-1$, this gives from equation (3.2),

$$u' = \frac{(au-b)(d-cu)-acv^2}{(d-cu)^2+c^2v^2} \quad \text{and} \quad v' = \frac{v}{(d-cu)^2+c^2v^2}. \tag{3.3}$$

If $v>0$ then $v'>0$ as $cu\neq d$. Therefore the upper half plane is preserved.

(2) **Parabolic Case**: Here we have $e_2^2=0$, this gives from equation (3.2),

$$u' = \frac{(au-b)}{(d-cu)} \quad \text{and} \quad v' = \frac{v}{(d-cu)^2}. \tag{3.4}$$

If $v>0$ then $v'>0$ as $cu\neq d$. Therefore the upper half plane is preserved.

(3) **Hyperbolic Case**: Here $e_2^2 = 1$, this gives from equation (3.2),

$$u' = \frac{(au-b)(d-cu)+acv^2}{(d-cu)^2-c^2v^2} \quad \text{and} \quad v' = \frac{v}{(d-cu)^2-c^2v^2}. \tag{3.5}$$

If $v > 0$ then v' may have an arbitrary sign. Therefore the upper half plane is not preserved.

Hence it can also be shown that in the hyperbolic case any point (u, v) with $v > 0$ can be mapped to an arbitrary point (u', v') with $v' \neq 0$. For example, the point $(0, 1)$ can be mapped to the point $(0, -1)$ by the Möbius transformation $\begin{pmatrix} -2 & \sqrt{5}e_1 \\ \sqrt{5}e_1 & 2 \end{pmatrix} \in SL_2(\mathbb{R})$. In other words $\begin{pmatrix} -2 & \sqrt{5}e_1 \\ \sqrt{5}e_1 & 2 \end{pmatrix} : e_2 \longrightarrow -e_2$.

Special Case: If $v = 0$, then the Möbius mapping (3.1) becomes

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : ue_1 \longrightarrow \frac{au-b}{d-cu}e_1,$$

regardless of the value of e_2^2. This means that on the boundary the value of e_2^2 becomes irrelevant. □

Remark 3.2. In the elliptic case we observe that divisors of zero exist if $v \neq 0$ and $d - cu \neq 0$. This reduces to $v \neq 0$ and $u \neq d/c$, under the assumption that $c \neq 0$. If $c = 0$, then $d - cu$ is non-zero, since d is non-zero. Hence it is irrelevant to our question of finding a non-zero denominator.

4. Compactification of $\mathbb{R}^a$

4.1. **Divisors of Zero:** We know that division is not possible when denominator is a divisor of zero ie., it is not invertible. For the elliptic case where $e_2^2 = -1$, the denominator of equation (3.3) vanishes when

$$\begin{aligned} & (d-cu)^2 + c^2v^2 = 0 \\ \Rightarrow \quad & d - cu = 0 \quad \text{and} \quad v = 0 \\ \Rightarrow \quad & u = d/c \quad \text{and} \quad v = 0, \end{aligned} \tag{4.1}$$

which is a single point. For the parabolic case where we have $e_2^2 = 0$, the denominator of equation (3.4) vanishes when

$$\begin{aligned} & (d-cu)^2 = 0 \\ \Rightarrow \quad & d - cu = 0 \\ \Rightarrow \quad & u = d/c, \end{aligned} \tag{4.2}$$

which is the equation of a line parallel to the V-axis. For the hyperbolic case where $e_2^2 = 1$, the denominator of equation (3.5) vanishes when

$$\begin{aligned} & (d-cu)^2 - c^2v^2 = 0 \\ \Rightarrow \quad & d - cu = \pm cv \\ \Rightarrow \quad & u = d/c \pm v, \end{aligned} \tag{4.3}$$

which is the equation of a light cone not, passing through the origin. Therefore the Möbius transformation (3.1) becomes infinite for these divisors of zero. As a result, we need to compactify our three EPH spaces with points at infinity which in turn can be parametrised with these divisors of zero.

4.2. **Projective Coordinates:** The standard way of nice representations of points at infinity, is done by using projective coordinates. It is a well-known fact that Möbius transformation can be linearised by transition into suitable projective space [23, Cha. 1]. Following [24, §4.2] we consider the one-dimensional projective space $\mathbb{P}_1(\mathbb{R}^a, \mathbb{E})$ and the point element of $\mathbb{P}_1(\mathbb{R}^a, \mathbb{E})$ represents the line $\{[tw_1, tz_1] \mid t \in \mathbb{E}\}$, where the planes $\mathbb{R}^a$ and $\mathbb{E}$ as is explained in Section 2. The symbol $[w_1, z_1]$, where $w_1 = u_1e_1 + v_1e_2 \in \mathbb{R}^a$ and $z_1 = x_1 + y_1e_{12} \in \mathbb{E}$, with $w_1, z_1 \neq (0,0)$, together with the equivalence relation $[w_1, z_1] \sim [tw_1, tz_1]$, where $t \in$ the even Clifford algebra $\mathbb{E}$, which is not a zero-divisor, are called *projective coordinates* or *homogeneous coordinates* of the point $p = [w_1, z_1]$.

Coordinates in the usual sense may be defined as follows. As long as z_1 is not a divisor of zero, the point $p = [w_1, z_1]$ may be uniquely written in the form $p = [w, 1]$, where $w = w_1/z_1 \in \mathbb{R}^a$. This is because w is a vector as the numerator w_1 belongs to the odd part $\mathcal{C}\ell(a)^-$ of $\mathcal{C}\ell(a)$, see Section 2 and the denominator z_1 belongs to the even part $\mathcal{C}\ell(a)^+$ of $\mathcal{C}\ell(a)$. Hence the whole fraction belongs to the odd part. Therefore we can write $w = ue_1 + ve_2 \in \mathbb{R}^a$. The points at infinity are represented by $[w_1, z_1]$, where z_1 is a divisor of zero.

4.3. **Compactification:** The natural action of $SL_2(\mathbb{R})$ on $(\mathbb{R}^a, \mathbb{E})$ induces an action on $\mathbb{P}_1(\mathbb{R}^a, \mathbb{E})$, which is transitive. In terms of projective or homogeneous coordinates $[w_1, z_1]$, the action is simply matrix multiplication as, just like a linear transformation of $\mathbb{R}^2$ is represented by a real 2×2 matrix, so a linear transformation of the space $(\mathbb{R}^a, \mathbb{E})$ is represented by a 2×2 matrix:

$$\begin{pmatrix} w_1 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} w_1' \\ z_1' \end{pmatrix} = \begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} \begin{pmatrix} w_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} aw_1 - be_1z_1 \\ ce_1w_1 + dz_1 \end{pmatrix}.$$

But if $[w_1, z_1]$ and $[w_1', z_1']$ are considered as the homogeneous coordinates in the space $(\mathbb{R}^a, \mathbb{E})$ of the point $p = (w_1/z_1)$ in $\mathbb{R}^a$ and its image point $p' = (w_1'/z_1')$, then the above linear transformation of $(\mathbb{R}^a, \mathbb{E})$ induces the following (non-linear) Möbius transformation of $\mathbb{R}^a$ given by:

$$p=\frac{w_1}{z_1}\mapsto p'=\frac{w_1'}{z_1'}=\frac{aw_1-be_1z_1}{ce_1w_1+dz_1}=\frac{ap-be_1}{ce_1p+d}=\frac{a(ue_1+ve_2)-be_1}{ce_1(ue_1+ve_2)+d}, \quad (4.4)$$

where $p=w_1/z_1=ue_1+ve_2\in\mathbb{R}^a$ provided z_1 is not a divisor of zero. Thus we see that the action on the projective space is the same as the action given by equation (3.1). Also the mapping given by equation (4.4) is defined when the denominator $(ce_1(ue_1+ve_2)+d)$ is not a divisor of zero. Otherwise (4.4) is interpreted as ∞ and the divisors of zero are given by equations (4.1) — (4.3). In the elliptic case, we can explicitly assign the point $[1:0]$ to ∞.

Indeed the initial space $\mathbb{R}^a$ is not a closed set under Möbius transformations. It is well-known that in the elliptic case the problem is solved by the compactification of $\mathbb{R}^e$ with a point ∞ at infinity. Thus in each EPH case the correct compactification is done by

$$\mathbb{R}^a\cup\{\text{points at infinity parametrised by divisors of zero in each case}\},$$

see [12] and [13] for more details. It is common to identify the compactification $\dot{\mathbb{R}}^e$ of the space $\mathbb{R}^e$ with a Riemann sphere. This model can be visualised by the stereographic projection [2, § 18.1.4]. A similar model can be provided for the parabolic and hyperbolic spaces as well [13]. Indeed the space $\mathbb{R}^a$ can be identified with a corresponding surface of constant curvature: the sphere ($e_2^2=-1$), the cylinder ($e_2^2=0$), or the one-sheet hyperboloid ($e_2^2=1$). The map of a surface to $\mathbb{R}^a$ is given by the polar projection, see [13, Fig. 1]. These surfaces provide "compact" model of the corresponding $\mathbb{R}^a$ in the sense that Möbius transformations which are lifted from $\mathbb{R}^a$ by the projection are not singular on those surfaces.

The hyperbolic case however has its own caveats which can be easily seen in the above cited paper, for example. A compactification of the hyperbolic space $\mathbb{R}^h$ by a light cone at infinity will indeed produce a closed Möbius invariant object. However it will not be satisfactory for some other reasons explained below.

The lack of invariance in the hyperbolic case (see Theorem 3.1) has many important consequences in seemingly different areas, for example:

Geometry: $\mathbb{R}^h$ is not split by the real axis into two disjoint pieces: there is a continuous path (through the light cone at infinity) from the upper half plane to the lower which does not cross the real axis (see the sin-like joined two sheets of the hyperbola in Figure 3(a)). In other words by [17] we cannot separate $\mathbb{R}^h$ into interior and exterior of the unit cycle (cf. Definition 4.4).

Physics: There is no Möbius invariant way to separate "past" and "future" parts of the light cone [25], i.e. there is a continuous family

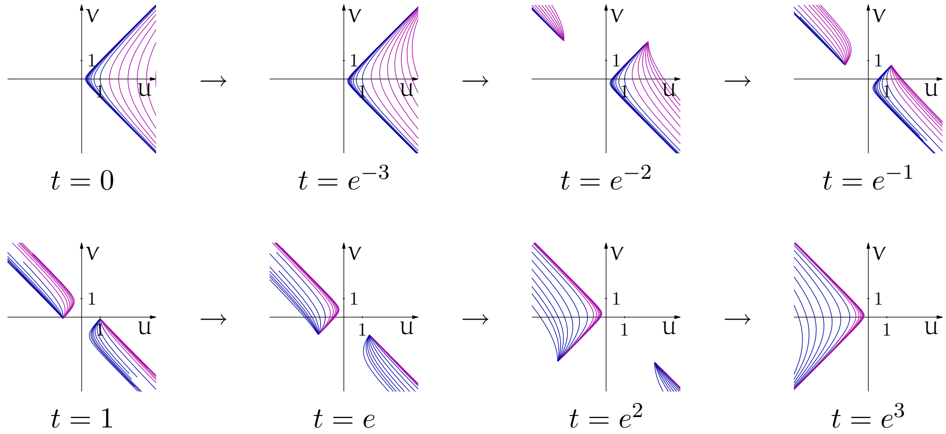

FIGURE 2. Eight frames from a continuous transformation from future to the past parts of the light cone.

of Möbius transformations reversing the arrow of time. For example, the family of matrices $\begin{pmatrix} 1 & -te_2 \\ te_2 & 1 \end{pmatrix}$, $t \in [0, \infty)$ provide the transformations and Figure 2 presents images for eight values of t. On this picture the positive direction of the U-axis is transformed at the end to the negative. This means we cannot separate the time axis into the "future" and the "past" halves, see [17].

Analysis: There is no possibility to split $L_2(\mathbb{R})$ space of functions into a direct sum of the Hardy-like space of functions having an analytic extension into the upper half plane and its non-trivial complement, i.e. any function from $L_2(\mathbb{R})$ has an "analytic extension" into the upper half plane, see [16]. In other words we cannot get a direct sum decomposition

$$L_2 = \mathbb{H}_2 \oplus \mathbb{H}_2^{\perp},$$

where $\mathbb{H}_2$ consists of a sort of "analytic function" and $\mathbb{H}_2^{\perp}$ is non-trivial.

This happens because Möbius transformations mix both sets in each case. All the above problems can be resolved in the following way, see [16, § A.2]. We take two copies $\mathbb{R}^h_+$ and $\mathbb{R}^h_-$ of $\mathbb{R}^h$, depicted by the squares $ACA'C''$ and $A'C'A''C''$ in Figure 3 correspondingly. The boundaries of these squares represent light cones at infinity. We glue $\mathbb{R}^h_+$ and $\mathbb{R}^h_-$ in such a way that the construction is conformally invariant. In other words the construction is invariant under the natural action of the Möbius transformations. That is achieved if in Figure 3 the letters A, A', A'' are identified, the letters B, B' are identified, etc. This aggregate denoted by $\widetilde{\mathbb{R}}^h$ is a two-fold cover of $\mathbb{R}^h$. The hyperbolic

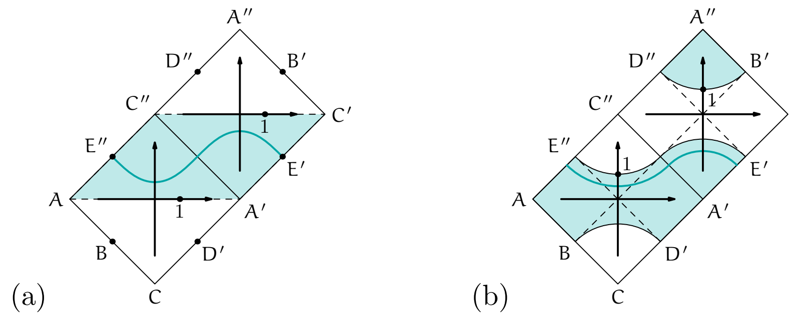

FIGURE 3. Hyperbolic objects in the double cover of $\mathbb{R}^h$:
(a) the "upper" half plane; (b) the unit cycle.

"upper" half plane in $\widetilde{\mathbb{R}}^h$ consists of the upper half plane in $\mathbb{R}^h_+$ and the lower one in $\mathbb{R}^h_-$, cf. Figure 3(a). In other words, if the Möbius transformation acts on a vector in the hyperbolic "upper" half plane in $\widetilde{\mathbb{R}}^h$ then the resulting image vector will also lie in the hyperbolic "upper" half plane in $\widetilde{\mathbb{R}}^h$.

A similar conformally invariant two-fold cover of the Minkowski space-time was constructed in [25, § III.4] in connection with the red shift problem in extragalactic astronomy. It is well-known that the Cayley transform maps the upper half plane to the unit disk and the "real" axis to the unit cycle (cf. Definition 4.4). In our case $\widetilde{\mathbb{R}}^h$, the two-fold cover of $\mathbb{R}^h$, is topologically equivalent to the object known as a *crosscap* [11, p. 117], see Figure 3.

Remark 4.1. (1) The hyperbolic orbit of the subgroup K in $\widetilde{\mathbb{R}}^h$ consists of two branches of the hyperbola passing through $(0, v)$ in $\mathbb{R}^h_+$ and $(0, -v^{-1})$ in $\mathbb{R}^h_-$, see Figure 3. As explained in Remark 2.15 in [18], they both have the same focal length.
(2) The "upper" half plane is bounded by two disjoint "real" axes denoted by AA' and $C'C''$ on Figure 3.

It may be worthwhile to state that for studying the hyperbolic Cayley transform, we need the conformal version of the hyperbolic unit disk. We define it in $\widetilde{\mathbb{R}}^h$ as follows:

$$\begin{aligned}\widetilde{\mathbb{D}} &= \{(ue_1 + ve_2) \mid l_h(ue_1 + ve_2) < -1,\ u \in \mathbb{R}^h_+\}\\ &\quad \cup \{(ue_1 + ve_2) \mid l_h(ue_1 + ve_2) > -1,\ u \in \mathbb{R}^h_-\},\end{aligned}$$

where the Minkowski metric l_h is defined in Lemma 4.2.

Lemma 4.2. *The following are lengths in the sense of Definition 4.3:*
(e) *In the elliptic case: the Euclidean metric* $l_e(ue_1 + ve_2) = u^2 + v^2$
(h) *In the hyperbolic case: the Minkowski metric* $l_h(ue_1 + ve_2) = u^2 - v^2$.

Definition 4.3. By [19] the *radius* r of a cycle is defined by

$$r^2 = \frac{l^2 - \sigma n^2 - km}{k^2}, \tag{4.5}$$

where as before $\sigma = -1, 0, 1$ for the elliptic, parabolic and hyperbolic cases. As usual the *diameter* of a cycle is two times its radius.

Definition 4.4. We use the word *unit cycle* to denote **one of the following**
(e) Circles whose radius is 1 in the elliptic case, i.e. $l_e(ue_1 + ve_2) = 1$;
(h) Rectangular hyperbolas with a vertical axis of symmetry whose radius is (-1) in the hyperbolic case, i.e. $l_h(ue_1 + ve_2) = -1$,

(where l_e, l_h as is defined in Lemma 4.2). The term *unit disk* is used to represent the inner part of circles and both inner and outer parts for rectangular hyperbolas.

Therefore the hyperbolic unit disk $\widetilde{\mathbb{D}}$ consists of all vectors in the inner part of the hyperbola in $\mathbb{R}^h_+$ and the outer part of the hyperbola in $\mathbb{R}^h_-$, see the shaded portion in Figure 3(b). It can be shown that $\widetilde{\mathbb{D}}$ is conformally invariant i.e.,

$$\begin{pmatrix} a & -be_1 \\ ce_1 & d \end{pmatrix} : ue_1 + ve_2 \mapsto u'e_1 + v'e_2,$$

where the vector $u'e_1 + v'e_2$ lies inside the disk $\widetilde{\mathbb{D}}$; it cannot go outside the hyperbolic unit disk $\widetilde{\mathbb{D}}$. Also the hyperbolic disk $\widetilde{\mathbb{D}}$ has a boundary $\widetilde{\mathbb{T}}$—the two copies of the unit cycles in $\mathbb{R}^h_+$ and $\mathbb{R}^h_-$. We call $\widetilde{\mathbb{T}}$ the *(conformal) unit cycle* in $\mathbb{R}^h$, see [16, § A.2] for more details. Figure 3 illustrates the geometry of the "upper" half plane as well as the conformal unit disk in $\widetilde{\mathbb{R}}^h$ conformally equivalent to it.

5. Conclusion

We have discussed about compactification and projective coordinates for all the three EPH cases. It results in- a sphere in the elliptic case, a cylinder in the parabolic and a crosscap in the hyperbolic case. With the help of earlier results, we also establish the isomorphism between the compactified upper half plane and the homogeneous spaces $SL_2(\mathbb{R})/K$ and $SL_2(\mathbb{R})/N$ i.e., the elliptic and parabolic cases. We also establish that the compactification of the entire hyperbolic plane without the "real axis" U i.e., $\mathbb{R}^h \setminus U$ is isomorphic to the homogeneous space $SL_2(\mathbb{R})/A$, i.e., the hyperbolic case.

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